

Evaluation of a logarithmic integral

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Abstract

We evaluate a logarithmic integral, for certain integer values of its parameters, in terms of the Riemann zeta function and the Stirling numbers of the first kind. Finally, we pose some related open problems.

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1 Introduction and main result

In their recent article [2] Xu and Yin evaluated, for integers $m, n > 1$ the logarithmic integral

$$I_{n,m} := \int_0^1 x^{n-1} \log^m(1-x) dx$$

and posed the open problem [2, Section 3, Eq. (3.1)] to compute $I_{n,m}$ for integers $m > 1$ and $n < 0$. We shall give a representation of $I_{-n,m}$ in terms of the Riemann zeta function and the Stirling numbers of the first kind.

Theorem 1.1 *For integers $m \in \mathbb{N}$ and $n = 0, 1, \dots, m-1$,*

$$I_{-n,m} = (-1)^m \frac{m!}{n!} \sum_{j=0}^n (-1)^{n-j} s(n, j) \zeta(m+1-j),$$

where $\zeta(z) = \sum_{k=1}^{\infty} 1/k^z$ ($\operatorname{Re} z > 1$) is the Riemann zeta function and $s(n, j)$ denote the Stirling numbers of the first kind.

Recall that the Stirling numbers of the first kind $s(n, j)$ are defined to be the coefficients occurring when expanding the falling factorial as a linear combination of powers (see [1, Eq. [5e] on page 213]):

$$z^n = \sum_{j=0}^n s(n, j) z^j,$$

where $z^0 = 1$, $z^n = z(z-1)\cdots(z-n+1)$, for $n \in \mathbb{N}$. Note that an immediate consequence is the representation (see [1, Eq. [5f] on page 213])

$$z^{\bar{n}} = \sum_{j=0}^n (-1)^{n-j} s(n, j) z^j, \quad (1)$$

for the rising factorial $z^{\bar{0}} = 1$, $z^{\bar{n}} = z(z-1)\cdots(z-n+1)$, for $n \in \mathbb{N}$. Note that $(-1)^{n-j} s(n, j) \geq 0$.

Proof. The change of variable $x = 1 - e^{-t}$ yields

$$I_{-n, m} = \int_0^\infty (1 - e^{-t})^{-n-1} (-t)^m e^{-t} dt.$$

Using the binomial theorem we have, for each $R > 0$,

$$\begin{aligned} & \int_0^R (1 - e^{-t})^{-n-1} (-t)^m e^{-t} dt \\ &= (-1)^m \sum_{k=0}^\infty \binom{n+k}{n} \int_0^R t^m e^{-(k+1)t} dt \\ &= (-1)^m \sum_{k=0}^\infty \binom{n+k}{n} \frac{1}{(k+1)^{m+1}} \int_0^{(k+1)R} t^m e^{-t} dt. \end{aligned}$$

Because $\int_0^{(k+1)R} t^m e^{-t} dt = m! - O(e^{-\gamma R})$ as $R \rightarrow +\infty$, with a constant $\gamma > 0$ which depends only on m , we conclude that

$$I_{-n, m} = (-1)^m m! \sum_{k=0}^\infty \binom{n+k}{n} \frac{1}{(k+1)^{m+1}}. \quad (2)$$

By virtue of Eq. (1), we have

$$\binom{n+k}{n} = \frac{1}{n!} (k+1)^{\bar{n}} = \frac{1}{n!} \sum_{j=0}^n (-1)^{n-j} s(n, j) (k+1)^j.$$

This leads to

$$I_{-n, m} = (-1)^m \frac{m!}{n!} \sum_{j=0}^n (-1)^{n-j} s(n, j) \sum_{k=0}^\infty \frac{1}{(k+1)^{m+1-j}}$$

which completes the proof. ■

We list $I_{-n,m}$ for the special cases $n = 0, 1, \dots, 4$:

$$\begin{aligned} I_{0,m} &= (-1)^m m! \zeta(m+1), \quad \text{for } m \geq 1, \\ I_{-1,m} &= (-1)^m m! \zeta(m), \quad \text{for } m \geq 2, \\ I_{-2,m} &= (-1)^m \frac{m!}{2} (\zeta(m-1) + \zeta(m)), \quad \text{for } m \geq 3, \\ I_{-3,m} &= (-1)^m \frac{m!}{6} (\zeta(m-2) + 3\zeta(m-1) + 2\zeta(m)), \quad \text{for } m \geq 4, \\ I_{-4,m} &= (-1)^m \frac{m!}{24} (\zeta(m-3) + 6\zeta(m-2) + 11\zeta(m-1) + 6\zeta(m)), \quad \text{for } m \geq 5. \end{aligned}$$

Using the well-known formula

$$\zeta(2n) = (-1)^{n-1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n},$$

where B_n denotes the n -th Bernoulli number, we give the special instances for $m = 4$:

$$\begin{aligned} I_{0,4} &= 24\zeta(5), \quad I_{-1,4} = 4\pi^4/15, \quad I_{-2,4} = 12\zeta(3) + 2\pi^4/15, \\ I_{-3,4} &= 2\pi^2/3 + 12\zeta(3) + 4\pi^4/45. \end{aligned}$$

Remark 1.2 Note that Eq. (2) is valid also for real values of n such that $0 < n < m$ if rewritten in the form

$$I_{-n,m} = (-1)^m m! \sum_{k=0}^{\infty} \binom{n+k}{k} \frac{1}{(k+1)^{m+1}}.$$

2 Open Problems

Compute the integrals

$$I_{n,m} = \int_0^1 x^{n-1} \log^m(1-x) dx,$$

for integers m and real parameters n , satisfying $m+n > 0$.

We list some instances determined by a computer algebra system:

- We have

$$\begin{aligned} I_{1/2,1} &= -4 + \log 16, \\ I_{1/2,2} &= -\frac{2}{3}\pi^2 + 16 - 16 \log 2 + 2 \log^2 4. \end{aligned}$$

Give an explicit representation of $I_{1/2,n}$.

- We have

$$\begin{aligned} I_{1/3,1} &= (\sqrt{3}\pi + 9 \log 3) / 2 - 9, \\ I_{1/4,1} &= 2\pi + 12 \log 2 - 16, \\ I_{1/5,1} &= \frac{\pi}{2} \sqrt{5(5 + 2\sqrt{5})} + \frac{25}{4} \log 5 + \frac{5}{2} \sqrt{5} \operatorname{arccoth} \sqrt{5} - 25. \end{aligned}$$

Give an explicit representation of $I_{1/k,1}$, for $k \in \mathbb{N}$.

- We close with the following problem: We have

$$\begin{aligned} I_{2,-1} &= -\log 2, \\ I_{3,-1} &= -\log(4/3), \\ I_{4,-1} &= -\log(32/27). \end{aligned}$$

Is it true that, for all integers $n \geq 2$,

$$I_{n,-1} = -\log q_n,$$

where q_n is a rational number? If so determine the explicit value of q_n .

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References

- [1] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, 1974.
- [2] Hua-Zhong Xu and Li Yin, *A logarithmic integral related to generalized harmonic numbers*, Int. J. Open Problems Compt. Math., Vol. 10, No. 2, June 2017, 1–4.