Int. J. Open Problems Compt. Math., Vol. 3, No. 5, December 2010
ISSN 1998-6262;
www.i-csrs.org

Numerical Solution of System of Linear Integral Equations by using Legendre Wavelets

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Abstract

In this paper, a direct method for numerical solution of linear Fredholm integral equations system by using Legendre wavelets is presented. Another method for solving Volterra type system of linear integral equations which uses zeros of Legendre wavelets for collocation points is introduced and used to reduce this type of system of integral equations to a system of algebraic equations.

Keywords: *System of linear integral equations, Legendre wavelets,Linear algebraic equations.*

1 Introduction

Consider the system of linear Fredholm and Volterra integral equations

$$U(x) = F(x) + \int_0^1 K(x,t)U(t)dt, \quad x \in [0,1] \tag{1}$$

and

$$U(x) = F(x) + \int_0^x K(x,t)U(t)dt, \quad x \in [0,1] \tag{2}$$

where

$$\begin{aligned} &U(x) = [u_1(x), u_2(x), ..., u_n(x)]^T, \\ &F(x) = [f_1(x), f_2(x), ..., f_n(x)]^T, \\ &K(x,t) = [k_{ij}(x,t)], \quad i,j = 1,2,\cdots,n. \end{aligned} \tag{3}$$

In system (1),(2) the known kernel $K(x,t)$ is continuous, the function $F(x)$ is given, and $U(x)$ is the solution to be determind.
There are several numerical methods for solving Eq.(1) and (2). For example homotopy perturbation method [1], Adomian decomposition method[2], rationalized Haar functions method [3], Block-Pulse functions mehod[4].
In recent years, wavelets have found their way into many different fields of science and engineerring. The uses of wavelets method for solving integral equations has considered by many authors: Legendre and Chebyshev wavelets method[5, 6], Wavelet Galerkin method[7].
In this study we want to solve system(1) by using Legendre wavelets. The method consist of reducing the solution of (1) to a system of linear algebraic equations and also a collocation method is used for solving system (2).The method consist of expanding the solution by Legendre wavelets with unkhown coefficients. For a suitable collocation points we choose zeros of Legendre wavelets. The properties of Legendre wavelets together with zeros of Legendre wavelets are then utilized to evaluate the unknown coefficients and to find an approximate solution to sysyem (2).

2 Properties of Legendre wavelets

2.1 Wavelets and Legendre wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet.When the dilation parameter a and the translation parameter b very continuously we have the following family of continuous wavelets as [8]

$$\psi_{a,b} = |a|^{\frac{-1}{2}}\psi(\frac{t-b}{a}), \quad a, b \in R, a \neq 0. \tag{4}$$

If we restrict the parameter a and b to discrete values: $a = a_0^{-k}, b = nb_0a_0^{-k}$, where $a_0 > 1, b_0 > 0$ and n, and k can assume any positive integers, we have the following family of discrete wavelets:

$$\psi_{k,n}(t) = |a_0|^{\frac{k}{2}}\psi(a_0^k t - nb_0), \tag{5}$$

where $\psi_{k,n}(t)$ form wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$, then $\psi_{k,n}(t)$ forms an orthonormal basis[8]. Legendre wavelets $\psi_{n,m}(t) = \psi(k, \hat{n}, m, t)$ have four argument;$\hat{n} = 2n-1, n = 1, 2, ..., 2^{k-1}, k$ can assume any positive integer, m is the order for Legendre polynomial and t is the normalized time. They are defind on the interval [0,1) as[9, 10]

$$\psi_{n,m}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}\ 2^{\frac{k}{2}}L_m(2^k t - \hat{n}) & for\ \frac{\hat{n}-1}{2^k} \leq t < \frac{\hat{n}+1}{2^k}, \\ 0 & \text{other wise.} \end{cases} \tag{6}$$

where $m = 0, 1, ..., M-1$ and $n = 1, 2, ..., 2^{k-1}$.The coefficient $\sqrt{m+\frac{1}{2}}$ is for orthonormality, the dilation parameter is $a = 2^{-k}$ and translation parameter is $b = \hat{n}2^{-k}$.Here, $L_m(t)$ are the well-known Legendre polynomials of order m, which are orthogonal with respect to the weight function $w(t) = 1$ and satisfy the following recursive formula:

$$\begin{aligned} &L_0(t) = 1, L_1(t) = t, \\ &L_{m+1}(t) = \frac{2m+1}{m+1} t L_m(t) - \frac{m}{m+1} L_{m-1}(t), \qquad m = 1, 2, 3, ... \end{aligned} \tag{7}$$

2.2 Function approximation

A function $f(t)$ defined over [0,1) may be expanded as

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(t), \tag{8}$$

where $c_{n,m} = (f(t), \psi_{n,m}(t))$, in which $(.,.)$ denotes the inner product.
If the infinite series in Eq.(8)is truncated, then Eq.(8) can be written as

$$f(t) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(t) = C^T \Psi(t), \tag{9}$$

where C and $\Psi(t)$ are $2^{k-1}M \times 1$ matrices given by

$$C = [c_{10}, c_{11}, ..., c_{1M-1}, c_{20}, c_{21}, ..., c_{2M-1}, ..., c_{2^{k-1}0}, c_{2^{k-1}1}, ..., c_{2^{k-1}M-1}]^T, \tag{10}$$

$$\begin{aligned} &\Psi(t) = [\psi_{10}(t), \psi_{11}(t), ..., \psi_{1M-1}(t), \\ &\psi_{20}(t), \psi_{21}(t), ..., \psi_{2M-1}(t), ..., \psi_{2^{k-1}0}(t), \psi_{2^{k-1}1}(t), ..., \psi_{2^{k-1}M-1}(t)]^T. \end{aligned} \tag{11}$$

Similarly a function $k(x,t) \in L^2([0,1] \times [0,1])$ may be approximated as:

$$k(x,t) \simeq \Psi^T(x) K \Psi(t), \tag{12}$$

where K is $2^{k-1}M \times 2^{k-1}M$ matrix, with

$$K_{ij} = (\psi_i(x), (k(x,t), \psi_j(t))). \tag{13}$$

The integration of the product of two Legendre wavelets vector function is obtained as:

$$I = \int_0^1 \Psi(t)\Psi^T(t)dt, \tag{14}$$

where I is an identity matrix.

3 System of linear integral equations

In this section, we use Legendre wavelets method for solving system of linear Fredholm and Volterra integral equations.

3.1 Legendre wavelets method for solving System of linear Fredholm integral equations:

Consider the system of linear Fredholm integral equations as follows:

$$U(x) = F(x) + \int_0^1 K(x,t)U(t)dt, \quad 0 \le x \le 1, \tag{15}$$

where

$$\begin{aligned} &U(x) = [u_1(x), u_2(x), ..., u_n(x)]^T, \\ &F(x) = [f_1(x), f_2(x), ..., f_n(x)]^T, \\ &K(x,t) = [k_{ij}(x,t)], \quad i,j = 1,2,\cdots,n. \end{aligned} \tag{16}$$

In Eq.(15) the functions K and F are given, and U is the vector function of the solution of system(15) that will be determind.Consider the ith equation of (15)

$$u_i(x) = f_i(x) + \int_0^1 \sum_{j=1}^{n} k_{ij}(x,t)u_j(t)dt, \quad i = 1,2,\cdots,n, \tag{17}$$

where $f_i \in L^2[0,1), k_{ij} \in L^2([0,1) \times [0,1))$, and u_i is an unknown function. We approximate f_i, u_i and k_{ij} by (6)-(9) as follows:

$$f_i(x) \simeq {F_i}^T\Psi(x), \quad u_i(x) \simeq {C_i}^T\Psi(x), \quad k_{ij}(x,t) \simeq \Psi^T(x)K_{ij}\Psi(t). \tag{18}$$

By substituting the above relation in (17) we have:

$$\begin{aligned} \Psi^T(x)C_i &= \Psi^T(x)F_i + \int_0^1 \sum_{j=1}^{n} \Psi^T(x)K_{ij}\Psi(t)\Psi^T(t)C_j dt \\ &= \Psi^T(x)F_i + \Psi^T(x)\sum_{j=1}^{n} K_{ij}(\int_0^1 \Psi(t)\Psi^T(t)dt)C_j \\ &= \Psi^T(x)F_i + \Psi^T(x)\sum_{j=1}^{n} K_{ij}C_j \end{aligned} \tag{19}$$

then we have the following linear system:

$$C_i = F_i + \sum_{j=1}^{n} K_{ij}C_j. \tag{20}$$

By solving this linear system, we can find the vector C_i, so

$$u_i(x) \simeq {C_i}^T\Psi(x), \quad i = 1,2,\cdots,n. \tag{21}$$

3.2 Legendre wavelets method for solving System of linear Volterra integral equations:

For solving the system of linear Volterra integral equations (2), we consider the ith equation of (17):

$$u_i(x) = f_i(x) + \int_0^x \sum_{j=1}^{n} k_{ij}(x,t)u_j(t)dt, \quad i = 1,2,\cdots,n, \tag{22}$$

where $f_i \in L^2[0,1), k_{ij} \in L^2([0,1) \times [0,1))$, and u_i is an unknown function. In order to use Legendre wavelets, we first approximate $u_i(x)$ as

$$u_i(x) = C_i^T \Psi(x), \tag{23}$$

where C and $\Psi(x)$ are defined similarly to Eq.(10) and (11). In view of Eqs. (22) and (23) we have

$$C_i^T \Psi(x) = f_i(x) + \int_0^x \sum_{j=1}^{n} k_{ij}(x,t)C_j^T \Psi(x)dt, \quad i = 1,2,\cdots,n. \tag{24}$$

Now we collocate Eq.(24) at $2^{k-1}M$ points.

$$C_i^T \Psi(x_s) = f_i(x_s) + \int_0^{x_s} \sum_{j=1}^{n} k_{ij}(x_s,t)C_j^T \Psi(x_s)dt, \quad i = 1,2,\cdots,n \tag{25}$$

Suitable collocation points are zeros of Chebyshev polynomial [11]

$$x_s = \cos(\frac{(2s+1)\pi}{2^k M}), \quad s = 1,2,\cdots,2^{k-1}M.$$

but in this method we choose zeros of Legendre wavelets for collocation points. For this purpose, we put $k = 1$ in Eq.(6), so the Legendre wavelet of order m is computed as follows

$$\psi_{1,m}(t) = \sqrt{m + \frac{1}{2}}\, 2^{\frac{1}{2}} L_m(2t-1) \tag{26}$$

where $\psi_{1,m}(t)$ are Legendre wavelet defined on the interval $[0,1)$ and $L_m(t)$ are the well-known Legendre polynomials. Eq.(25) gives $(n2^{k-1}M)$ linear equations which can be solved for the vector $C_i,\ i = 1,2,\cdots,n$ in Eq.(23).

4 Illustrative example

To demonstrate the effectiveness of the method,here we consider some system of linear Fredholm and Volterra integral equations. The Legendre wavelets are defined only for $t \in [0,1]$, we take $a = 0, b = 1$. The computations associated with the examples were performed using *Mathematica*.

Example 4.1 *Consider the following system of Fredholm integral equations [12]:*

$$\begin{cases} u_1(x) = f_1(x) + \int_0^1 \frac{x+t}{3}(u_1(t) + u_2(t))dt, \\ \\ u_2(x) = f_2(x) + \int_0^1 xt(u_1(t) + u_2(t))dt, \end{cases} \tag{27}$$

where $f_1(x) = \frac{x}{18} + \frac{17}{36}$ and $f_2(x) = x^2 - \frac{19}{12}x + 1$. We apply the presented method in this paper and solve Eq.(27) with $k = 1$ and $M = 3$. We obtain

$$\begin{aligned} u_1(x) &= (\frac{3}{2})\psi_{10}(x) + (\frac{1}{2\sqrt{3}})\psi_{11} + (0)\psi_{12} = x + 1, \\ u_2(x) &= (\frac{4}{3})\psi_{10}(x) + (\frac{1}{2\sqrt{3}})\psi_{11}(x) + (\frac{1}{6\sqrt{5}})\psi_{12}(x) = x^2 + 1, \end{aligned} \tag{28}$$

which is the exact solution.

Example 4.2 *Consider the following system of Fredholm integral equations [13]:*

$$\begin{cases} u_1(x) = f_1(x) - \int_0^1 t\ cosx\ u_1(t)dt - \int_0^1 x\ sint\ u_2(t)dt, \\ \\ u_2(x) = f_2(x) - \int_0^1 e^{x\ t^2} u_1(t)dt - \int_0^1 (x+t)u_2(t))dt, \end{cases} \tag{29}$$

with $f_1(x) = \frac{cosx}{3} + \frac{xsin^2 1}{2} + x$ and $f_2(x) = \frac{e^x - 1}{2x} + cosx + (x+1)sin1 + cos1 - 1$ and with exact solution $u_1(x) = x, u_2(x) = cosx$.We apply the Legendre wavelets approach and solved Eq.(29).Table (1) presents values of $u_1(x), u_2(x)$ using the present method with $k = 1, M = 5$ together with the exact values.

Table 1

x_i	Exact $u_1(x_i)$	LW $u_1(x_i)$	Exact $u_2(x_i)$	LW $u_2(x_i)$
0.0	0	0	1	1.037920
0.2	0.2	0.199999	0.9800670	0.964591
0.4	0.4	0.400000	0.9210610	0.929857
0.6	0.6	0.600000	0.8253360	0.834141
0.8	0.8	0.800000	0.6967070	0.681227
1.0	1	1	0.5403023	0.578255

Example 4.3 *Consider the following system of Volterra integral equations [14]:*

$$\begin{cases} u_1(x) = f_1(x) + xu_2(x) + 2\int_0^x (xu_1(t) + u_2(t))dt, \\ \\ u_2(x) = f_2(x) - \frac{1}{2}(x+x^2)u_1(x) - \frac{1}{2}\int_0^x (u_1(t) - u_2(t))dt \end{cases} \tag{30}$$

where $f_1(x) = -x^2 - \frac{2}{3}x^4$ and $f_2(x) = x - \frac{1}{4}x^2 + \frac{2}{3}x^3 + \frac{1}{2}x^4$. We apply the presented method for solving Eq.(30) with $k = 1$ and $M = 3$. For this system we get

$$\begin{aligned}
u_1(x) &= (\frac{594749\sqrt{5} - 837205\sqrt{3}}{3(5778\sqrt{3} - 11545\sqrt{5})(-35 + 11\sqrt{15})})\psi_{10}(x) \\
&+ (\frac{594749\sqrt{15} - 2511615}{6(5778\sqrt{3} - 11545\sqrt{5})(-35 + 11\sqrt{15})})\psi_{11}(x) \\
&+ (\frac{594749 - 167441\sqrt{15}}{6(5778\sqrt{3} - 11545\sqrt{5})(-35 + 11\sqrt{15})})\psi_{12}(x) \\
&= x^2, \\
u_2(x) &= (\frac{1}{2})\psi_{10}(x) + (\frac{17334 - 11545\sqrt{15}}{6(5778\sqrt{3} - 11545\sqrt{5})})\psi_{11}(x) + (0)\psi_{12}(x) \\
&= x, \qquad (31)
\end{aligned}$$

which is the exact solution.

Example 4.4 *Consider the following system of Volterra integral equations [15]:*

$$\begin{cases} u_1(x) = f_1(x) + \int_0^x (te^t u_2(t) + u_1(t))dt, \\ u_2(x) = f_2(x) + \int_0^x (-te^t u_1(t) - u_2(t))dt \end{cases} \qquad (32)$$

where $f_1(x) = 1 - \frac{x^2}{2}$ and $f_2(x) = 1 + \frac{x^2}{2}$. The exact solution of (32) are $u_1(x) = e^x$ and $u_2(x) = e^{-x}$. We apply the Legendre wavelets approach and solve Eq.(32). Table (2) presents values of $u_1(x), u_2(x)$ using the present method with $k = 1, M = 5$ together with the exact values.

Table 2

x_i	Exact $u_1(x_i)$	LW $u_1(x_i)$	Exact $u_2(x_i)$	LW $u_2(x_i)$
0.0	1	1.00005	1	0.999979
0.2	1.221400	1.22139	0.818731	0.818734
0.4	1.491820	1.49184	0.670320	0.670314
0.6	1.822120	1.82210	0.548812	0.548818
0.8	2.225540	2.22555	0.449329	0.449326
1.0	2.718282	2.71822	0.367879	0.367898

5 Conclusions

In this paper, Legendre Wavelets method was applied for solving system of linear integral equations. The present method reduced the system of integral equations to a system of linear algebraic equations. Illustrative examples were given to demonstrate the validity and applicability of the technique.

6 Open Problem

In this paper, we discuss solving system of linear integral equations by using Legendre Wavelets method. The method can be examine for solving fractional integral equations.

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