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Monotonicity, Convexity and Inequalities for the Generalized Complete (p, q)-Elliptic Integrals

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Abstract

In this paper, we discuss monotonicity, convexity about the functions involving the generalized complete (p, q)-elliptic integrals, and establish some new inequalities.

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1 Introduction

The classical Legendre's complete elliptic integrals of the first and the second kind are defined by

$$\begin{cases} \kappa = \kappa(r) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-r^2\sin(t)^2}} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-r^2t^2)}}, \\ \varepsilon = \varepsilon(r) = \int_0^{\pi/2} \sqrt{1-r^2\sin(t)^2} = \int_0^1 \sqrt{\frac{1-r^2t^2}{1-t^2}}dt, \\ \kappa(0) = \frac{\pi}{2} = \varepsilon(0), \quad \kappa(1) = \infty, \quad \varepsilon(1) = 1, \\ \kappa' = \kappa'(r) = \kappa(r'), \quad \varepsilon' = \varepsilon'(r) = \varepsilon(r'), \end{cases}$$

respectively, where $0 < r < 1$ and $r' = \sqrt{1-r^2}$. These integrals not only play an important theoretical role in mathematical branches such as complex power system, analytic number theory, geometric function theory, Ramanujan's modular equation and low-dimensional topologies, but also have a wide range of applications in mathematics, theoretical physics, information science and so on [1, 2, 3]. In the 1990s, Carlson and Gustafson established asymptotic properties of complete elliptic integrals. Meanwhile, Qiu and his collaborators [4, 5, 6] gave the monotonicity and convexity of these integrals and other series of new analytical properties and inequalities. Subsequently, the elliptic integrals have been extensive interest of several researchers. They have been extending the research on elliptic integrals to generalized elliptic integrals and gave the series of new properties of special functions closely related to complete elliptic integral.

Generalized elliptic integral is the generalizations of complete elliptic integral. Numerous results show that the applications of generalized elliptic integrals are more and more prominent in analytic number theory, quasi-conformal analysis, hyperbolic geometry, the theory of mean values and other fields. For example, it has been proved that the transform of Schwarz-Christoffel that transforms the upper half plane to the topological quadrilateral is closely related to the generalized elliptic integral [7, 8]. The generalization of some properties about complete elliptic integrals to generalized elliptic integrals has become a hot topic that researchers are concerned with in this field.

Among several kinds of generalized elliptic integral, the generalized (p, q)-elliptic integrals attracts more attention of researchers. For all $p, q \in (1, \infty)$,$r \in (0, 1)$ and $r' = (1 - r^p)^{\frac{1}{p}}$, the generalized complete (p, q)-elliptic integrals of the first and the second kind are defined by

$$K_{p,q}(r) = \int_0^{\pi_{p,q}/2} (1 - r^q sin_{p,q}^q t)^{1/p-1} dt, K'_{p,q} = K'_{p,q}(r) = K_{p,q}(r'),$$

and

$$E_{p,q}(r) = \int_0^{\pi_{p,q}/2} (1 - r^q sin_{p,q}^q t)^{1/p} dt, E'_{p,q} = E'_{p,q}(r) = E_{p,q}(r'),$$

respectively. Here, the function $\sin_{p,q}$ is known as the generalized sine function with two parameters $p, q > 1$ in the literature[10]-[17], and defined as the inverse function of

$$\text{arcsin}_{p,q}(x) = \int_0^x (1 - t^p)^{-\frac{1}{q}} dt, 0 < x < 1.$$

These functions play an important role in the so-called one-dimensional (p, q)-Laplacian problem (See [19])

$$-\Delta_p u = -(|u'|^{p-2}u')' = \lambda |u|^{q-2}u, u(0) = u(1) = 0, p, q > 1$$

Also the generalized $\pi_{p,q}$ denotes the half-period of $\sin_{p,q}$ and it is defined as

$$\pi_{p,q} = 2\,\mathrm{arcsin}_{p,q}(1) = \frac{2}{q}B(1-\frac{1}{p},\frac{1}{q}),$$

which is the generalized version of the celebrated formula of π proved by Salamin and Brent in 1976. Here $B(.,.)$ denotes the classical beta function. More details about the generalized complete (p,q)-elliptic integrals can be seen in literature [9].

Moreover, the generalized complete (p,q)-elliptic integrals can be expressed in terms of the hypergeometric functions as follows:

$$K_{p,q}(r) = \frac{\pi_{p,q}}{2}F(\frac{1}{q},1-\frac{1}{p};1-\frac{1}{p}+\frac{1}{q};r^p),$$

and

$$E_{p,q}(r) = \frac{\pi_{p,q}}{2}F(\frac{1}{q},-\frac{1}{p};1-\frac{1}{p}+\frac{1}{q};r^p),$$

where $F(a,b;c;r)$ denotes the hypergeometric function which has the infinite series representation

$$F(a,b;c;r) = {}_2F_1(a,b;c;r) = \sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{r^n}{n!}, |r|<1,$$

for given complex numbers a,b and c with $c \neq 0,-1,-2,\cdots$, where$(a)_0 = 1$ and $(a,n) = \prod_{k=0}^{n-1}(a+k)$ is shifted factorial function or the Appell symbol. For $p=q$, we note $K_{p,q} = K_p$. It is easy to see that $K_2 = \kappa$ and $E_2 = \varepsilon$.

ALzer and Richards [22] studied the monotonicity and convexity properties of the function

$$\Delta(r) = \frac{\varepsilon-(1-r^2)\kappa}{r^2} - \frac{\varepsilon'-r^2\kappa'}{1-r^2}.$$

Recently, Bhayo and Yin [23] extend their results to the counterpart function involving the generalized complete (p,q)-elliptic integrals of the first and second kinds

$$\Delta_{p,q}(r) = \frac{E_{p,q}-(1-r^p)K_{p,q}}{r^p} - \frac{E'_{p,q}-r^pK'_{p,q}}{1-r^p}.$$

While, in 1998, Anderson, Qiu and Vamanamurthy [24] have given the monotonicity and convexity of the similar function

$$f(r) = \frac{\varepsilon-r'^2\kappa}{r^2}\cdot\frac{r'^2}{\varepsilon'-r\kappa'}$$

by the theorem below.

Theorem 1.1.[23, Theorem 1.14] The function $f(r)$ is increasing and convex from $(0,1)$ onto$(\pi/4, 4/\pi)$. In particular,

$$\frac{\pi}{4} < f(r) < \frac{\pi}{4} + (\frac{4}{\pi} - \frac{\pi}{4})$$

for $r \in (0,1)$. These two inequalities are sharp as $r \to 0$, while the second inequality is also sharp as $r \to 1$.

Very recently, Huang, Tan and Zhang[27] generalized Theorem 1.1 to generalized elliptic integrals

$$K_a(r) = \tfrac{\pi}{2} F\left(a, 1-a; 1; r^2\right),$$
$$E_a(r) = \tfrac{\pi}{2} F\left(a-1, 1-a; 1; r^2\right).$$

It is natural to extend the above result to the generalized complete (p,q)-elliptic integrals. So we generalize their function $f(r)$ by

$$f_{p,q}(r) = \frac{E_{p,q} - r'^p K_{p,q}}{r^p} \cdot \frac{r'^p}{E'_{p,q} - rK'_{p,q}} \tag{1.1}$$

This paper discusses the case of $p = q$, i.e. we give the monotonicity and convexity of the function

$$f_{p,p}(r) = \frac{E_{p,p} - r'^p K_{p,p}}{r^p} \cdot \frac{r'^p}{E'_{p,p} - rK'_{p,p}}, \tag{1.2}$$

where

$$K_{p,p}(r) = K_p(r) = \frac{\pi_p}{2} F(\frac{1}{p}, 1 - \frac{1}{p}; 1; r^p),$$

and

$$E_{p,p}(r) = E_p(r) = \frac{\pi_p}{2} F(\frac{1}{p}, -\frac{1}{p}; 1; r^p).$$

If $p = q = 2$, our results return to above Theorem 1.1, which has been showed in [24].

2 Lemmas

In the section, we shall prove severally useful lemmas.

Lemma 2.1 *[23, Lemma2.1] Let*

$$H_{a,b}(r) = \frac{\pi_{1/b,1/a}}{2r^{1/b}}[F(a, -b; 1 + a - b; r^{1/b} - (1 - r^{1/b})F(a, 1 - b; 1 + a - b; r^{1/b})].$$

For $a, b, r \in (0,1)$, we have

$$H_{a,b}(r) = \frac{(1-b)\pi_{1/b,1/a}}{2(1+a-b)} F(a, 1-b; 2+a-b; r^{1/b}).$$

Lemma 2.2 *[23, Lemma2.3] For $p, q > 1$ and $r \in (0, 1)$, we have*

$$H_{1/q,1/p}(0) = \frac{(1-\frac{1}{p})\pi_{p,q}}{2(1+\frac{1}{q}-\frac{1}{p})}, H_{1/q,1/p}(1) = 1.$$

Lemma 2.3 *[25, Theorem 3.12] For $a, b, c > 0, r \in (0, 1)$, let$u = u(r) = F(a-1, b; c; r)$,$v = v(r) = F(a, b; c; r)$, $u_1 = u(1-r), v_1 = v(1-r)$. Then*

$$r\frac{du}{dr} = (a-1)(v-u), \tag{2.1}$$

$$r(1-r)\frac{dv}{dr} = (c-a)u + (a-c+br)v, \tag{2.2}$$

Lemma 2.4

$$\frac{dK_{p,q}}{dr} = \frac{\frac{p}{q}[E_{p,q} - (1-r^p)K_{p,q}]}{r(1-r^p)}, \tag{2.3}$$

$$\frac{dE_{p,q}}{dr} = \frac{E_{p,q} - K_{p,q}}{r}, \tag{2.4}$$

$$\frac{d(K_{p,q} - E_{p,q})}{dr} = (\frac{p}{q} - 1)\frac{E_{p,q} - K_{p,q}}{r} + \frac{\frac{p}{q}r^{p-1}E_{p,q}}{1-r^p}, \tag{2.5}$$

$$\frac{d(E_{p,q} - r'^p K_{p,q})}{dr} = (1-\frac{p}{q})\frac{E_{p,q} - K_{p,q}}{r} + (p - \frac{p}{q})r^{p-1}K_{p,q}. \tag{2.6}$$

Proof. Formulas 2.3 and 2.4 follow from Lemma 2.3, while 2.5 and 2.6 follow from 2.3 and 2.4.

Remark 2.1. In Lemma 2.4, if $p = q$, we can conclude the results as below:

$$\frac{dK_p}{dr} = \frac{E_p - (1-r^p)K_p}{r(1-r^p)}, \tag{2.7}$$

$$\frac{dE_p}{dr} = \frac{E_p - K_p}{r}, \tag{2.8}$$

$$\frac{d(K_p - E_p)}{dr} = \frac{r^{p-1}E_p}{1-r^p}, \tag{2.9}$$

$$\frac{d(E_p - r'^p K_p)}{dr} = (p-1)r^{p-1}K_p. \tag{2.10}$$

Hence for $p = q = 2$, the above conclusions are the classical derivative formulas of the Legendre's complete elliptic integrals of the first and the second kinds.

Lemma 2.5 *[25, Lemma2.3] Let $I \subset \mathbb{R}$ be an interval, and let $f, g : I \to (0, \infty)$. If both f, g are convex and increasing(decreasing), then the product $f \cdot g$ is convex.*

Lemma 2.6 *[26, Lemma2.1] For $-\infty < a < b < \infty$, let $f, g : [a, b] \to \mathbb{R}$ be continuous and differentiable on $[a, b]$. Let $g'(x) \neq 0$ on $[a, b]$. If $f'(x)/g'(x)$ is increasing(decreasing) on $[a, b]$, then so are*

$$\frac{f(x) - f(a)}{g(x) - g(a)} \text{ and } \frac{f(x) - f(b)}{g(x) - g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

3 Main results

Theorem 3.1 *For $p, q \in (1, \infty)$, the function $\frac{E_{p,q}(r) - r'^p K_{p,q}(r)}{r^p}$ is strictly increasing and strictly convex from $(0, 1)$ onto $\left(\frac{(1-\frac{1}{p})\pi_{p,q}}{2(1+\frac{1}{q}-\frac{1}{p})}, 1\right)$.*

Proof. Let

$$H(r) = \frac{[E_{p,q}(r) - r'^p K_{p,q}(r)]}{r^p}.$$

Applying the Lemma 2.1 and letting $a = \frac{1}{q}, b = \frac{1}{p}$, we can easily find

$$H(r) = H_{1/q,1/p}(r) = \frac{(1 - \frac{1}{p})\pi_{p,q}}{2(1 + \frac{1}{q} - \frac{1}{p})} F(\frac{1}{q}, 1 - \frac{1}{p}; 2 + \frac{1}{q} - \frac{1}{p}; r^p).$$

Applying the derivative formula for the hypergeometric function

$$\frac{d}{dr} F(a, b; c; r) = \frac{ab}{c} F(a + 1, b + 1; c + 1; r),$$

we can obtain

$$H'(r) > 0, H''(r) > 0.$$

Combining with Lemma 2.2, we can get the results. The proof is complete.

Theorem 3.2 *The function $f_{p,q}(r)$ in 1.1 is increasing from $(0, 1)$ onto $(\frac{(1-\frac{1}{p})\pi_{p,q}}{2(1+\frac{1}{q}-\frac{1}{p})}, \frac{2(1+\frac{1}{q}-\frac{1}{p})}{(1-\frac{1}{p})\pi_{p,q}})$. As $r \to 0$ and $r \to 1$, the following inequality holds true:*

$$\frac{(1 - \frac{1}{p})\pi_{p,q}}{2(1 + \frac{1}{q} - \frac{1}{p})} + \alpha r < f_{p,q}(r) < \frac{(1 - \frac{1}{p})\pi_{p,q}}{2(1 + \frac{1}{q} - \frac{1}{p})} + \beta r,$$

for $r \in (0, 1)$ with the best constant $\alpha = 0, \beta = \frac{2(1+\frac{1}{q}-\frac{1}{p})}{(1-\frac{1}{p})\pi_{p,q}} - \frac{(1-\frac{1}{p})\pi_{p,q}}{2(1+\frac{1}{q}-\frac{1}{p})}$.

Proof. Rewrite

$$H(r)=H_{1/q,1/p}(r)=\frac{(1-\frac{1}{p})\pi_{p,q}}{2(1+\frac{1}{q}-\frac{1}{p}}F(\frac{1}{q},1-\frac{1}{p};2+\frac{1}{q}-\frac{1}{p};r^p).$$

We have

$$f_{p,q}(r)=H(r)\cdot\frac{1}{H(r')}.$$

By Theorem 3.1, $H(r)$ and $\frac{1}{H(r')}$ are positive and increasing functions on $(0,1)$. This implies that the function $f_{p,q}(r)$ is increasing on $(0,1)$. The proof is complete.

Theorem 3.3 *The function$f_{p,p}(r)$ in 1.2 is increasing and convex from* $(0,1)$ *onto* $(\frac{(1-\frac{1}{p})\pi_p}{2},\frac{2}{(1-\frac{1}{p})\pi_p})$. *For all* $r\in(0,1)$, *we have*

$$\frac{(1-\frac{1}{p})\pi_p}{2}+\alpha_1 r<f_{p,p}(r)<\frac{(1-\frac{1}{p})\pi_p}{2}+\beta_1 r,$$

with the best constants $\alpha_1=0$, *and* $\beta_1=\frac{2}{(1-\frac{1}{p})\pi_p}-\frac{(1-\frac{1}{p})\pi_p}{2}$.

Proof. Take $p=q$ in Theorem 3.2 and it is obvious that the function $f_{p,q}(r)=f_{p,p}(r)=\frac{E_{p,p}-r'^pK_{p,p}}{r^p}\cdot\frac{r'^p}{E'_{p,p}-rK'_{p,p}}$ is increasing on $(0,1)$. In addition, the part $H(r)=H_{1/p,1/p}(r)=\frac{E_{p,p}-r'^pK_{p,p}}{r^p}$ is strictly convex by Theorem 3.1. Then we can get the convexity of the function $f_{p,p}(r)$ by Lemma 2.5 if it is proved to be true that $\frac{1}{H_{1/p,1/p}(r')}$ is convex on $(0,1)$.

According to Lemma 2.4 and (2.7)-(2.10), we have

$$\left(\frac{1}{H_{1/p,1/p}(r)}\right)'=\left(\frac{r^p}{E_{p,p}-r'^pK_{p,p}}\right)'=\frac{g_1(r)}{g_2(r)},$$

where

$$g_1(r)=p(E_{p,p}-K_{p,p})+r^pK_{p,p}\text{ and }g_2(r)=\frac{[E_{p,p}-r'^pK_{p,p}]^2}{r^{p-1}}.$$

Easy computation yields

$$\frac{g_1'(r)}{g_2'(r)}=\frac{r^{2p-1}}{(1-r^p)(E_{p,p}-r'^pK_{p,p})}.$$

Hence taking into account

$$\begin{aligned}y(x)&=(1-x)(E_{p,p}-(1+x)K_{p,p})\\&=(1-x)(F(-\frac{1}{p},\frac{1}{p};1;x)-(1+x)F(1-\frac{1}{p},\frac{1}{p};1;x)),\end{aligned}$$

we get

$$
\begin{aligned}
y(x) &= (1-x)(\sum_{n=0}^{\infty}\frac{(-\frac{1}{p},n)(\frac{1}{p},n)}{n!}\cdot\frac{x^n}{n!}-(1+x)\sum_{n=0}^{\infty}\frac{(1-\frac{1}{p},n)(\frac{1}{p},n)}{n!}\cdot\frac{x^n}{n!})\\
&= (1-x)(\sum_{n=0}^{\infty}[\frac{(-\frac{1}{p},n)(\frac{1}{p},n)}{n!}-\frac{(1-\frac{1}{p},n)(\frac{1}{p},n)}{n!}]\frac{x^n}{n!}\\
&\quad -\sum_{n=0}^{\infty}\frac{(1-\frac{1}{p},n)(\frac{1}{p},n)}{n!}\cdot\frac{x^{n+1}}{n!})\\
&= (1-x)\sum_{n=1}^{\infty}\frac{(-n)(2n+\frac{1}{p}-1)(1-\frac{1}{p},n-1)(\frac{1}{p},n-1)}{n!}\cdot\frac{x^n}{n!}\\
&= \sum_{n=1}^{\infty}\frac{(-n)(2n+\frac{1}{p}-1)(1-\frac{1}{p},n-1)(\frac{1}{p},n-1)}{n!}\cdot\frac{x^n}{n!}\\
&\quad -\sum_{n=1}^{\infty}\frac{(-n)(2n+\frac{1}{p}-1)(1-\frac{1}{p},n-1)(\frac{1}{p},n-1)}{n!}\cdot\frac{x^{n+1}}{n!}\\
&= [-2+(1-\frac{1}{p})]x+\sum_{n=2}^{\infty}\frac{(1-\frac{1}{p},n-2)(\frac{1}{p},n-2)}{n!\cdot n!}\cdot\epsilon\cdot x^n
\end{aligned}
$$

where $\epsilon = 2n^2(n-2)+\frac{2}{p^2}n^2+(2+\frac{1}{p}-\frac{1}{p^2})(1-\frac{1}{p})n$, and we can conclude that $\epsilon > 0$ for $p,q\in(1,\infty)$ and $n\geq 2$.

We notice that

$$\frac{g_1'(r)}{g_2'(r)} = \frac{r^{2p-1}}{y(r^p)},$$

then

$$\frac{y(r^p)}{r^{2p-1}} = \sum_{n=2}^{\infty}\frac{(1-\frac{1}{p},n-2)(\frac{1}{p},n-2)}{n!\cdot n!}\cdot\epsilon\cdot r^{p(n-2)+1}-\frac{[1+\frac{1}{p}]}{r^{p-1}}.$$

It is esaily to see that $\frac{y(r^p)}{r^{2p-1}}$ is increasing, so $\frac{g_1'(r)}{g_2'(r)}$ is decreasing. we can get $(\frac{1}{H_{1/p,1/p}(r)})'$ is decreasing from Lemma 2.6. Then $(\frac{1}{H_{1/p,1/p}(r')})'$ is increasing. It concludes that $\frac{1}{H_{1/p,1/p}(r')}$ is convex on$(0,1)$. According to Lemma 2.5, we get that function$f_{p,p}(r) = \frac{E_{p,p}-r'^pK_{p,p}}{r^p}\cdot\frac{r'^p}{E'_{p,p}-rK'_{p,p}}$ in 1.2 is convex. This completes the proof.

Remark 3.1 If $p = q = 2$, the above Theorem 3.3 can be reduced to Theorem 1.1 [23, Theorem 1.14].

4 Open Problem

Opem problem 4.1. Very recently, Takeuchi [18] gave a new complete $(p, q, r)-$ elliptic integrals with three parameters. These integrals are defined by

$$K_{p,q,r}(k) := \int_0^1 \frac{dt}{(1-t^q)^{\frac{1}{p}}(1-k^q t^q)^{\frac{1}{r}}} \tag{4.1}$$

and

$$E_{p,q,r}(k) := \int_0^1 \frac{1-k^q t^{q\,1/r^*}}{1-t^{q\,\frac{1}{p}}} dt, \tag{4.2}$$

where$p \in \mathbb{P}^* := (-\infty, 0) \cup (1, \infty], q, r \in (1, \infty)$ and $1/r + 1/r^* = 1$.

It is natural how to generalize main results of the paper to new complete $(p, q, r)-$ elliptic integrals with three parameters?

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