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An invariant subspace method for solving a class of fractional diffusion-wave problems

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Abstract

In this article, we present a reliable numerical method for solving fractional diffusion-wave equations of order $2\alpha \in (1,2]$***. It is based on an invariant subspaces method. It is an easy and efficient method to investigate the solution of this type of problems. The fractional derivative is described in the Caputo sense. Numerical and theoretical results are presented.***

Keywords: *Fractional diffusion-wave equation, An invariant subspaces method, Caputo derivative.*
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1 Introduction

Fractional diffusion-wave equations (FDWE) have several applications in physics and engineering such as electromagnetic, acoustic and mechanical responses [1]-[2]. Several numerical methods have been used to solve such problems. For example, finite-difference methods is used to discretize the fractional derivatives [3]-[7]. Low-order finite-elements have also used to solve it [8]-[10]. Spectral methods have been investigated to solve such problems. For example, spectral Legendre collocation scheme, sinc function approximation for time and space respectively is used in [11]. Mridula [12] solved it using matrix method and in [13], authors investigate the solvability and stability by the matrix method and fractional Fourier method. In [14], a spectral tau algorithm based on Jacobi operational matrix is presented for solving FDWE. Bhrawy [15] developed a numerical method for solving FDWE using spectral shifted Legendre Gauss-Lobatto collocation method. Recently, the Legendre-collocation methods and convergence analysis for nonlinear Volterra type integral equations in [16] and Jacobi spectral-collocation method for fractional integro-differential equations in [17, 18]. Spectral methods based on the Petrov-Galerkin formulation were also constructed for fractional integro-differential equations in [19] and Volterra integral equations with weakly singular kernel in [20]. An invariant subspaces method is used to solve this type of problems [21]-[25].

In this paper, we present a reliable solution of time-fractional non-homogeneous equation of the form

$$D_t^{2\alpha} w(x,t) = a w_{xx}(x,t) + b w_x(x,t) + c w(x,t) + g(x,t), \quad \frac{1}{2} < \alpha < 1$$

subject to

$$\begin{aligned} w(x,0) &= \phi(x), \\ D_t^{\alpha} w(x,0) &= \psi(x), \end{aligned}$$

where $\phi(x)$, $\psi(x)$, and $g(x,t)$ are continuous functions, $a > 0, b$, and $c \leq 0$ are contents. The derivative in the above equation is the Caputo derivative. The Caputo derivative of order $\rho > 0$ is defined by

$$D_t^{\rho} w(x,t) = \frac{1}{\Gamma(n-\rho)} \int\limits_0^t \frac{\partial^2 w(x,s)}{\partial s^2} (t-s)^{n-1-\rho} ds, t > 0$$

where $\Gamma(.)$ denotes the Gamma function and n is the smallest integer less than or equal ρ. We use the following property of the Caputo derivative in this paper which is

$$D_t^{\rho} t^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\rho+\nu+1)} t^{\nu-\rho}$$

when $\nu \geq \rho$ and $D_t^{\rho} t^{\nu} = 0$ when $0 \leq \nu < \rho$. We organize the paper as follows. Analysis of the proposed method is presented in section 2. In section 3, three numerical examples are presented. In section 4; drawn conclusions are presented.

2 Analysis of the method

Consider the following class of time-fractional non-homogeneous equation of the form

$$D_t^{2\alpha} w(x,t) = a w_{xx}(x,t) + b w_x(x,t) + c w(x,t) + g(x,t), \quad \frac{1}{2} < \alpha < 1 \tag{1}$$

subject to

$$\begin{aligned} w(x,0) &= \phi(x) \\ D_t^{\alpha} w(x,0) &= \psi(x) \end{aligned} \tag{2}$$

where $a > 0, b, c \leq 0$ are constants, $\phi(x)$, $\psi(x)$, and $g(x,t)$ are continues functions. Further conditions on g will be added later. Let

$$W_n = Span\{c_1(t), c_2(t), ..., c_n(t)\}$$

where $c_1(t), c_2(t), ..., c_n(t)$ are n linearly independent functions. Let

$$g(x,t) = \sum_{i=1}^{n} c_i(t) g_i(x).$$

A finite dimensional linear subspace W_n is said to be invariant with respect to a differential operator ℓ if $\ell W_n \subseteq W_n$ i.e.; $\ell w \in W_n$, for all $w \in W_n$ where

$$\ell w = D_t^{2\alpha} w.$$

Let the proposed problem admits an invariant subspace W_n. Then, there exists n functions $\phi_1, \ \phi_2, \ ..., \ \phi_n$ such that

$$\ell\left(\sum_{i=1}^{n} c_i(t) f_i(x)\right) = \sum_{i=1}^{n} \phi_i(f_1(x), f_2(x), ..., f_n(x)) \ c_i(t),$$

where $\{\phi_i\}_{i=1}^{n}$ are the expansion coefficients of $\ell(w) \in W_n$ in the basis $\{c_i\}$. Thus, the solution of the proposed problem has the form

$$w(x,t) = \sum_{i=1}^{n} c_i(t) f_i(x)$$

where the coefficients $f_1(x), f_2(x), ..., f_n(x)$ satisfy a system of ordinary differential equations

$$a f_i''(x) + b f_i'(x) + c f_i(x) + g_i(x) = \phi_i(f_1(x), f_2(x), ..., f_n(x)), \ i = 1, 2, ..., n. \tag{3}$$

Using the initial conditions in Equation (2), we get two more equations

$$\begin{aligned} w(x,0) &= \phi(x) = \sum_{i=1}^{n} c_i(0) f_i(x), \\ D_t^{\alpha} w(x,0) &= \psi(x) = \sum_{i=1}^{n} D_t^{\alpha} c_i(0) f_i(x). \end{aligned} \tag{4}$$

We solve systems 3 and 4 to find $f_1(x), f_2(x), ..., f_n(x)$.

Theorem 2.1 *Let W_n be the linear space spanned by n linearly independent functions $\{c_1(t), c_2(t), ..., c_n(t)\}$ and assume that W_n is invariant under the operator $D_t^{2\alpha}$. Then, there exists n functions $\phi_1, \phi_2, ..., \phi_n$ such that*

$$D_t^{2\alpha}\left(\sum_{i=1}^{n} c_i(t) f_i(x)\right) = \sum_{i=1}^{n} c_i(t) \phi_i(f_1(x), f_2(x), ..., f_n(x))$$

where $\{\phi_1, \phi_2, ..., \phi_n\}$ is the expansion coefficients of $D_t^{2\alpha} w \in W_n$ in the basis $\{c_1(t), c_2(t), ..., c_n(t)\}$. Thus, the solution of Equation (1)-(2) has the form

$$w(x,t) = \sum_{i=1}^{n} c_i(t) f_i(x)$$

where $\{f_1(x), f_2(x), ..., f_n(x)\}$ satisfy the following system

$$a f_i''(x) + b f_i'(x) + c f_i(x) + g_i(x) = \phi_i(f_1(x), f_2(x), ..., f_n(x)), \;\; i = 1, 2, ..., n.$$

Proof. Direct substitution implies that

$$\begin{aligned} D_t^{2\alpha}\left(\sum_{i=1}^{n} c_i(t) f_i(x)\right) &= \sum_{i=1}^{n} c_i(t) \phi_i(f_1(x), f_2(x), ..., f_n(x)) \\ &= \sum_{i=1}^{n} \left(a f_i''(x) + b f_i'(x) + c f_i(x) + g_i(x)\right) c_i(t). \end{aligned}$$

Since $c_1(t), c_2(t), ..., c_n(t)$ are linearly independent, we get

$$a f_i''(x) + b f_i'(x) + c f_i(x) + g_i(x) = \phi_i(f_1(x), f_2(x), ..., f_n(x)), \;\; i = 1, 2, ..., n.$$

■

3 Numerical Results

In this section, we present three of our examples to show the efficiency of the proposed method.

Example 1 *Consider the following fractional diffusion-wave equation*

$$\begin{aligned} D_t^{2\alpha} w(x,t) &= w_{xx}(x,t) - 2w_x(x,t) - 4w(x,t) \\ &+ \frac{15\sqrt{\pi}}{8\Gamma(\frac{7}{2}-2\alpha)} t^{\frac{5}{2}-2\alpha}(1+x^2) + t^{\frac{5}{2}}(4x^2+4x+2), \quad \frac{1}{2} < \alpha < 1 \end{aligned} \tag{5}$$

subject to

$$\begin{aligned} w(x,0) &= 0 \\ D_t^{\alpha} w(x,0) &= 0. \end{aligned} \tag{6}$$

Let

$$W_2 = Span\{1, t^{\frac{5}{2}}\}. \tag{7}$$

Then,

$$w(x,t) = c_0(x) + c_1(x) t^{\frac{5}{2}}. \tag{8}$$

Applying the first initial condition, we reach to the fact that $w(x,0) = c_0(x) = 0$. *Now, applying the operator* $D_t^{2\alpha}$ *on Equation (8) to lead to*

$$D_t^{2\alpha} w(x,t) = c_1(x) \frac{\Gamma(\frac{7}{2})}{\Gamma(\frac{7}{2}-2\alpha)} t^{\frac{5}{2}-2\alpha}. \tag{9}$$

Then, we substitute both Equations (8) and (9) in Equation (5) and we collect the coefficients of same t^i *and set them to be zero. This will result in the following system*

$$\begin{aligned} 0 &= c_1''(x) - 2c_1'(x) - 4c_1(x) + 4x^2 + 4x + 2, \\ 0 &= c_1(x) - 1 - x^2. \end{aligned} \tag{10}$$

Hence,

$$c_1(x) = 1 + x^2.$$

Therefor, we get the exact solution

$$w(x,t) = t^{\frac{5}{2}} \left(1 + x^2\right).$$

Example 2 *Consider the following fractional diffusion-wave equation*

$$D_t^{2\alpha} w(x,t) = w_{xx}(x,t) + \left(\frac{6}{\Gamma(4-2\alpha)} t^{3-2\alpha} - t^3\right) e^x, \quad \frac{1}{2} < \alpha < 1 \tag{11}$$

subject to

$$\begin{aligned} w(x,0) &= 0, \\ D_t^{\alpha} w(x,0) &= 0. \end{aligned} \tag{12}$$

Let

$$W_2 = Span\{1, t^3\}. \tag{13}$$

Then,

$$w(x,t) = c_0(x) + c_1(x)t^3. \tag{14}$$

Applying the first initial condition, we reach to the fact that $w(x,0) = c_0(x) = 0$*. Now, applying the operator* $D_t^{2\alpha}$ *on Equation (14) to lead to*

$$D_t^{2\alpha} w(x,t) = c_1(x) \frac{\Gamma(4)}{\Gamma(4-2\alpha)} t^{3-2\alpha}. \tag{15}$$

Then, we substitute both Equations (15) and (14) in Equation (11) and we collect the coefficients of same t^i *and set them to be zero. This will result in the following system*

$$\begin{aligned} 0 &= c_1''(x), \\ 0 &= c_1(x) - e^x. \end{aligned} \tag{16}$$

Hence,

$$c_1(x) = e^x.$$

Therefor, we get the exact solution

$$w(x,t) = t^3 e^x.$$

Example 3 *Consider the following fractional diffusion-wave equation*

$$\begin{aligned} D_t^{2\alpha} w(x,t) &= w_{xx}(x,t) - w(x,t) - 12x^2 + xt + t^3 - x^4 \\ &- \frac{6}{\Gamma(4-2\alpha)} t^{3-2\alpha} - \frac{2}{\Gamma(3-2\alpha)} t^{2-2\alpha} \sin(x), \quad \frac{1}{2} < \alpha < 1 \end{aligned} \tag{17}$$

subject to

$$\begin{aligned} w(x,0) &= x^4, \\ D_t^{\alpha} w(x,0) &= 0. \end{aligned} \tag{18}$$

Let

$$W_4 = Span\{1, t, t^2, t^3\}. \tag{19}$$

Then,

$$w(x,t) = c_0(x) + c_1(x)t + c_2(x)t^2 + c_3(x)t^3. \tag{20}$$

Applying the first initial condition, we reach to the fact that $w(x,0) = c_0(x) = x^4$. *Now, applying the operator* $D_t^{2\alpha}$ *on equation (20) to lead to*

$$D_t^{2\alpha} w(x,t) = c_2(x)\frac{\Gamma(3)}{\Gamma(3-2\alpha)}t^{2-2\alpha} + c_3(x)\frac{\Gamma(4)}{\Gamma(4-2\alpha)}t^{3-2\alpha}. \tag{21}$$

Then, we substitute both Equations (21) and (20) in Equation (17) and we collect the coefficients of same t^i *and set them to be zero. This will result in the following system*

$$\begin{aligned} 0 &= c_1''(x) - c_1(x) + x, \\ 0 &= c_2''(x) - c_2(x), \\ 0 &= c_3''(x) - c_3(x) + 1, \\ 0 &= c_2(x) + \sin(x), \\ 0 &= c_3(x) + 1. \end{aligned}$$

Hence,

$$\begin{aligned} c_1(x) &= x + k_1\ e^{x} + k_2\ e^{-x}, \\ c_2(x) &= -\sin(x), \\ c_3(x) &= -1, \end{aligned}$$

where $k_1,\ k_2$ *are free constants. Therefor, we get the exact solution*

$$w(x,t) = x^4 + (x + k_1\ e^{x} + k_2\ e^{-x})t - \sin(x)t^2 - t^3.$$

4 Conclusions

In this paper, we consider the following class of time-fractional non-homogeneous equation of the form

$$D_t^{2\alpha} w(x,t) = a w_{xx}(x,t) + b w_x(x,t) + c w(x,t) + g(x,t), \quad \frac{1}{2} < \alpha < 1 \tag{22}$$

subject to

$$\begin{aligned} w(x,0) &= \phi(x) \\ D_t^{\alpha} w(x,0) &= \psi(x) \end{aligned} \tag{23}$$

where $a > 0, b, c \leq 0$ are constants, $\phi(x)$, $\psi(x)$, and $g(x,t)$ are continues functions. We assume the solution

$$w(x,t) \in Span\{c_1(t), c_1(t), ..., c_n(t)\}.$$

Then, we substitute the series solution in the proposed problem and we collect the coefficients of same $c_i(t)$ and set them to be zero. We produce a system of algebraic differential equation to find the coefficient functions. We present three of our example. We get the exact solution in them. This shows that the proposed numerical method is an efficient. In addition, the proposed method is easy to impanelment and its complexity is cheap. We will end this section with the following open problem:

Open Problem

How to generalize this technique for the nonlinear problems?

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

References

[1] R.R. Nigmatullin, To the theoretical explanation of the universal response, Phys. Solid State B 123 (1984) 739–745.

[2] R.R. Nigmatullin, The realization of the generalized transfer equation in a medium with fractal geometry, Phys. Solid State B 133 (1986) 425–430.

[3] N.H. Sweilam, T.A.R. Assiri, Numerical simulations for the space–time variable order nonlinear fractional wave equation, J. Appl. Math. (2013) 7. Article ID 586870.

[4] F. Liu, M.M. Meerschaert, R.J. McGough, P. Zhuang, Q. Liu, Numerical methods for solving the multi-term timefractional wave-diffusion equation, Fract.Calc. Appl. Anal. 16 (1) (2013) 9–25.

[5] T.A.M. Langlands, B.I. Henry, The accuracy and stability of an implicit solution method for the fractional diffusion equation, J. Comput. Phys. 205 (2) (2005) 719–736.

[6] Y. Lin, C. Xu, Finite difference/spectral approximations for the time-fractional diffusion equation, J. Comput. Phys. 225 (2) (2007) 1533–1552.

[7] C.M. Chen, F. Liu, K. Burrage, Finite difference methods and a Fourier analysis for the fractional reaction-subdiffusion equation, Appl. Math. Comput. 198 (2) (2008) 754–769.

[8] G.J. Fix, J.P. Roop, Least squares finite-element solution of a fractional order two-point boundary value problem, Comput. Math. Appl. 48 (7–8) (2004) 1017–1033.

[9] J.P. Roop, Computational aspects of FEM approximation of fractional advection dispersion equations on bounded domains in R2, J. Comput. Appl. Math. 193 (1) (2006) 243–268.

[10] W.H. Deng, Finite elementmethod for the space and time fractional Fokker–Planck equation, SIAM J. Numer. Anal. 47 (1) (2008) 204–226.

[11] H. Osama, S.F. Fadhel, G.S. Mohammed, Numerical solution for the time-fractional diffusion-wave equations by using Sinc-Legendre collocation method, Math. Theory Model. 5 (1) (2015) 49–57.

[12] M. Garg, P. Manohar, Matrix method for numerical solution of space–time fractional diffusion-wave equations with three space variables, Afrika Mat. 25 (2014) 161–181.

[13] H.F. Ding, C.P. Li, Numerical algorithms for the fractional diffusion-wave equation with reaction term, Abstr. Appl. Anal. (2013) 15. Article ID 493406.

[14] A.H. Bhrawy, E.H. Doha, D. Baleanu, S.S. Ezz-Eldien, A spectral tau algorithm based on Jacobi operational matrix for numerical solution of time fractional diffusion-wave equations, J. Comput. Phys. 293 (2015) 142–156.

[15] A.H. Bhrawy, A new legendre collocation method for solving a two-dimensional fractional diffusion equation, Abstr. Appl. Anal. (2014) 10. Article ID 636191.

[16] Y. Yang, Y.P. Chen, Y.Q. Huang, W. Yang, Convergence analysis of Legendre-collocation methods for nonlinear Volterra type integral Equations, Adv. Appl. Math. Mech. 7 (1) (2015) 74–88.

[17] Y. Yang, Y. Chen, Y. Huang, Spectral-collocation method for fractional Fredholm integro-differential equations, J. Korean Math. Soc. 51 (1) (2014) 203–224.

[18] Y. Yang, Y.P. Chen, Y.Q. Huang, Convergence analysis of the Jacobi spectral-collocation method for fractional integro-differential equations, Acta Math. Sci. 34B (3) (2014) 673–690.

[19] Y. Yang, Jacobi spectral Galerkin methods for Volterra integral equations with weakly singular kernel, Bull. Korean Math. Soc. 53 (1) (2016) 247–262.

[20] Y. Yang, Jacobi spectral Galerkin methods for fractional integro-differential equations, Calcolo 52 (4) (2015) 519–542.

[21] Galaktionov VA , Svirshchevskii SR . Exact solutions and invariant subspaces of nonlinear partial differential equations in mechanics and physics. Lon- don: Chapman and Hall/CRC; 2007.

[22] Gazizov RK , Kasatkin AA . Construction of exact solutions for fractional order differential equations by invariant subspace method. Comput Math Appl 2013;66:576–84.

[23] Sahadevan R , Bakkyaraj T . Invariant subspace method and exact solutions of certain nonlinear time fractional partial differential equations. Fract Calc Appl Anal 2015;18:146–62.

[24] Harris PA , Garra R . Analytic solution of nonlinear fractional burgers-type equation by invariant subspace method. Nonlinear Stud 2013;20(4):471–81.

[25] Harris PA , Garra R . Nonlinear time-fractional dispersive equations. Commun Appl Indus Math 2014;6(1).