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On triple sequence spaces of Bernstein operator of χ^3 of rough λ-statistical convergence in probability of random variables defined by Musielak-Orlicz function

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Abstract

We introduced the triple sequence spaces of Bernstein polynomials of χ^3 of rough $\lambda-$ statistical convergence in probability of random variables and discuss general properties of among these sequence spaces defined by Musielak-Orlicz function.

Keywords: *Rough $\lambda-$ Statistical Convergence, Triple Sequences, $\chi-$ Sequence, Bernstein Polynomials.*

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1 Introduction

The idea of rough convergence was introduced by Phu [12], who also introduced the concepts of rough limit points and roughness degree. The idea of rough convergence occurs very naturally in numerical analysis and has interesting applications. Aytar [1] extended the idea of rough convergence into rough statistical convergence using the notion of natural density just as usual convergence was extended to statistical convergence. Pal et al. [11] extended the notion of rough convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence.

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}\,(\mathbb{C})$, where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by *Sahiner et al. [13, 14]*, *Esi et al. [2, 3, 4, 5]*, *Dutta et al. [6]*,*Subramanian et al. [15]*, *Esi et al. [19, 20, 21]*, *Debnath et al. [7]* and many others.

A triple sequence $x = (x_{mnk})$ is said to be triple analytic if

$$\sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple analytic sequences are usually denoted by Λ^3. A triple sequence $x = (x_{mnk})$ is called triple gai sequence if

$$((m+n+k)!\,|x_{mnk}|)^{\frac{1}{m+n+k}} \to 0 \text{ as } m,n,k \to \infty.$$

The space of all triple gai sequences are usually denoted by χ^3.

Let K be a subset of the set of positive integers $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and let us denote the set $K_{ik\ell} = \{(m,n,k) \in K : m \geq i, n \leq j, k \leq \ell\}$. Then the natural density of K is given by

$$\delta\,(K) = \lim_{i,j,\ell\to\infty} \frac{|K_{ij\ell}|}{ij\ell},$$

where $|K_{ij\ell}|$ denotes the number of elements in $K_{ij\ell}$.

The Bernstein operator of order (r,s,t) is given by

$$B_{rst}(f,x) = \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} f\left(\frac{mnk}{rst}\right)\binom{r}{m}\binom{s}{n}\binom{t}{k} x^{m+n+k}(1-x)^{(m-r)+(n-s)+(k-t)}$$

where f is a continuous (real or complex valued) function defined on $[0,1]$.

Throughout the paper, $\mathbb{R}$ denotes the real of three dimensional space with metric (X,d). Consider a triple sequence of Bernstein polynomials $(B_{mnk}\,(f,x))$ such that $(B_{mnk}\,(f,x)) \in \mathbb{R}$, $m,n,k \in \mathbb{N}$.

Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(B_{rst}\,(f,x))$ is said to be statistically convergent to $0 \in \mathbb{R}$, written as $st - \lim\, x = 0$, provided that the set

$$K_\epsilon := \left\{(m,n,k) \in \mathbb{N}^3 : |B_{mnk}\,(f,x) - f\,(x)| \geq \epsilon\right\}$$

has natural density zero for any $\epsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence of Bernstein polynomials. i.e., $\delta\,(K_\epsilon) = 0$. That is,

$$\lim_{rst\to\infty} \frac{1}{rst}\,|\{(m,n,k) \leq (r,s,t) : |B_{mnk}\,(f,x) - (f,x)| \geq \epsilon\}| = 0.$$

In this case, we write $\delta - \lim B_{mnk}\,(f,x) = f\,(x)$ or $B_{mnk}\,(f,x) \to^{S_B} f\,(x)$.

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers, χ_A- the characteristic function of $A \subset \mathbb{N}$, $\mathbb{R}$ the set of all real numbers. A subset A of $\mathbb{N}$ is said to have asymptotic density $d\left(A\right)$ if

$$d\left(A\right) = \lim_{ij\ell \to \infty} \frac{1}{ij\ell} \sum_{m=1}^{i} \sum_{n=1}^{j} \sum_{k=1}^{\ell} \chi_A\left(K\right).$$

2 Preliminaries

Definition 2.1 *An Orlicz function ([see [8]) is a function* $M : [0, \infty) \to [0, \infty)$ *which is continuous, non-decreasing and convex with* $M\left(0\right) = 0, M\left(x\right) > 0,$ *for* $x > 0$ *and* $M\left(x\right) \to \infty$ *as* $x \to \infty$*. If convexity of Orlicz function* M *is replaced by* $M\left(x + y\right) \leq M\left(x\right) + M\left(y\right)$*, then this function is called modulus function.*

Lindenstrauss and Tzafriri ([9]) used the idea of Orlicz function to construct Orlicz sequence space.

A sequence $g = \left(g_{mn}\right)$ *defined by*

$$g_{mn}\left(v\right) = \sup\left\{\left|v\right| u - \left(f_{mnk}\right)\left(u\right) : u \geq 0\right\}, m, n, k = 1, 2, \ldots$$

is called the complementary function of a Musielak-Orlicz function f*. For a given Musielak-Orlicz function* f*, [see [10]] the Musielak-Orlicz sequence space* t_f *is defined as follows*

$$t_f = \left\{x \in w^3 : I_f\left(\left|x_{mnk}\right|\right)^{1/(m+n+k)} \to 0\, as m, n, k \to \infty\right\},$$

where I_f *is a convex modular defined by*

$$I_f\left(x\right) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}\left(\left|x_{mnk}\right|\right)^{1/(m+n+k)}, x = \left(x_{mnk}\right) \in t_f.$$

We consider t_f *equipped with the Luxemburg metric*

$$d\left(x, y\right) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}\left(\frac{\left|x_{mnk}\right|^{1/(m+n+k)}}{mnk}\right)$$

is an extended real number.

Definition 2.2 *Let* X, Y *be a real vector space of dimension* w*, where* $n \leq m$*. A real valued function* $d_p(x_1, \ldots, x_n) = \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p$ *on* X *satisfying the following four conditions:*

(i) $\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = 0$ *if and only if* $d_1(x_1, 0), \ldots, d_n(x_n, 0)$ *are linearly dependent,*

(ii) $\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p$ *is invariant under permutation,*

(iii) $\|(\alpha d_1(x_1,0),\ldots,d_n(x_n,0))\|_p = |\alpha|\,\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p, \alpha \in \mathbb{R}$

(iv) $d_p((x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)) = (d_X(x_1,x_2,\ldots,x_n)^p + d_Y(y_1,y_2,\ldots,y_n)^p)^{1/p}$ *for* $1 \leq p < \infty$*; (or)*

(v) $d((x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)) := \sup\{d_X(x_1,x_2,\ldots,x_n), d_Y(y_1,y_2,\ldots,y_n)\}$, *for* $x_1,x_2,\ldots,x_n \in X$, $y_1,y_2,\ldots,y_n \in Y$ *is called the p product metric of the Cartesian product of n metric spaces (see [16]).*

Definition 2.3 *Let* $\eta = (\lambda_{abc})$ *be a non-decreasing sequence of positive real numbers tending to infinity and* $\lambda_{111} = 1$ *and* $\lambda_{a+b+c+3} \leq \lambda_{a+b+c+3} + 1$*, for all* $a,b,c \in \mathbb{N}$*. The collection of all such triple sequences of Bernstein polynomials of* λ *is denoted by* $\Im$*.*

The generalized de la Vallée-Poussin means are defined by

$$t_{abc}\left(B_{mnk}\left(f,x\right)\right) = \lambda_{abc}^{-1} \sum_{m,n,k \in I_{abc}} B_{mnk}\left(f,x\right),$$

where $I_{abc} = [abc - \lambda_{abc} + 1, abc]$*. A triple sequence space of Bernstein polynomials of* $(B_{mnk}(f,x))$ *is said to* $(V,\lambda)-$ *summable to a real number* $f(x)$ *if* $t_{abc}(B_{mnk}(f,x)) \rightarrow f(x)$*, as* $abc \rightarrow \infty$*.*

Definition 2.4 *Let* f *be a continuous function defined on the closed interval* $[0,1]$*. A triple sequence of Bernstein polynomials* $(B_{mnk}(f,x))$ *of real numbers is said to strong* (V,λ) *summable (or shortly :* $[V,\lambda]-$ *convergent) to* $f(x)$ *if* $\lim \frac{1}{\lambda_{abc}} \sum_{m\in I_a} \sum_{n \in I_b} \sum_{k\in I_c} |B_{mnk}(f,x), f(x)| = 0$*. In this case write* $B_{mnk}(f,x) \rightarrow^{[V,\lambda]} f(x)$*.*

Definition 2.5 *Let* f *be a continuous function defined on the closed interval* $[0,1]$*. A triple sequence of Bernstein polynomials* $(B_{mnk}(f,x))$ *of real numbers is said to be* $\lambda-$ *statistically convergent (or shortly:* $S_\lambda-$ *convergent) to* $f(x)$ *if for any* $\epsilon > 0$, $\lim_{abc\rightarrow\infty} \frac{1}{\lambda_{abc}} |\{(m,n,k) \in I_{abc} : |B_{mnk}(f,x), f(x)| \geq \epsilon\}| = 0$*. In this case we write* $S_\lambda - \lim B_{mnk}(f,x) = f(x)$ *or by* $B_{mnk}(f,x) \rightarrow^{S_\lambda} f(x)$*.*

Now we introduce the following main definition:

Definition 2.6 *Let* f *be a continuous function defined on the closed interval* $[0,1]$*. A triple sequence of Bernstein polynomials* $(B_{mnk}(f,x))$ *of real numbers of random variables and* α *be non negative real number is said to be rough* $[V,\lambda]-$ *summable in probability to* $B_{mnk}(f,x) : W \times W \times W \rightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ *with respect to the roughness of degree* α *(or shortly:* $\alpha - [V,\lambda]-$ *summable in probability) to* $f(x)$ *if for any* $\epsilon > 0$,

$$\lim_{abc\rightarrow\infty} \frac{1}{\lambda_{abc}} \sum_{m\in I_a} \sum_{n\in I_b} \sum_{k\in I_c} P\left(\left|B_{mnk}\left(f,x\right), f\left(x\right)\right| \geq \alpha + \epsilon\right) = 0.$$

In this case we write $B_{mnk}(f,x) \rightarrow_{\alpha}^{[V,\lambda]^P} f(x)$. *The class of all rough* $[V,\lambda]-$ *summable triple sequence spaces of Bernstein polynomials of random variables in probability will be denoted simply by* $\alpha\,[V,\lambda]^P$.

Definition 2.7 *Let* f *be a continuous function defined on the closed interval* $[0,1]$. *A triple sequence of Bernstein polynomials* $(B_{mnk}(f,x))$ *of real numbers of random variables and* α *be non negative real number is said to be rough* $\lambda-$ *statistically convergent in probability to* $B_{mnk}(f,x) : W \times W \times W \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ *with respect to the roughness of degree* α *(or shortly:* $\alpha - \lambda-$ *statistically convergent in probability) to* $f(x)$ *if for any* $\epsilon, \delta > 0$, $\lim_{abc\to\infty} \frac{1}{\lambda_{abc}} \left|\{(m,nk) \in I_{abc} : P\left(|B_{mnk}(f,x), f(x)| \geq \alpha+\epsilon\right) \geq \delta\}\right| = 0$. *In this case we write* $B_{mnk}(f,x) \rightarrow_{\alpha}^{S_\lambda^P} f(x)$. *The class of all* $\alpha - \lambda-$ *statistically convergent triple sequence spaces of Bernstein polynomials of random variables in probability will be denoted simply by* αS_λ^P.

Note 2.8 *Let* f *be a continuous function defined on the closed interval* $[0,1]$. *A triple sequence of Bernstein polynomials* $(B_{mnk}(f,x))$ *of real numbers and* M *be an Musielak-Orlicz function,*

$$\left\|\chi^3_{M}, \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|_p = \left[M_{mnk}\left(\left\|\mu_{mnk}(X), \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|_p\right)\right],$$

where $\mu_{mnk}(X) = \left(\left((m+n+k)!B_{mnk}(f,x)\right)^{1/(m+n+k)}, f(x)\right)$.

3 Main results

Theorem 3.1 *Let* f *be a continuous function defined on the closed interval* $[0,1]$. *A triple sequence of Bernstein polynomials* $(B_{mnk}(f,x))$ *of real numbers of random variables are equivalent:*

(i) $\left\|\chi^3_M(B_{mnk}(f,x)), \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|_p$ *is* $\alpha-[V,\lambda]-$ *summable in probability to* $f(x)$.

(ii) $\left\|\chi^3_M(B_{mnk}(f,x)), \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|_p$ *is* $\alpha-\lambda-$ *statistically convergent in probability to* $f(x)$.

Proof. Similar to the proof of Theorem (3.1) in (see [18]). ■

Theorem 3.2 *Let* f *be a continuous function defined on the closed interval* $[0,1]$. *A triple sequence of Bernstein polynomials* $(B_{mnk}(f,x))$ *of real numbers. If* $\left\|\chi^3_M(B_{mnk}(f,x)), \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|_p \rightarrow_{\alpha}^{S^P} f(x)$ *and* $\left\|\chi^3_M(B_{mnk}(f,y)), \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|_p \rightarrow_{\beta}^{S^P} f(y)$ *then*

$$P\left(\left|\left[M_{mnk}\left(\left\|\mu_{mnk}(X), \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|_p\right)\right]\right| \geq \alpha+\epsilon\right) = 0.$$

Proof. Similar to the proof of Theorem (3.1) in (see [17]). ■

Theorem 3.3 *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ of real numbers. If $\lambda \in \Im$ is such that $\frac{\lambda_{abc}}{(abc)} = 1$ then $\alpha S^P_\lambda \subset \alpha S^P$.*

Proof. Let $0 < \eta < 1$ be given. Since $\lim_{abc \to \infty} \frac{\lambda_{abc}}{(abc)} = 1$, we can choose $(u,v,w) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $\left|\frac{\lambda_{abc}}{(abc)} - 1\right| < \frac{\eta}{2}$ for all $(a,b,c) > (u,v,w)$. Now, for $\epsilon, \delta > 0$

$$
\begin{aligned}
&\frac{1}{abc}\Big|\Big\{(mnk) \le (abc) : \\
&P\Big(\Big|\Big[M_{mnk}\Big(\|\mu_{mnk}(X),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|_p\Big)\Big]\Big| \ge \alpha+\epsilon\Big) \ge \delta\Big\}\Big| \\
&= \frac{1}{abc}\Big|\Big\{(mnk) \le (abc) - \lambda_{abc} : \\
&P\Big(\Big|\Big[M_{mnk}\Big(\|\mu_{mnk}(X),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|_p\Big)\Big]\Big| \ge \alpha+\epsilon\Big) \ge \delta\Big\}\Big| \\
&= \frac{1}{abc}\Big|\Big\{(mnk) \in I_{abc} : \\
&P\Big(\Big|\Big[M_{mnk}\Big(\|\mu_{mnk}(X),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|_p\Big)\Big]\Big| \ge \alpha+\epsilon\Big) \ge \delta\Big\}\Big| \\
&\le \frac{(abc)-\lambda_{abc}}{(abc)} + \frac{1}{abc}\Big|\Big\{(mnk) \in I_{abc} : \\
&P\Big(\Big|\Big[M_{mnk}\Big(\|\mu_{mnk}(X),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|_p\Big)\Big]\Big| \ge \alpha+\epsilon\Big) \ge \delta\Big\}\Big| \\
&\le 1 - \Big(1 - \frac{\eta}{2}\Big) + \frac{1}{abc}\Big|\Big\{(mnk) \in I_{abc} : \\
&P\Big(\Big|\Big[M_{mnk}\Big(\|\mu_{mnk}(X),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|_p\Big)\Big]\Big| \ge \alpha+\epsilon\Big) \ge \delta\Big\}\Big| \\
&= \frac{\eta}{2} + \frac{\lambda_{abc}}{abc}\frac{1}{\lambda_{abc}}\Big|\Big\{(mnk) \in I_{abc} : \\
&P\Big(\Big|\Big[M_{mnk}\Big(\|\mu_{mnk}(X),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|_p\Big)\Big]\Big| \ge \alpha+\epsilon\Big) \ge \delta\Big\}\Big| \\
&< \frac{\eta}{2} + \frac{1}{\lambda_{abc}}\Big|\Big\{(mnk) \in I_{abc} : \\
&P\Big(\Big|\Big[M_{mnk}\Big(\|\mu_{mnk}(X),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|_p\Big)\Big]\Big| \ge \alpha+\epsilon\Big) \ge \delta\Big\}\Big|
\end{aligned}
$$

holds for all $(a,b,c) \ge (u,v,w)$. ■

Theorem 3.4 *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ of real numbers and $\alpha S^P \subset \alpha S^P_\lambda$ if and only if $\lim \frac{\lambda_{abc}}{(abc)} > 0$.*

Proof. Let $\lim_{abc\to\infty}\frac{\lambda_{abc}}{(abc)} > 0$. Then for $\epsilon, \delta > 0$, we have

$$\frac{1}{abc}\left|\Big\{(mnk) \leq (abc):\right.$$
$$\left. P\left(\left|\left[M_{mnk}\left(\left\|\mu_{mnk}(X), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\right\|_p\right)\right]\right| \geq \alpha+\epsilon\right) \geq \delta\Big\}\right|$$
$$\geq \frac{1}{abc}\left|\Big\{(mnk) \in I_{abc}:\right.$$
$$\left. P\left(\left|\left[M_{mnk}\left(\left\|\mu_{mnk}(X), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\right\|_p\right)\right]\right| \geq \alpha+\epsilon\right) \geq \delta\Big\}\right|$$
$$= \frac{\lambda_{abc}}{abc}\frac{1}{\lambda_{abc}}\left|\Big\{(mnk) \in I_{abc}:\right.$$
$$\left. P\left(\left|\left[M_{mnk}\left(\left\|\mu_{mnk}(X), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\right\|_p\right)\right]\right| \geq \alpha+\epsilon\right) \geq \delta\Big\}\right|.$$

Taking limit $abc \to \infty$ we get

$$\left\|\chi^3_M(B_{mnk}(f,x)), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\right\|_p \to^{S^P}_{\alpha} f(x)$$
$$\Longrightarrow \left\|\chi^3_M(B_{mnk}(f,x)), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\right\|_p \to^{S^P_{\lambda}}_{\alpha} f(x).$$

Conversely, let

$$\lim_{abc\to\infty}\frac{\lambda_{abc}}{(abc)} = 0$$

then we can choose a subsequence $(a_u, b_v, c_w)_{u,v,w\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ such that $\frac{\lambda_{a_u,b_v,c_w}}{a_u,b_v,c_w} < \frac{1}{uvw}$ for all $u, v, w \in \mathbb{N}$. Define a triple sequence spaces of Bernstein polynomials of $(B_{mnk}(f,x))$ of random variables whose probability density function is

$$\mu_{abc}(X) = \begin{cases} 1, & \text{if } 0 < X < 1 \\ 0, & \text{otherwise, where } (abc) \in I_{abc} \text{ for some } (uvw) \in \mathbb{N} \end{cases}$$

Let $0 < \epsilon, \delta < 1$. Then

$$P\left(\left|\left[M_{mnk}\left(\left\|\mu_{mnk}(X), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\right\|_p\right)\right]\right| \geq 1+\epsilon\right)$$
$$= \begin{cases} 1, & \text{if } (abc) \in I_{abc} \text{ for some } (uvw) \in \mathbb{N} \\ \left(1-\frac{\epsilon}{2}\right)^n, & \text{otherwise.} \end{cases}$$

We have

$$\frac{1}{\lambda_{abc}}\left|\Big\{(mnk) \in I_{abc}:\right.$$
$$\left. P\left(\left|\left[M_{mnk}\left(\left\|\mu_{mnk}(X), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\right\|_p\right)\right]\right| \geq 1+\epsilon\right) \geq \delta\Big\}\right|$$
$$= \begin{cases} 1, & \text{if } (abc) \in I_{abc} \text{ for some } (uvw) \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}.$$

Hence $\left\|\chi^3_M(B_{mnk}(f,x)), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\right\|_p \not\to^{S^P_{\lambda}}_{\alpha}$. ■

4 Open Problem

We introduced triple sequence spaces of Bernstein polynomials of rough $\lambda-$ statistical convergence in probability of random variables with respect sequence of Musielak-Orlicz function. It is open problem that these spaces of Bernstein polynomials whether Banach spaces or not.

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