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Solutions and Stabilities for a $2D-$Non Homogeneous Lane-Emden Fractional System

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Abstract

In this work, we are concerned with a two dimension fractional Lane Emden differential system with right hand side depending on an unknown vector function. Using Banach contraction principle on an appropriate product Banach space, we establish some results on the existence and uniqueness of solutions. The existence of at least one solution of the considered problem is also studied. Some notions of Ulam type stabilities are presented and illustrated. At the end, an example is discussed.

Keywords: *Caputo derivative, fixed point, existence, Lane-Emden system of two dimension, uniqueness.*
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1 Introduction

Recently fractional calculus started to attract serious attention in a lot of scientific areas, such as mathematics, biology and engineering. For getting a better understanding of the theory we suggest the reader to address the

following papers [12, 14, 15] and the reference therein. It is also important to mention how relevant it is to do research on fractional deferential equations. Nowadays many branches of science and technology are making use of this theory (see [4, 13, 18]). The existence and uniqueness of solutions for nonlinear fractional differential equations was studied by many scholars. For getting further information the reader can address the following papers [6, 9, 20].
Along with that it is needed to mention that the Ulam type stabilities for fractional differential problems are useful for solving practical problems in biology, economics and mechanics. The examples of the application of this theory can be found in [1, 3, 11].
Now we would like to bring to the attention the Lane-Emden model, which serves as the basis for our research.
It is generally known that the Lame-Emden equations are found in a few models of mathematical physics and astrophysics, such as aspects of stellar structure, isothermal gas spheres and thermionic currents [5]. The Lane-Emden equation has the following form:

$$x^{''}(t)+\frac{a}{t}x^{'}(t)+f(t,x(t))=g(t),\ t\in[0,1],$$

with the initial conditions:

$$x(0)=A,\ x^{'}(0)=B,$$

where A and B are constants, f,g are continuous real valued functions. This equation and the problems related to it has occupied the minds of a number of researchers. For getting further information the reader is recommended to turn to [8, 16, 19].
In [10] the authors studied coupled Lane-Emden equations arising in catalytic diffusion reaction by reproducing kernel Hilbert space method while giving consideration to the following problem:

$$\begin{cases} u''(x)+\frac{K_1}{x}u'(x)=f_1(u,v), 0<x\leq 1 \\ \\ v''(x)+\frac{K_2}{x}v'(x)=f_2(u,v), 0<x\leq 1, \end{cases}$$

subject to the initial conditions

$$\begin{array}{c} u'(0)=0, u(1)=\alpha_1, \\ v'(0)=0, v(1)=\alpha_2, \end{array}$$

where K_1 and K_2 are constants, $f_1(u,v)$ and $f_2(u,v)$ are analytic functions in u and v. Such boundary value problems arise in catalytic diffusion reaction.
A. Akgl, M. Inc, E. Karatas, D. Baleanu applied the reproducing kernel method in [2] and suggested a numerical study for the following Lane-Emden problem:

$$\begin{cases} D^{\alpha}y(t)+\frac{k}{t^{\alpha-\beta}}D^{\beta}y(t)+f(t,y(t))=g(t),\ t\in[0,1], \\ k\geq 0,\ 1<\alpha\leq 2,\ 0<\beta\leq 1, \end{cases}$$

with the initial conditions:

$$y(0) = A,\ y^{'}(0) = B,$$

where A and B are constants, f is a continuous real valued function and $g \in C([0,1])$.

Most recently, in [7] Z. Dahmani and M.Z. Sarikaya studied the following generalized Lane Emden system:

$$\begin{cases} D^{\beta_1}(D^{\alpha_1} + b_1 g_1(t))\, x_1(t) + f_1(t, x_1(t), x_2(t)) = h_1(t),\ 0 < t < 1, \\ D^{\beta_2}(D^{\alpha_2} + b_2 g_2(t))\, x_2(t) + f_2(t, x_1(t), x_2(t)) = h_2(t),\ 0 < t < 1, \\ x_k(0) = 0, D^{\alpha} x_k(1) + b_k g_k(1) x_k(1) = 0, \end{cases}$$

where $0 < \beta_k < 1, 0 < \alpha_k < 1, b_k \geq 0, k = 1,2$ and the derivatives D^{β_k} and D^{α_k} are in the sense of Caputo.
Motivated by the above work, this paper considers a more general system of Lane Emden type by injecting the unknown functions (solutions) not only on the left hand side of the system, but on right hand side of the problem too. This injection makes the problem very difficult to study, since basically the problem is singular. So let us consider the following problem:

$$\begin{cases} D^{\beta_1}(D^{\alpha_1} + b_1 g_1(t))\, x_1(t) + f_1(t, x_1(t), x_2(t)) = \omega_1 S_1(t, x_1(t), x_2(t)),\ 0 < t < 1, \\ D^{\beta_2}(D^{\alpha_2} + b_2 g_2(t))\, x_2(t) + f_2(t, x_1(t), x_2(t)) = \omega_2 S_2(t, x_1(t), x_2(t)),\ 0 < t < 1, \\ x_k(0) = 0, D^{\alpha} x_k(1) + b_k g_k(1) x_k(1) = 0, \end{cases} \tag{1}$$

where $0 < \beta_k < 1, 0 < \alpha_k < 1, b_k \geq 0, 0 < \omega_k < \infty, k = 1,2$ and the derivatives D^{β_k} and D^{α_k} are in the sense of Caputo. The functions $f_k : [0,1]\times$ are continuous, $g_k :]0,1] \to [0,+\infty)$ is continuous and singular at $t = 0$.

2 Preliminaries

Definition 2.1 *The Riemann-Liouville integral operator [14]:*

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,\ \alpha > 0,\ t \geq 0, \tag{2}$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-x} x^{\alpha-1} dx$,
and the Caputo fractional derivative D^{α}

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds,\ n-1 < \alpha < n. \tag{3}$$

We need the following auxiliary results [12]:

Lemma 2.2 *For* $\alpha > 0$, *the general solution of the fractional differential equation* $D^{\alpha}x(t) = 0$ *is given by*

$$x(t) = \sum_{j=0}^{n-1} c_j t^j, \tag{4}$$

where $c_j \in \mathbb{R}$, $j = 0, ..., n-1$, $n = [\alpha] + 1$.

Lemma 2.3 *Let* $\alpha > 0$. *We have*

$$J^{\alpha}D^{\alpha}x(t) = x(t) + \sum_{j=0}^{n-1} c_j t^j, \tag{5}$$

where $c_j \in \mathbb{R}$, $j = 0, 1, ..., n-1$, $n = [\alpha] + 1$.

Lemma 2.4 *Let* $q > p > 0, g \in L^1([a,b])$. *Then* $D^p J^q g(t) = J^{q-p} g(t), t \in [a,b]$.

Lemma 2.5 *Let* E *be a Banach space and let's assume that* $T : E \to E$ *is a completely continuous operator. If the set* $V := \{x \in E : x = \mu T x,\ 0 < \mu < 1\}$ *is bounded, then* T *has a fixed point in* E.

To give the integral representation of (1), we need to prove the following auxiliary result:

Lemma 2.6 *Let* $H_1, H_2 \in C([0,1], \mathbb{R})$. *Then, the problem*

$$\begin{cases} D^{\beta_1}\left(D^{\alpha_1} + b_1 g_1(t)\right) x_1(t) = H_1(t),\ t \in [0,1], \\ D^{\beta_2}\left(D^{\alpha_2} + b_2 g_2(t)\right) x_2(t) = H_2(t),\ t \in [0,1] \end{cases} \tag{6}$$

associated with the conditions

$$x_k(0) = 0, D^{\alpha} x_k(1) + b_k g_k(1) x_k(1) = 0, k = 1, 2 \tag{7}$$

has a unique solution (x_1, x_2) *given by:*

$$x_k(t) = J^{\alpha_k+\beta_k} H_k(t) - b_k J^{\alpha_k} g_k(t) x_k(t) - J^{\beta_k} H_k(1) \frac{t^{\alpha_k}}{\Gamma(\alpha_k + 1)}. \tag{8}$$

Proof. We use Lemma 2.3 to obtain:

$$x_1(t) = \int_0^t \frac{(t-\tau)^{\alpha_1 - 1}}{\Gamma(\alpha_1)} \left(\int_0^{\tau} \frac{(\tau - s)^{\beta_1 - 1}}{\Gamma(\beta_1)} H_1(s)\, ds - b_1 g_1(t) x_1(\tau) \right) d\tau - c_1 J^{\alpha_1}(1) - c_2. \tag{9}$$

Then, by (7) we obtain:

$$c_2 = 0, c_1 = J^{\beta_1} H_1(1).$$

With the same arguments we obtain the component $x_2(t)$.
Lemma 2.6 is thus proved. ■

Let us now introduce the Banach space $\left(X \times X, \|(u,v)\|_{X\times X}\right)$, with $\|(u,v)\|_{X\times X} = \max\{\|u\|_X, \|v\|_X\}$ and $X := C([0,1],\mathbb{R}),\ ||.||_X = ||.||_\infty$.

3 Main results

3.1 Existence and Uniqueness

We prove the following theorem.

Theorem 3.1 *Let* $g_1, g_2 :]0,1] \to [0,+\infty)$ *be continuous,* $\lim_{t\to 0} g_1(t) = \lim_{t\to 0} g_2(t) = \infty$. *Suppose that there exist* $0 < \lambda_1, \lambda_2 < 1, t \longmapsto (t^{\lambda_1} g_1(t), t^{\lambda_2} g_2(t))$ *are continuous on* $[0,1]$. *If*

$$\begin{aligned} |f_1(t,x_1,x_2) - f_1(t,y_1,y_2)| &\le \sum_{j=1}^{2} L_i |x_j - y_j|, \\ |f_2(t,x_1,x_2) - f_2(t,y_1,y_2)| &\le \sum_{j=1}^{2} L'_i |x_j - y_j|, \end{aligned} \tag{10}$$

$$|S_1(t,x_1,x_2) - S_1(t,y_1,y_2)| \le \sum_{j=1}^{2} R_i |x_j - y_j|,$$

$$|S_2(t,x_1,x_2) - S_2(t,y_1,y_2)| \le \sum_{j=1}^{2} R'_i |x_j - y_j|,$$

for all $t \in [0,1], (x_1,x_2), (y_1,y_2) \in \mathbb{R}^2$,

then the problem (1) *has a unique solution on* $[0,1]$ *provided that*

$$D_1 + D_2 + 2L(K_1 + K_2) + 2R(\omega_1 K_1 + \omega_2 K_2) < 1, \tag{11}$$

where

$$D_1 = b_1 M_1 \frac{\beta(\alpha_1, 1-\lambda_1)}{\Gamma(\alpha_1)}$$
$$D_2 = b_2 M_2 \frac{\beta(\alpha_2, 1-\lambda_2)}{\Gamma(\alpha_2)}$$
$$K_1 = \frac{1}{\Gamma(\alpha_1+\beta_1+1)} + \frac{1}{\Gamma(\alpha_1+1)\Gamma(\beta_1+1)}$$
$$K_2 = \frac{1}{\Gamma(\alpha_2+\beta_2+1)} + \frac{1}{\Gamma(\alpha_2+1)\Gamma(\beta_2+1)}$$

and $L := \max\{ L_1, L_2, L_1', L_2'\}$, $R := \max\{ R_1, R_2, R_1', R_2'\}$, $M_k := Max_{t\in[0,1]} |t^{\lambda_k} g_k(t)|$, $k = 1, 2.$

Proof. Let us consider the operator $T : X \times X \to X \times X$ defined by

$$T(x_1, x_2) := \Big(T_1(x_1, x_2), T_2(x_1, x_2)\Big), \tag{12}$$

where

$$T_k(x_1, x_2)(t) := J^{\alpha_k+\beta_k}(\omega_k S_k(x_1, x_2)(t) - f_k(x_1, x_2)(t)) - b_k J^{\alpha_k} g_k(t)\, x_k(t)$$
$$-J^{\beta_k}(\omega_k S_k(x_1, x_2)(1) - f_k(x_1, x_2)(1)) \tfrac{t^{\alpha_k}}{\Gamma(\alpha_k+1)}, k = 1, 2. \tag{13}$$

We need to prove that T is contractive.
Let $(x_1, x_2), (y_1, y_2) \in X \times X$. We have

$$T_1(x_1, x_2)(t) - T_1(y_1, y_2)(t) = J^{\alpha_1+\beta_1}(\omega_1 S_1(x_1, x_2)(t) - f_1(x_1, x_2)(t)) - b_1 J^{\alpha_1}(g_1(t)\, x_1(t))$$
$$-J^{\beta_1}(\omega_1 S_1(x_1, x_2)(1) - f_1(x_1, x_2)(1)) \tfrac{t^{\alpha_1}}{\Gamma(\alpha_1+1)} - \Big(J^{\alpha_1+\beta_1}(\omega_1 S_1(x_1, x_2)(t) - f_1(y_1, y_2)(t))$$
$$-b_1 J^{\alpha_1}(g_1(t)\, y_1(t)) - J^{\beta_1}(\omega_1 S_1(x_1, x_2)(1) - f_1(y_1, y_2)(1)) \tfrac{t^{\alpha_1}}{\Gamma(\alpha_1+1)}\Big). \tag{14}$$

Some easy techniques allow us to write

$$|T_1(x_1, x_2)(t) - T_1(y_1, y_2)(t)|$$
$$\leq |\omega_1 J^{\alpha_1+\beta_1}(S_1(y_1, y_2)(t) - S_1(x_1, x_2)(t))| + \tfrac{\omega_1}{\Gamma(\alpha_1+1)} |J^{\beta_1}(S_1(y_1, y_2)(1) - S_1(x_1, x_2)(1))|$$
$$+|J^{\alpha_1+\beta_1}(f_1(y_1, y_2)(t) - f_1(x_1, x_2)(t))| + \tfrac{1}{\Gamma(\alpha_1+1)} |J^{\beta_1}(f_1(y_1, y_2)(1) - f_1(x_1, x_2)(1))|$$
$$+b_1 M_1 |x_1 - y_1|(t) J^{\alpha_1} t^{-\lambda_1}. \tag{15}$$

Thanks to the conditions on f_1, S_1 and $t^{\lambda_1} g_1$, we obtain

$$\begin{aligned} & \|T_1(x_1,x_2) - T_1(y_1,y_2)\|_X \\ \le\ & \omega_1\Big(\tfrac{1}{\Gamma(\alpha_1+\beta_1+1)} + \tfrac{1}{\Gamma(\alpha_1+1)\Gamma(\beta_1+1)}\Big)\Big(R_1\|x_1-y_1\| + R_2\|x_2-y_2\|\Big) \\ & \Big(\tfrac{1}{\Gamma(\alpha_1+\beta_1+1)} + \tfrac{1}{\Gamma(\alpha_1+1)\Gamma(\beta_1+1)}\Big)\Big(L_1\|x_1-y_1\| + L_2\|x_2-y_2\|\Big) \\ & + b_1 M_1 \|x_1-y_1\| \tfrac{\beta(\alpha_1, 1-\lambda_1)}{\Gamma(\alpha_1)}. \end{aligned} \tag{16}$$

Therefore,

$$\begin{aligned} & \|T_1(x_1,x_2) - T_1(y_1,y_2)\|_X \\ \le\ & \Big(b_1 M_1 \tfrac{\beta(\alpha_1,1-\lambda_1)}{\Gamma(\alpha_1)} + 2(L+\omega_1 R)\big(\tfrac{1}{\Gamma(\alpha_1+\beta_1+1)} + \tfrac{1}{\Gamma(\alpha_1+1)\Gamma(\beta_1+1)}\big)\Big)\|(x_1-y_1, x_2-y_2)\|_{X\times X}. \end{aligned} \tag{17}$$

With the same arguments as before we can write

$$\begin{aligned} & \|T_2(x_1,x_2) - T_2(y_1,y_2)\|_X \\ \le\ & \Big(b_2 M_2 \tfrac{\beta(\alpha_2,1-\lambda_2)}{\Gamma(\alpha_2)} + 2(L+\omega_2 R)\big(\tfrac{1}{\Gamma(\alpha_2+\beta_2+1)} + \tfrac{1}{\Gamma(\alpha_2+1)\Gamma(\beta_2+1)}\big)\Big)\|(x_1-y_1, x_2-y_2)\|_{X\times X}. \end{aligned} \tag{18}$$

Using these two inequalities we get

$$\begin{aligned} & \|T(x_1,x_2) - T(y_1,y_2)\|_{X\times X} \\ \le\ & \Big(D_1 + D_2 + 2L(K_1+K_2) + 2R(\omega_1 K_1 + \omega_2 K_2)\Big)\|(x_1-y_1, x_2-y_2)\|_{X\times X}. \end{aligned} \tag{19}$$

Since $D_1 + D_2 + 2L(K_1+K_2) + 2R(\omega_1 K_1 + \omega_2 K_2) < 1$, we can state that the operator T is contractive. Thus the theorem is proved. ■ Thus the theorem is proved.

3.2 Existence

In the case where $0 < \lambda_k \le \alpha_k < 1$, we present the following theorem:

Theorem 3.2 *For $k = 1, 2$, suppose that $g_k :]0,1] \to [0,+\infty)$ are continuous, $\lim_{t\to 0} g_k(t) = \infty$, and there exist $\lambda_k; 0 < \lambda_k \le \alpha_k < 1, t \longmapsto t^{\lambda_k} g_k(t)$ are continuous on $[0,1]$. Assume that $f_k : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ and $S_k : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ are bounded respectively by I_k and Q_k. Then the problem (1) has at least one solution on $[0,1]$.*

Proof. We will prove the theorem through the following steps:

The continuity of the functions $f_k, S_k, t^{\lambda_k} g_k, k = 1,2$ implies that T is continuous on $X \times X$.

Step 2: The operator T is completely continuous:
We define the set $\Omega_r := \left\{(x_1, x_2) \in X \times X,\ \|(x_1, x_2)\|_{X\times X} \leq r\right\}$, where $r > 0$.
For $(x_1, x_2) \in \Omega_r$, we obtain

$$||T_k\left(x_1, x_2\right)||_X \leq \frac{I_k + \omega_k Q_k}{\Gamma(\alpha_k + \beta_k + 1)} + \frac{r b_k M_k \beta(\alpha_k, 1-\lambda_k)}{\Gamma(\alpha_k)} + \frac{I_k + \omega_k Q_k}{\Gamma(\beta_k + 1)\Gamma(\alpha_k + 1)}. \tag{20}$$

This is to say that

$$||T\left(x_1, x_2\right)||_{X\times X} \leq max_{k=1,2}\left(\frac{I_k+\omega_k Q_k}{\Gamma(\alpha_k+\beta_k+1)} + \frac{r b_k M_k \beta(\alpha_k, 1-\lambda_k)}{\Gamma(\alpha_k)} + \frac{I_k+\omega_k Q_k}{\Gamma(\beta_k+1)\Gamma(\alpha_k+1)}\right). \tag{21}$$

Hence, the operator T maps bounded sets into bounded sets in $X \times X$.

Step 3: Equi-continuity of $T(\Omega_r)$:
For $t_1, t_2 \in [0,1]$; $t_1 < t_2$, and $(x_1, x_2) \in \Omega_r$, we have:

$$||T_k\left(x_1, x_2\right)(t_2) - T_k\left(x_1, x_2\right)(t_1)||_X$$

$$\leq \frac{(I_k+\omega_k Q_k)(t_2^{\alpha_k+\beta_k} - t_2^{\alpha_k+\beta_k})}{\Gamma(\alpha_k+\beta_k+1)} + \frac{r b_k M_k \Gamma(1-\lambda_k)(t_2^{\alpha_k-\lambda_k} - t_2^{\alpha_k-\lambda_k})}{\Gamma(\alpha_k-\lambda_k+1)\Gamma(\alpha_k)} + \frac{(I_k+\omega_k Q_k)(t_2^{\alpha_k} - t_1^{\alpha_k})}{\Gamma(\beta_k+1)\Gamma(\alpha_k+1)} := C_k. \tag{22}$$

In these inequalities the right hand sides are independent of x_1, x_2 and tend to zero as t_1 tends to t_2.
In view of the results obtained in steps $2, 3$ and according to Arzela-Ascoli theorem, it is seen that T is completely continuous.

Step 4: The set

$$\Omega := \left\{(x_1, x_2) \in X \times X; (x_1, x_2) = \lambda T\left(x_1, x_2\right), 0 < \lambda < 1\right\} \tag{23}$$

is bounded:
Let $(x_1, x_2) \in \Omega$, then $(x_1, x_2) = \lambda T\left(x_1, x_2\right)$, for some $0 < \lambda < 1$. Hence, for $t \in [0,1]$, we have:

$$x_1\left(t\right) = \lambda T_1\left(x_1, x_2\right)(t), x_2\left(t\right) = \lambda T_2\left(x_1, x_2\right)(t). \tag{24}$$

Thus,

$$\|(x_1, x_2)\|_{X\times X} = \lambda \left\|T(x_1, x_2)\right\|_{X\times X}. \tag{25}$$

Since the functions f_k and S_k are bounded, then by (22) we obtain

$$\|(x_1, x_2)\|_{X\times X} \leq \lambda(C_1 + C_2). \tag{26}$$

Consequently, Ω is bounded.
As a conclusion of Schaefer fixed point theorem, we deduce that T has at least one fixed point, which is a solution of (1). ■

3.3 $\Delta-$Ulam Stabilities

In this section, we will focus our attention on the $\Delta-$Ulam-Hyers and generalized $\Delta-$Ulam-Hyers stabilities for the problem (1) . We start with the following definitions:

Definition 3.3 *The problem* (1) *is* $\Delta-$ *Ulam-Hyers stable, if there exists a real number* $R > 0$, *such that for each* $\epsilon_k > 0, k = 1, 2$ *and for for each solution* $(x_1, x_2) \in X \times X$ *of the inequalities*

$$\left|D^{\beta_k}\left(D^{\alpha_k} + b_k g_k\left(t\right)\right) x_k\left(t\right) + f_k\left(t, x_1\left(t\right), x_2\left(t\right)\right) - \omega_k S_k\left(t, x_1\left(t\right), x_2\left(t\right)\right)\right| < \epsilon_k, \quad t \in [0, 1], \tag{27}$$

there exists a solution $(y_1, y_2) \in X \times X$ *of* (1) ,
such that

$$\|(x_1, x_2) - (y_1, y_2)\|_{X\times X} < \Delta + (\epsilon_1 + \epsilon_2)R. \tag{28}$$

Definition 3.4 *The problem* (1) *is* $\Delta-$*generalized Ulam-Hyers stable, if there exists an increasing function* $Z \in C\left(\mathbb{R}^+, \mathbb{R}^+\right), Z(0) = \Delta$, *such that for all* $\epsilon_k > 0$, *and for each solution* $(x_1, x_2) \in X \times X$ *of* (27), *there exists a solution* $(y_1, y_2) \in X \times X$ *of* (1) *(with the same conditions as in* (1)*), such that*

$$\|(x_1, x_2) - (y_1, y_2)\|_{X\times X} < Z(\epsilon_1 + \epsilon_2). \tag{29}$$

Let us consider the equation (1) and the inequalities (27). We prove the following stability result:

Theorem 3.5 *Let the assumptions of Theorem* 3.1 *hold. If the inequality*

$$1 - 2[L + R(\omega_1 + \omega_2)]\Big(J^{\alpha_1+\beta_1}(1) + J^{\alpha_2+\beta_2}(1)\Big) - \Big(\beta(\alpha_1, 1-\lambda_1)b_1 M_1 + \beta(\alpha_2, 1-\lambda_2)b_2 M_2\Big) > 0 \tag{30}$$

is valid, then problem (1) *is* $\Delta-$ *Ulam-Hyers stable in the generalized sense.*

Proof. By Theorem 3.1, the problem (1) has a unique solution $(y_1, y_2) \in X \times X$. Let (x_1, x_2) be a solution of (28). By definition, we can state that there exist l_k (depending on (x_1, x_2)) that satisfy $|l_k(t)| \leq \epsilon_k$, such that

$$x_k(t) = \int_0^t \frac{(t-\tau)^{\alpha_k - 1}}{\Gamma(\alpha_k)} \left(\int_0^\tau \frac{(\tau - s)^{\beta_k - 1}}{\Gamma(\beta_k)} (\omega_k S_k - f_k + l_k)(s) ds - b_k g_k(t) x_k(\tau) \right) d\tau$$
$$-c_k J^{\alpha_k}(1) - d_k, c_k, d_k \in \mathbb{R}. \tag{31}$$

So, we have

$$|x_k(t) - y_k(t)| \leq |J^{\alpha_k+\beta_k}(\omega_k S_k(x_1,x_2)(t) - f_k(x_1,x_2)(t) + l_k(t)) - b_k J^{\alpha_k} g_k(t) x_k(t) - c_k J^{\alpha_k} - d_k$$
$$-\Big(J^{\alpha_k+\beta_k}(\omega_k S_k(y_1,y_2)(t) - f_k(y_1,y_2)(t))$$
$$-b_k J^{\alpha_k} g_k(t) y_k(t) - J^{\beta_k}(\omega_k S_k(y_1,y_2)(1) - f_k(y_1,y_2)(1)) \tfrac{t^{\alpha_k}}{\Gamma(\alpha_k+1)} \Big)|. \tag{32}$$

Therefore,

$$|x_k(t) - y_k(t)|$$
$$\leq \omega_k |J^{\alpha_k+\beta_k}(S_k(y_1,y_2)(t) - S_k(x_1,x_2)(t)) + ||J^{\alpha_k+\beta_k}(f_k(y_1,y_2)(t) - f_k(x_1,x_2)(t))|$$
$$+\tfrac{1}{\Gamma(\alpha_k+1)} |J^{\beta_k}(f_k(y_1,y_2)(1) + \omega_k J^{\beta_k} S_k(y_1,y_2)(1))|$$
$$+b_k M_k |x_k(t) - y_k(t)| J^{\alpha_k} t^{-\lambda_k} + |c_k| J^{\alpha_k}(1) + |d_k| + \epsilon_k J^{\alpha_k+\beta_k}(1). \tag{33}$$

Consequently,

$$||x_k - y_k||_X$$
$$\leq 2[L + R\omega_k] J^{\alpha_k+\beta_k}(1) ||x_k - y_k|| + \tfrac{1}{\Gamma(\alpha_k+1)} \Big(|J^{\beta_k} f_k(y_1,y_2)(1)| + \omega_k |J^{\beta_k} S_k(y_1,y_2)(1)| \Big)$$
$$+\beta(\alpha_k, 1-\lambda_k) b_k M_k ||x_k - y_k|| + |c_k| J^{\alpha_k}(1) + |d_k| + \epsilon_k J^{\alpha_k+\beta_k}(1). \tag{34}$$

Adding these two inequalities (for $k=1,2,$) we obtain

$$\begin{aligned}
&||(x_1,x_2)-(y_1,y_2)||_{X\times X}\\
&\le 2R\Big(\omega_1 J^{\alpha_1+\beta_1}(1)+\omega_2 J^{\alpha_2+\beta_2}(1)\Big)||(x_1,x_2)-(y_1,y_2)||_{X\times X}\\
&+2L\Big(J^{\alpha_1+\beta_1}(1)+J^{\alpha_2+\beta_2}(1)\Big)||(x_1,x_2)-(y_1,y_2)||_{X\times X}\\
&+\tfrac{1}{\Gamma(\alpha_1+1)}\Big(|J^{\beta_1}f_1(y_1,y_2)(1)|+|\omega_1 J^{\beta_1}S_1(y_1,y_2)(1)|\Big)\\
&+\tfrac{1}{\Gamma(\alpha_2+1)}\Big(|J^{\beta_2}f_2(y_1,y_2)(1)|+\omega_2|J^{\beta_2}S_2(y_1,y_2)(1)|\Big)\\
&+\Big(\beta(\alpha_1,1-\lambda_1)b_1M_1+\beta(\alpha_2,1-\lambda_2)b_2M_2\Big)||(x_1,x_2)-(y_1,y_2)||_{X\times X}\\
&+|c_1|J^{\alpha_1}(1)+|d_1|+|c_2|J^{\alpha_2}(1)+|d_2|+\epsilon\Big(J^{\alpha_1+\beta_1}(1)+J^{\alpha_2+\beta_2}(1)\Big).
\end{aligned} \tag{35}$$

Consequently, we deduce that

$$\begin{aligned}
&\Big[1-2R\Big(\omega_1 J^{\alpha_1+\beta_1}(1)+\omega_2 J^{\alpha_2+\beta_2}(1)\Big)-2L\Big(J^{\alpha_1+\beta_1}(1)+J^{\alpha_1+\beta_1}(1)\Big)-\\
&\Big(\beta(\alpha_1,1-\lambda_1)b_1M_1+\beta(\alpha_2,1-\lambda_2)b_2M_2\Big)\Big]\\
&\times||(x_1,x_2)-(y_1,y_2)||_{X\times X}\\
&\le \tfrac{1}{\Gamma(\alpha_1+1)}\Big(|J^{\beta_1}f_1(y_1,y_2)(1)|+|\omega_1 J^{\beta_1}S_1(y_1,y_2)(1)|\Big)\\
&+\tfrac{1}{\Gamma(\alpha_2+1)}\Big(|J^{\beta_2}f_2(y_1,y_2)(1)|+\omega_2 J^{\beta_1}S_2(y_1,y_2)(1)|\Big)\\
&+|c_1|J^{\alpha_1}(1)+|d_1|+|c_2|J^{\alpha_2}(1)+|d_2|+\epsilon\Big(J^{\alpha_1+\beta_1}(1)+J^{\alpha_2+\beta_2}(1)\Big).
\end{aligned} \tag{36}$$

Thanks to (36), we deduce that (1) is $\Delta-$Ulam-Hyers stable. Hence, the problem (1) is $\Delta-$generalized Ulam-Hyers stable. ■

3.4 Illustrations

We consider the following system

$$\begin{cases} D^{\frac{3}{4}}\left(D^{\frac{3}{4}}+g_1(t)\right)x_1(t)+f_1(t,x_1(t),x_2(t))=\frac{1}{2}S_1(t,x_1(t),x_2(t)),\ 0<t<1, \\ D^{\frac{3}{4}}\left(D^{\frac{3}{4}}+g_2(t)\right)x_2(t)+f_2(t,x_1(t),x_2(t))=\frac{1}{4}S_2(t,x_1(t),x_2(t)),\ 0<t<1, \\ x_k(0)=0, D^{\alpha}x_k(1)+b_kg_k(1)x_k(1)=0, \end{cases} \tag{37}$$

where,

$$\begin{aligned} f_1(t,u,v) &= t^2(\cos u.v),\ f_2(t,u,v)=t^2(\sin\ u.v), \\ S_1(t,u,v) &= \sin(u)\sin(v)+t,\ S_2(t,u,v)=\cos(tv)\sin(tv), \\ g_1(t) &= \frac{1}{\sqrt[2]{t}},\ g_2(t)=\frac{1}{\sqrt[3]{t}} \end{aligned}$$

and $\lambda_1=\frac{1}{2}\ \lambda_2=\frac{1}{3}$. Then the problem (1) has at least one solution on $[0,1]$.

4 Open Problem

We end this paper by proposing the following open questions:
What will happen if the function g admits an arbitrary singularity on the whole $t-$positive real line? What about the stability of the associated Lane-Emden system in this case?

References

[1] S. Abbas, M. Benchohra: *Uniqueness and Ulam stabilities results for partial fractional differential equations with not instantaneous impulses,* Applied Mathematics and Computation, Vol. 257(C), pp. 190-198, 2015.

[2] A. Akgl, M. Inc, E. Karatas, D. Baleanu: *Numerical solutions of fractional differential equations of Lane-Emden type by an accurate technique*, Advances in Difference Equations, Vol. 2015(1), pp.220-232, 2015.

[3] T. Amele, Z. Dahmani: *The hight order Lane-Emden fractional differential system: Existence, uniqueness and Ulam type stabilities* , Kragujevac Journal of Mathematics, Vol. 40(2), pp: 238-259. 2016.

[4] G. Baumann: *Fractional Calculus: Application and Results,* Biophysical Journal, Vol. 56(5), pp. 955-66, 2017.

[5] C.V. Bert, H. Zeng: *Analysis of axial vibration of compound bars by differential transformation method,* J. Sound Vib., Vol. 275, pp. 641-647, 2004.

[6] A. Cabada, S. Dimitrijevi, T. Tomovi, S. Aleksi: *The existence of a positive solution for nonlinear fractional differential equations with integral boundary value conditions,* Mathematical Methods in the Applied Sciences, Vol. 40(6), 2017.

[7] Z. Dahmani, M.Z. Sarikaya: *On a generalized Lane-Emden fractional differential system and its* $\Delta-$*stability*, Accepted on IASR-JARDCS of Dynamical and Control Systems. Accepted End 2015.

[8] J. Dvila, L. Dupaigne, J. Wei: *On the fractional Lane-Emden equation,* Transactions of the American Mathematical Society, Vol. 369(9), 2014.

[9] H. Fenga, C. Zhai: *Existence and uniqueness of positive solutions for a class of fractional differential equation with integral boundary conditions,* Nonlinear Analysis Modelling and Control, Vol. 22(2), pp. 160172, 2017.

[10] T. Hao, S. Song: *Solving coupled Lane-Emden equations arising in catalytic diffusion reactions by reproducing kernel Hilbert space method*, International Conference on Applied Science and Engineering Innovation, pp. 529-536, 2015.

[11] A. Khan, K Shah, Y. Li, T.S. Khan: *Ulam Type Stability for a Coupled System of Boundary Value Problems of Nonlinear Fractional Differential Equations*, Journal of Function Spaces, Vol. 2, pp. 1-8, 2017.

[12] V. Lakshmikantham, A.S. Vatsala: *Basic Theory Of Fractional Defferential Equations*, Nonlinear Anal, pp. 2677-2682, 2008.

[13] J. Nowakowski, P. Ostalczyk, D. Sankowski: *Application of fractional calculus for modeling of tow-phase gas/liquid flow system*, IAPGOS, Vol. 7(1), pp. 42-45, 2017.

[14] I. Podlubny: *Fractional Differential Equations*, Academic Press, New York, 1999.

[15] J. Sabatier, OP. Agrawal, JAT. Machado: *Advances in fractional calculus. Theoretical developments and applications in physics and engineering,* Biochemical Journal, Vol. 361(1), pp. 97-103, 2017.

[16] S. Srivastava: *On the physical validity of Lane-Emden equation of index 5,* Math. Stud., Vol. 34, pp. 1926, 1966.

[17] S.M. Ulam, *Problems in Modern Mathematics,* Wiley, 1964.

[18] X.J. Yang, J.A. Tenreiro Machado, C. Cattani, F. Gao: *On a fractal LC-electric circuit modeled by local fractional calculus,* Commun Nonlinear Sci Numer Simulat, Vol. 47, pp. 200206, 2017.

[19] KLS. Yip, TK. Chan, PT. Leung: *Perturbative solution to the Lane-Emden equation: an eigenvalue approach,* Monthly Notices of the Royal Astronomical Society, pp. 1-17, 2016.

[20] H. Zhang, Y. Li, W. Lu: *Existence and uniqueness of solutions for a coupled system of nonlinear fractional differential equations with fractional integral boundary conditions,* J. Nonlinear Sci. Appl., Vol. 9, pp. 2434-2447, 2016.