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A new Bohr-Mollerup type theorem related to gamma function with two parameters

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Abstract

In the note, we give a new Bohr-Mollerup type theorem related to gamma function with two parameters. The main result reads as follows: If $F : (0, \infty) \to (0, \infty)$ ***is logarithmically convex on*** $(0, \infty)$ ***and satisfies the functional equation***

$$F(x + k) = \frac{pkx}{x + pk + k} F(x), x \in (0, \infty); F(k) = 1,$$

for $k > 0$***. Then*** F ***is the*** (p, k)***-gamma function.***

Keywords: (p, k)*-gamma function; Bohr-Mollerup type theorem; logarithmically convex.*

2010 Mathematics Subject Classification: *33B15.*

1 Introduction

It is well-known that the theorem of H. Bohr and J . Mollerup[4] characterizes the Euler gamma function as the uniquely defined log-convex solution f :

$(0,\infty) \to (0,\infty)$ and satisfies the functional equation

$$f(x+1) = xf(x), x \in (0,\infty); f(1) = 1.$$

In 2007, Diaz and Pariguan[3] defined the Γ_k function for $k > 0$, by

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, x \in \mathbf{C} \setminus k\mathbf{Z}^-, \tag{1}$$

where $(x)_{n,k} = x(x+k)(x+2k)...(x+(n-1)k)$. The above definition is a generalization of classical gamma function. Based on definition the Γ_k function, they gave a generalization of theorem of H. Bohr and J . Mollerup. Very recently, K. Nantomah, E. Prempeh and S. B. Twum[6] introduced a new two parameters definition of gamma function as follows:

$$\Gamma_{p,k}(x) = \frac{(p+1)!k^{p+1}(pk)^{\frac{x}{k}-1}}{(x)_{p,k}}, x > 0 \tag{2}$$

where $(x)_{p,k} = x(x+k)(x+2k)...(x+pk)$. They call a $(p;k)$-analogue of the Gamma function, and also provide some identities and inequalities involving this new function. It is easily known that the function $\Gamma_{p,k}(x)$ satisfies the following properties(See definition 2.1 in [6]):

1. $\Gamma_{(p,k)}(x+k) = \dfrac{pkx}{x+pk+k}\Gamma_{(p,k)}(x)$.

2. $\Gamma_{(p,k)}(k) = 1$.

3. $\Gamma_{(p,k)}(x)$ is log-convex, for $x \in (0,\infty)$.

It is noting that K. Nontoman gave a $(p;k)$-analogue of the celebrated Bohr-Mollerup theorem in [5]. The object of this note is to give a new proof of the celebrated Bohr-Mollerup theorem in [5].

2 Main results

Theorem 2.1 *Suppose that* $F : (0,\infty) \to (0,\infty)$ *is a solution of the functional equation*

$$F(x+k) = \frac{pkx}{x+pk+k}F(x), x \in (0,\infty); F(k) = 1,$$

for $k > 0$*. and* F *is logarithmically convex on an interval* $(0,\infty)$*. Then* $F(x) = \Gamma_{(p,k)}(x)$*.*

Proof. Suppose $F(x) \neq \Gamma_{(p,k)}(x)$. Putting $x_1 = nk$, $x_2 = nk + k$, $x_3 = x + nk + x, x \in (0,k)$ and $x_4 = nk + 2k$, we have $x_1 < x_2 < x_3 < x_4$. Since F is logarithmically convex, we easily obtain

$$\frac{\log F(x_2) - \log F(x_1)}{x_2 - x_1} \leq \frac{\log F(x_3) - \log F(x_2)}{x_3 - x_2} \leq \frac{\log F(x_4) - \log F(x_x)}{x_4 - x_2}. \tag{3}$$

That is

$$\frac{1}{k}\log\frac{F(pk+k)}{F(pk)} \leq \frac{1}{x}\log\frac{F(pk+k+x)}{F(pk+k)} \leq \frac{1}{k}\log\frac{F(pk+2k)}{F(pk+k)}. \tag{4}$$

Using the functional equation $F(x+k) = \frac{pkx}{x+pk+k}F(x)$, we get

$$\frac{x}{k}\log F\left(\frac{p^2k^2}{2pk+k}\right) \leq \log\left(\lambda\frac{(x)_{p,k}}{p!k^p}F(x)\right) \leq \frac{x}{k}\log F\left(\frac{pk(pk+k)}{2(pk+k)}\right),$$

where

$$\lambda = \frac{(pk+pk+k)((p-1)k+pk+k)\cdots(k+pk+k)}{[x+pk+pk+k][x+(p-1)k+pk+k]\cdots(x+pk+k)}.$$

So, we have

$$\begin{aligned}&\log\left(\lambda\tfrac{(x)_{p,k}}{p!k^p}F(x)\right) - \tfrac{x}{k}\log F\left(\tfrac{p^2k^2}{2pk+k}\right)\\ &\leq \tfrac{x}{k}\log F\left(\tfrac{pk(pk+k)}{2(pk+k)}\right) - \tfrac{x}{k}\log F\left(\tfrac{p^2k^2}{2pk+k}\right).\end{aligned}$$

It may rewrite

$$0 \leq \log\left(\lambda\frac{(x)_{p,k}}{(1+p)!k^{p+1}(pk)^{\frac{x}{k}-1}}F(x)\right) \leq \log\left(\frac{1}{\lambda}\frac{p}{p+1}\frac{1}{2^{\frac{x}{k}}}\right) < 0,$$

where $\lim\limits_{p\to\infty}\frac{1}{\lambda}\frac{p}{p+1} = 1$. A contradiction. The proof is complete.

3 Further Comments and Open Problem

Recently, Bhayo and Yin[[1, 2]] studied generalized convexity and concavity, and proved the following results:

Theorem 3.1 ([1, Theorem 1, p138]) *Let $f : I \to (0,\infty)$ be a continuous function and $I \subseteq (0,\infty)$, then*
(1) $L(f(x), f(y)) \geq (\leq) f(L(x,y))$,
(2) $L(f(x), f(y)) \geq (\leq) f(A(x,y))$,
when f is increasing and log-convex(concave).

Theorem 3.2 ([2, Theorem 1, p6]) *Let $f : I \to (0, \infty)$ and $I \subseteq (0, \infty)$. Then the following inequality holds true:*

$$I(f(x), f(y)) \geq f(I(x, y))$$

$$(I(f(x), f(y)) \leq f(A(x, y)))$$

If the function $f(x)$ is a continuously differentiable, increasing and log-convex (concave).

Here, $I(x, y), L(x, y)$ and $A(x, y)$ are defined by

$$I(x, y) = \frac{1}{e} \left(\frac{x^x}{y^y} \right)^{\frac{1}{x-y}}, \quad x \neq y,$$

$$L(x, y) = \frac{x - y}{\log x - \log y}, \quad x \neq y,$$

and $A(x, y) = \frac{x+y}{2}$ respectively. Let $f : I \to (0, \infty)$ be continuous, where I is a sub-interval of $(0, \infty)$. Let M and N be the means defined above, then we call that the function f is MN-convex (concave) if

$$f(M(x, y)) \leq (\geq) N(f(x), f(y)) \quad for \quad all \quad x, y \in I.$$

Theorem 3.1 and 3.2 imply that increasing log-convex means LL-convex and II-convex. So we propose an open problem:

Open problem 3.1 If $g : (0, \infty) \to (0, \infty)$ is LL-convex(or II-convex, or GG-convex) on an interval $(0, \infty)$, and satisfies the functional equation

$$g(x + k) = \frac{pkx}{x + pk + k} g(x), x \in (0, \infty); g(k) = 1, \tag{5}$$

for $k > 0$. Then is g the (p, k) - gamma function?

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