Int. J. Open Problems Compt. Math., Vol. 3, No. 5, December 2010
ISSN 1998-6262; Copyright ©ICSRS Publication, 2010
www.i-csrs.org

The Invariant Subspace Problem and Its Main Developments

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Abstract

The famous mathematician and computer scientist J. von Neumann initiated the research of the invariant subspace problem and its applications. In this paper, we discus the invariant subspace problem and its main developments. In particular, we discus some open sub-problem of the invariant subspace problem.

Keywords: *Banach space, Hilbert space, bounded liner operator, invariant subspace.*

In [42] (p.100), the famous mathematician P. R. Halmos said: "one of the most important, most difficult, and most exasperating unsolved problems of operator theory is the problem of invariant subspace". So far, people have only obtained partial results on this question (see the references cited here and others).

The question is simple state as following: Does every bounded liner operator on a a Banach space have a nontrivial invariant closed subspace? "Nontrivial" means different from both 0 and the whole space; "invariant" means that the operator maps it into itself.

As stated in [12], it was the famous mathematician and computer scientist J. von Neumann who initiated the research of the invariant subspace problem and its applications.

We consider the five part separately:

1 The affirmative answer to some classes of operators

(1) Compact operators and the V. Lomonosov technique

In 1935, J. von Neumann proved that every compact operator on a Hilbert space has a nontrivial invariant closed subspace (cf. [12]), where the orthogonal projection is used.

In 1954, N. Aronszajn and K. T. Smith proved in [12] that every compact operator on a Banach space has a nontrivial invariant closed subspace.

But it was not until 1966 that any substantial progress was made. In that year, A. R. Bernstein and A. Robinson proved in [17] that every polynomially compact operator has a nontrivial invariant closed subspace, where the result is proved using nonstandard analysis. P. R. Halmos studied this paper, and before long he published a proof of this result written in the language of ordinary analysis in [43].

In 1973, V. Lomonosov astounded the mathematical world by proving in [56] that every bounded linear operator on a Banach space which commutes with a nonzero compact operator has a nontrivial invariant closed subspace, where the Schauder fixed point theorem is used.

In 2005, M. Liu [52] proved that the converse proposition of the famous Lomonosov Theorem [56] is true, and obtain some new necessary and sufficient condition for the invariant closed subspace problem.

(2) Subnormal operators and the S. Brown technique

In 1978, S. Brown proved in [21] that every subnormal operator has a nontrivial invariant closed subspace, where the analytic functional calculus and functional algebra are used.

In 1987, S. Brown proved in [22] that every hyponormal operator with the thick spectrum has a nontrivial invariant closed subspace.

In 1990, J. Eschmeier and B. Prunaru proved in [35] that every subdecomposable operator on Banach space with the thick spectrum has a nontrivial invariant closed subspace. In 1994, H. Mohebi and M. Radjabalipour [61] proved various invariant closed subspace theorems on reflexive Banach spaces by weakening decomposability condition of the operator and strengthening the thickness condition of the spectrum .

(3) Contraction operators and the dual algebra

In 1988, S. Brown, B. Chevreau, and C. Pearcy proved in [23] that every contraction operator on a Hilbert space with spectrum containing the unit circle has a nontrivial invariant closed subspace.

In 2004, C. Ambrozie and V. Müller proved in [8] that every polynomially bounded operator T on a Hilbert space such that the spectrum of T contains the unit circle has a nontrivial invariant closed subspace.

2 The affirmative answer to some single operators

In 1949, A. Beruling [20] gave a general from of invariant closed subspaces for the unilateral shift operator T:

$$\mathrm{Lat}(T) = \{\phi \in H^2 : \phi \text{ is a inner function}\}.$$

In 1957, G. K. Kalish [44] gave a general from of invariant closed subspaces for the Volterra operator V:

$$\mathrm{Lat}(V) = M_\alpha : 0 \leq \alpha \leq 1,$$

where $M_\alpha = \{f : f \in L^2[0,1], \text{and} f(t) = 0 \text{ for } 0 \leq \ t \leq \alpha, \text{a.e.}\}$.

3 The negative answer to non-reflexive Banach spaces

In 1987, P. Enflo [33] was the first to construct a continuous operator on a non-reflexive Banach space without a nontrivial invariant closed subspace.

In 1984, C. J. Read [69] presented an example of a continuous operator on l_1 without a nontrivial invariant closed subspace.

However, there are no known examples of operators without nontrivial invariant closed subspaces acting on a reflexive Banach space , and in particular, on a Hilbert space. Furthermore, there seems to be no evidence pertaining to what should be an expected answer for the operators acting on a Hilbert space, and the experts in the field have different opinions on it.

4 Some open sub-problems

(1) Open Problem 1. Does every subnormal operator has a nontrivial hyperinvariant closed subspace?

Kindly Link 1. In 2007, C. Foias, I. B. Jung, E. Ko, and C. Pearcy [39] showed that a special class of subnormal operators has a nontrivial hyperinvariant closed subspace.

(2) Open Problem 2. Does every hyponormal operator has a nontrivial invariant closed subspace?

Kindly Link 2. In 1987, S. Brown proved in [22] that every hyponormal operator with the thick spectrum has a nontrivial invariant closed subspace.

(3) Open Problem 3. Does every positive operator on a Banach lattice has a nontrivial invariant closed subspace?

Kindly Link 3. It is will known that Read's operator (see [72]) has no nontrivial invariant closed subspace, but the modulus of Read's operator has nontrivial invariant closed subspace (see [82]). This result provide the evidence to expect to affirmative answer to Open problem 3. In 2009, G. Sirotkin [80] gave an matrix (operator) on l_1 which has no nontrivial invariant closed subspace, and which has all non-negative entries but one. This example make us closer to the negative answer to Open problem 3.

(4) Open Problem 4. Does every bounded liner operator on a reflexive Banach space has a nontrivial invariant closed subspace?

Kindly Link 4. All known operators having no nontrivial invariant closed subspace act on reflexive Banach spaces.

(5) Open Problem 5. Does the adjoint of every bounded linear operator on a Banach space has a nontrivial invariant closed subspace?

Kindly Link 5. In 2004, C. Ambrozie and V. Müller [8] showed that the adjoint of every polynomially bounded operator on a Banach space whose spectrum contains the unit circle has a nontrivial invariant closed subspace. Moreover, if the answer to Open Problem 5 is affirmative, then the answer to Open Problem 4 is also affirmative.

5 Related subject of research

(1) the invariant closed ideal problem

As stated above, it was in 1954 that N. Aronszajn and K. T. Smith [12] solved the invariant subspace problem of compact operators, but it was not until 1986 that people solved the invariant closed ideal problem of a special kind of compact operators. To be more specific:

In 1986, B. de Pagter [63] proved the long standing conjecture that every positive quasinilpotent compact operator has a nontrivial invariant closed ideal.

(2) Hypercyclic operator

It is well known that interest in cyclic operators arises from the invariant closed subspace problem. In fact, an operator T has no nontrivial invariant closed subspace if and only if each non-zero vector is cyclic for T. Similarly, an operator T has no nontrivial invariant closed subset if and only if each nonzero vector is hypercyclic for T.

S. Rolewicz [74] was the first to isolate the concept of hypercyclicty.

C. J. Read [71] constructed a Banach space operator such that every nonzero vector is hypercyclic, and so this operator has no nontrivial invariant closed subspace.

S. I. Ansari [10] proved that there is a hypercyclic operator on each Banach space.

ACKNOWLEDGEMENTS. The research was supported by the Natural Science Foundation of P. R. China (No.10771039).

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