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Generalized involute and evolute curve-couple in Euclidean space

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Abstract

Evolute curves were studied by some researchers in 4 dimensional Euclidean space. However the special characters of the curve are not considered which is a research gap in this technique. In this study a kind of generalized involute and evolute curve-couple is considered in 4 dimensional Euclidean space. The necessary and sufficient condition for the curve possessing generalized involute as well as evolute is obtained.

Keywords: *Evolute, Involute curves, Mate curves.*
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1 Introduction

Many researchers have developed extensive significant research contribution in the field of general theory of the curves in Euclidean space and Minkowski space. Exiting literature highlight sufficient intellectual depth over vital aspects of local geometry as well as their global geometry. Characterization of a regular curve challenge researchers with on demand interesting issue in the theory of curves of Euclidean space. Researchers tend to develop characterization of the curve with relation to Frenet vectors of the curve to determine the size and particular shape of the curve by principal curvatures k_1 and k_2 [1]. The involute of a given curve is a well-known concept in Euclidean 3-space [11], while the idea of a string involute is due to C Huygens, who is well-known for his work in optics and he discovered involutes while trying to build a more accurate clock (1968). In classical differential geometry an evolute of a curve is defined as the locus of the centers of curvatures of the curve, which is the envelope of the normal of the given curve. In [5], an Involute of a given curve is a curve to which all tangents of a given curve are normal. This research has defined equation of enveloping curve of the family of normal planes for space curve. Research contribution in [13] provides the concept of parallel curves means if evolute exist then the evolute of parallel arc exist, furthermore, evolute coincides with evolute. In our research contribution we provide the evolution of similar existing concepts. In [9], further research work validates that the curve is composed of two arcs with common evolute. The common evolute of two arcs must be a curve with one and only one tangent in each direction. Our reseach contribution also utilize the concepts of moving fame along a front, which is basically proposed in [6]. In general, evolute of regular curve has singularities and these points corresponds to vertices. According research contribution presented in [12], evolute frenet apparatus can be formed by Involute apparatus in four dimensional Euclidean space by this way another orthonormal of the same space is obtained. Evolute curves and their characterization were studied by some researchers in Euclidean space [2, 3, 4, 16, 14, 15, 10, 8, 7] as well as in Minkowski space. In this paper we consider Evolute curves in Euclidean space with respect to casual character of the plane spanned by tangent and the first binormal of the curve.

2 Preliminaries

Consider the Euclidean space (E^4, G) where $E^4 = \{x = (x_1, x_2, x_3, x_4)|x_i \in R\}$ and $G = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$. For any $U = (u_1, u_2, u_3, u_4)$ and $V = (v_1, v_2, v_3, v_4) \in T_xE$, we denote

$$U \cdot V = G(U, V) = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4.$$

Let I be an open interval in R and $\beta : I \to E^4$ be a regular curve in E^4 parameterized by the arc-length s and $\{T, N, B_1, B_2\}$ be the moving Frenet Frame along β, consisting of tangent vector T, the principal normal vector N, the first binormal vector B_1 and the second binormal vector B_2 respectively, so that $T \wedge N \wedge B_1 \wedge B_2$ coincides with the standard orientation of E^4. Then

$$T \cdot T = N \cdot N = B_1 \cdot B_1 = B_2 \cdot B_2 = 1,$$
$$T \cdot N = T \cdot B_1 = T \cdot B_2 = N \cdot B_1 = N \cdot B_2 = B_1 \cdot B_2 = 0.$$

From [13] the Frenet-Serret Formula for β in E^4 is given by

$$\begin{pmatrix} T' \\ N' \\ B_1' \\ B_2' \end{pmatrix} = \begin{pmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B_1 \\ B_2 \end{pmatrix}. \tag{1}$$

We introduce some terminologies used in this paper. At any point of β, the plane spanned by $\{T, B_1\}$ is called *the* $(0,2)$*-tangent plane* of β. The plane spanned by $\{N, B_3\}$ is called *the* $(1,3)$*-normal plane* of β.

Let $\beta : I \to E^4$ and $\beta^* : I \to E^4$ be two regular curves in E^4 where s is the arc-length parameter of β. Denote $s^* = f(s)$ to be the arc-length parameters of β^*. For any $s \in I$, if the $(0,2)$-tangent plane of β at $\beta(s)$ of coincides with the $(1,3)$-normal plane at $\beta^*(s)$ of β^*, then β^* is called *the* $(0,2)$*-involute curve of* β in E^4 and β is called *the* $(1,3)$*-evolute curve of* β^* in E^4.

3 The $(0,2)$-involute curve of a given curve in E^4

In this section, we proceed to study the existence and expression of the $(0,2)$-involute curve of a given curve in E^4.

Let $\beta : I \to E^4$ be a regular curve with arc-length parameter s so that k_1, k_2 and k_3 are not zero. Suppose that $\beta^* : I \to E^4$ is the $(0,2)$-involute curve of β. Denote $\{T^*, N^*, B_1^*, B_2^*\}$ to be the Frenet Frame along β^* and k_1^*, k_2^* and k_3^* to be the curvatures of β^*. Then

$$\mathrm{span}\{T, B_1\} = \mathrm{span}\{N^*, B_2^*\}, \qquad \mathrm{span}\{N, B_2\} = \mathrm{span}\{T^*, B_1^*\}. \tag{2}$$

Moreover, β^* can be expressed as

$$\beta^*(s) = \beta(s) + a(s)T(s) + b(s)B_1, \tag{3}$$

where $a(s)$ and $b(s)$ are C^∞ functions on I.

Differentiating (3) with respect to s and using the Frenet formula (1), we get

$$f'T^* = (1 + a')T + b'B_1 + (ak_1 - bk_2)N + bk_3B_2, \tag{4}$$

Taking inner product on both-sides of (4) with T and B_1 respectively, we get $1 + a' = 0$ and $b' = 0$, which implies that b is constant and $a = a_0 - s$ where a_0 is the integration constant. So (4) turns into

$$f'T^* = (ak_1 - bk_2)N + bk_3B_2. \tag{5}$$

Denote

$$\delta = \frac{ak_1 - bk_2}{f'}, \qquad \gamma = \frac{bk_3}{f'}. \tag{6}$$

Then (5) turns into

$$T^* = \delta N + \gamma B_2, \quad \delta^2 + \gamma^2 = 1. \tag{7}$$

Case 1: $b \neq 0$. In this case, $\gamma \neq 0$. Denote $\delta/\gamma = t_1$. Then $\delta = t_1\gamma$ and

$$ak_1 - bk_2 = bt_1k_3, \quad f' = b\gamma^{-1}k_3, \quad \gamma^2 = 1/(1+t_1^2). \tag{8}$$

Differentiating (7) with respect to s and using the Frenet formula (1), we get

$$f'k_1^*N^* = \delta' N - \delta k_1 T + \gamma' B_2 + (\delta k_2 - \gamma k_3)B_1. \tag{9}$$

Taking inner product on both-sides of (9) with N and B_2 respectively, we get $\delta' = 0$ and $\gamma' = 0$ which implies that δ and γ are constants. So (9) turns into

$$f'k_1^*N^* = -\delta k_1 T + (\delta k_2 - \gamma k_3)B_1. \tag{10}$$

Denote

$$c = -\frac{\gamma t_1 k_1}{f'k_1^*}, \qquad e = \frac{\gamma(t_1k_2 - k_3)}{f'k_1^*}. \tag{11}$$

Then (10) turns into

$$N^* = cT + eB_1, \quad c^2 + e^2 = 1. \tag{12}$$

Denote $e/c = t_2$. Then $e = t_2c$ and

$$t_1(t_2k_1 + k_2) = k_3, \qquad c^2 = 1/(1+t_2^2). \tag{13}$$

From the first equations of (8) and (13), we have

$$\tau := \frac{k_2}{k_1} = \frac{a/b - t_1^2t_2}{1+t_2^2}, \qquad \frac{k_3}{k_1} = t_1(\tau + t_2). \tag{14}$$

Denote $\gamma/c = t_3$. Then $\gamma = t_3c$. From (11), we have

$$f'k_1^* = -t_1t_3k_1, \qquad t_3^2 = \frac{1+t_2^2}{1+t_1^2}. \tag{15}$$

Differentiating (12) with respect to s and using the Frenet formula (1), we get

$$-f'k_1^*T^* + f'k_2^*B_1^* = c'T + (ck_1 - ek_2)N + e'B_1 + ek_3B_2. \tag{16}$$

Taking inner product on both-sides of (16) with T and B_1 respectively, we get $c' = 0$ and $e' = 0$, which implies that c and e are constants. In this case, (16) turns into

$$f'k_2^*B_1^* = f'k_1^*T^* + c(k_1 - t_2k_2)N + ct_2k_3B_2. \tag{17}$$

Substituting (7) and (15) into (17), we obtain

$$f'k_2^*B_1^* = ck_1(t_2\tau + t_2^2 - t_3^2)(-N + t_1B_2). \tag{18}$$

From (18), we may choose that

$$B_1^* = -\gamma N + \delta B_2, \qquad f' k_2^* = t_3^{-1} k_1 (t_2 \tau + t_2^2 - t_3^2). \tag{19}$$

Differentiating (19) about s and using the Frenet formula (1), we get

$$-f' k_2^* N^* + f' k_3^* B_2^* = \gamma k_1 T - (\gamma k_2 + \delta k_3) B_1,$$

from which we obtain

$$f' k_3^* B_2^* = (c f' k_2^* + \gamma k_1) T + (e f' k_2^* - \gamma k_2 - \delta k_3) B_1 = -t_3^{-1} k_1 (\tau + t_2)(-eT + cB_1). \tag{20}$$

From (20), we may choose that

$$B_2^* = -eT + cB_1, \qquad f' k_3^* = -t_3^{-1} k_1 (\tau + t_2). \tag{21}$$

Summarizing the above discussions, we obtain the following

Theorem 3.1 *Let $\beta : I \to E^4$ be a regular curve with arc-length parameter s so that k_1, k_2 and k_3 are not zero. If β possesses the $(0,2)$-involute mate curve $\beta^*(s) = \beta(s) + (a_0 - s)T(s) + bB_1(s)$ with $b \neq 0$, then k_1, k_2 and k_3 satisfy*

$$\frac{k_2}{k_1} = \tau, \qquad \frac{k_3}{k_1} = t_1(\tau + t_2), \qquad \tau = \frac{a_0 - s - b t_1^2 t_2}{b(1 + t_2^2)}, \tag{22}$$

where a_0, b, t_1 and t_2 are given constants. Moreover, the three curvatures of β^* are given by

$$k_1^* = -\frac{c t_3^2}{b(\tau + t_2)}, \quad k_2^* = \frac{c(t_2 \tau + t_2^2 - t_3^2)}{b t_1 (\tau + t_2)}, \quad k_3^* = -\frac{c}{b t_1},$$

where $c \neq 0$. The associated Frenet Frame are given by

$$T^* = c t_3 (t_1 N + B_2), \quad N^* = c(T + t_2 B_1), \quad B_1^* = c t_3 (-N + t_1 B_2), \quad B_2^* = c(-t_2 T + B_1).$$

Case 2: $b = 0$. In this case, (3) turns into

$$\beta^*(s) = \beta(s) + (a_0 - s)T(s). \tag{23}$$

Differentiating (23) with respect to s and using the Frenet formula (1), we get

$$f' T^* = (a_0 - s) k_1 N, \tag{24}$$

from which we may assume that

$$f' = (s - a_0) k_1, \qquad T^* = -N. \tag{25}$$

Differentiating the second equation of (25) about s and using the Frenet formula (1), we get

$$f'k_1^*N^* = k_1T - k_2B_1. \tag{26}$$

Suppose that

$$N^* = cT + eB_1, \quad c = \frac{k_1}{f'k_1^*}, \quad e = -\frac{k_2}{f'k_1^*}, \quad c^2 + e^2 = 1. \tag{27}$$

It follows that

$$\frac{k_2}{k_1} = -\frac{e}{c}. \tag{28}$$

Differentiating (27) about s, we obtain that c and e are constant and

$$f'k_2^*B_1^* = f'k_1^*T^* + (ck_1 - ek_2)N + ek_3B_2 = -e\Big(\frac{e}{c}k_1 + k_2\Big)N + ek_3B_2 = ek_3B_2. \tag{29}$$

We suppose that

$$B_1^* = -B_2, \qquad f'k_2^* = -ek_3. \tag{30}$$

Differentiating (30) about s, we obtain

$$f'k_3^*B_2^* = f'k_2^*N^* + k_3B_1 = -k_3[ceT - (1-e^2)B_1] = -ck_3(eT - cB_1). \tag{31}$$

We suppose that $T^* \wedge N^* \wedge B_1^* \wedge B_2^* = T \wedge N \wedge B_1 \wedge B_2$. Then

$$B_2^* = -eT + cB_1, \quad f'k_3^* = ck_3. \tag{32}$$

Summarizing the above discussions, we obtain the following

Theorem 3.2 *Let $\beta : I \to E^4$ be a regular curve with arc-length parameter s so that k_1, k_2 and k_3 are not zero. If β possesses the $(0,2)$-involute mate curve $\beta^*(s) = \beta(s) + (a_0 - s)T(s)$, then k_1 and k_2 satisfy*

$$ek_1 + ck_2 = 0, \tag{33}$$

where a_0, c and e are given constants. Moreover, the three curvatures of β^* are given by

$$k_1^* = \frac{1}{c(s-a_0)}, \quad k_2^* = \frac{-ek_3}{(s-a_0)k_1}, \quad k_3^* = \frac{ck_3}{(s-a_0)k_1}.$$

The associated Frenet Frame are given by

$$T^* = -N, \quad N^* = cT + eB_1, \quad B_1^* = -B_2, \quad B_2^* = -eT + cB_1.$$

Remark 3.3 From theorems 3.1 and 3.2, we can see that the above two cases are quite different with each other.

4 The $(1,3)$-evolute curve of a given curve in E^4

In this section, we proceed to study the existence and expression of the $(1,3)$-evolute curve of a given curve in E^4.

Let $\beta : I \to E^4$ be a regular curve with arc-length parameter s so thatk_1, k_2 and k_3 are not zero. Let $\beta^* : I \to E^4$ be the $(1,3)$-evolute curve of β. Denote $\{T^*, N^*, B_1^*, B_2^*\}$ to be the Frenet Frame along β^* and k_1^*, k_2^* and k_3^* to be the curvatures of β^*. Then

$$\mathrm{span}\{T, B_1\} = \mathrm{span}\{N^*, B_2^*\}, \qquad \mathrm{span}\{N, B_2\} = \mathrm{span}\{T^*, B_1^*\}. \tag{34}$$

Moreover, β^* can be expressed as

$$\beta^*(s) = \beta(s) + u(s)N(s) + v(s)B_2, \tag{35}$$

where $u(s)$ and $v(s)$ are C^∞ functions on I.

Differentiating (35) with respect to s and using the Frenet formula (1), we get

$$f'T^* = (1 - uk_1)T + u'N + v'B_2 + (uk_2 - vk_3)B_1. \tag{36}$$

Taking inner product on both-sides of (36) with T and B_1 respectively, we get

$$f'T^* = u'N + v'B_2, \quad u = \frac{1}{k_1}, \quad v = \frac{k_2}{k_1k_3}. \tag{37}$$

Denote

$$x = \frac{u'}{f'}, \qquad y = \frac{v'}{f'}. \tag{38}$$

Then (37) turns into

$$T^* = xN + yB_2, \quad x^2 + y^2 = 1. \tag{39}$$

Differentiating (39) with respect to s and using the Frenet formula (1), we get

$$f'k_1^*N^* = x'N - xk_1T + y'B_2 + (xk_2 - yk_3)B_1. \tag{40}$$

Taking inner product on both-sides of (40) with N and B_2 respectively, we get $x' = 0$ and $y' = 0$ which implies that x and y are constants. From (38), we obtain

$$u = xf + u_0 = \frac{1}{k_1}, \qquad v = yf + v_0 = \frac{k_2}{k_1k_3}. \tag{41}$$

Moreover, (40) turns into

$$f'k_1^*N^* = -xk_1T + (xk_2 - yk_3)B_1. \tag{42}$$

Denote

$$w = -\frac{xk_1}{f'k_1^*}, \qquad z = \frac{xk_2 - yk_3}{f'k_1^*}. \tag{43}$$

Then (42) turns into

$$N^* = wT + zB_1, \qquad f'k_1^* = -w^{-1}xk_1, \qquad w^2 + z^2 = 1. \tag{44}$$

Moreover, we have

$$zxk_1 + wxk_2 - wyk_3 = 0. \tag{45}$$

Case 1: $z \neq 0$. Differentiating (44) about s and using (1), we obtain

$$-f'k_1^*T^* + f'k_2^*B_1^* = w'T + (wk_1 - zk_2)N + z'B_1 + zk_3B_2. \tag{46}$$

Taking inner product on both-sides of (46) with T and B_1 respectively, we get $w' = 0$ and $z' = 0$, which implies that w and z are constants. In this case, (46) turns into

$$f'k_2^*B_1^* = \Big(\frac{w^2 - x^2}{w}k_1 - zk_2\Big)N + \Big(zk_3 - \frac{xy}{w}k_1\Big)B_2. \tag{47}$$

Denote

$$\eta = (f'k_2^*)^{-1}\Big(\frac{w^2 - x^2}{w}k_1 - zk_2\Big), \quad \zeta = (f'k_2^*)^{-1}\Big(zk_3 - \frac{xy}{w}k_1\Big). \tag{48}$$

Then (47) turns into

$$B_1^* = \eta N + \zeta B_2, \qquad \eta^2 + \zeta^2 = 1. \tag{49}$$

Since $T^* \perp B_1^*$, it follows from (39) and (49) that $\eta/\zeta = -y/x$, which implies that

$$xk_1 + xk_2 - yk_3 = 0. \tag{50}$$

From (45) and (50), we can see that

$$xk_2 - yk_3 = -xk_1, \qquad (z - w)xk_1 = 0. \tag{51}$$

Since $z \neq 0$, it follows from (51) that $z = w$. Hence (47) turns into

$$B_1^* = -yN + xB_2, \qquad f'k_2^* = -\frac{y}{w}k_1 + \frac{z}{x}k_3. \tag{52}$$

Differentiating (52) about s and using (1), we get

$$-f'k_2^*N^* + f'k_3^*B_2^* = yk_1T - (yk_2 + xk_3)B_1,$$

from which we obtain

$$f'k_3^*B_2^* = f'k_2^*N^* + yk_1T - (yk_2 + xk_3)B_1 = -\frac{z^2}{x}k_3(-T + B_1). \tag{53}$$

It follows that from (53) that

$$B_2^* = -zT + wB_1, \qquad f'k_3^* = -\frac{z}{x}k_3. \tag{54}$$

Summarizing the above discussions, we obtain the following

Theorem 4.1 *Let $\beta : I \to E^4$ be a regular curve with arc-length parameter s so that k_1, k_2 and k_3 are not zero.If β possesses the $(1,3)$-evolute mate curve*

$$\beta^*(s) = \beta(s) + \frac{1}{xk_1(s)}\Big[xN(s) + yB_2(s)\Big] - \frac{1}{k_3(s)}B_2(s),$$

then k_1, k_2 and k_3 satisfy $xk_1 + xk_2 - yk_3 = 0$, where x and y are given constants. Moreover, the three curvatures of β^* are given by

$$k_1^* = -\sqrt{2}(xk_1)/f', \quad k_2^* = \sqrt{2}[k_3/(2x) - yk_1]/f', \quad k_3^* = -\sqrt{2}k_3/(2xf'), \quad f' = \Big(1/xk_1\Big)'. \tag{55}$$

The associated Frenet Frame are given by

$$T^* = xN+yB_2, \quad N^* = (T+B_1)/\sqrt{2}, \quad B_1^* = -yN+xB_2, \quad B_2^* = (-T+B_1)/\sqrt{2}.$$

Case 2: $z = 0$**.** In this case, we may suppose that

$$N^* = T, \qquad f'k_1^* = -xk_1, \qquad xk_2 - yk_3 = 0. \tag{56}$$

Moreover, we have from (41) and the third equation of (56) that

$$u = x(f + f_0) = \frac{1}{k_1}, \qquad v = y(f + f_0) = \frac{y}{xk_1}.$$

Differentiating (56) about s and using (1), we get

$$-f'k_1^*T^* + f'k_2^*B_1^* = k_1N.$$

It follows that we may choose

$$B_1^* = -yN + xB_2, \qquad f'k_2^* = -yk_1. \tag{57}$$

Differentiating (57) about s, using (1) and the third equation of (56), we get

$$B_2^* = B_1, \qquad f'k_3^* = -(yk_2 + xk_3) = x^{-1}k_3. \tag{58}$$

Summarizing the above discussions, we obtain the following

Theorem 4.2 *Let $\beta : I \to E^4$ be a regular curve with arc-length parameter s so that k_1, k_2 and k_3 are not zero. If β possesses the $(1,3)$-evolute mate curve*

$$\beta^*(s) = \beta(s) + \frac{1}{xk_1(s)}\Big[xN(s) + yB_2(s)\Big],$$

then k_2 and k_3 satisfy $xk_2 - yk_3 = 0$, where x and y are given constants. Moreover, the three curvatures of β^* are given by

$$k_1^* = -xk_1/f', \qquad k_2^* = -yk_1/f', \qquad k_3^* = x^{-1}k_3/f', \qquad f' = \Big(\frac{1}{xk_1}\Big)'. \quad (59)$$

The associated Frenet Frame are given by

$$T^* = xN + yB_2, \quad N^* = T, \quad B_1^* = -yN + xB_2, \quad B_2^* = B_1.$$

5 Main results

In this section a kind of generalized involute and evolute curve-couple is considered in 4 dimensional Euclidean space. We obtained 4 theorems for a curve possessing generalized involute as well as evolute curve.

6 Open Problem

In this study a kind of generalized involute and evolute curve-couple is considered in 4 dimensional Euclidean space. The necessary and sufficient condition for the a curve possessing generalized involute as well as evolute curve is obtained. This kind of work maybe possible for Minkowski space-time which will be more interesting.

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