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Dynamic Process with Viscous Dissipation in Thermo-Viscoelasticity

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Abstract

We consider a mathematical model which describes the dynamic evolution of a thermo-viscoelastic linear body with taking into account the effects of internal forces which generate a non linear viscous dissipative function. We derive a variational formulation of the system consisting of a motion equation, and energy equation. An existence result of weak solutions was obtained in an appropriate function space.

Keywords: *Viscous dissipation, thermo-viscoelasticity.*
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1 Introduction

The constitutive laws with internal variables describe the behaviour of real bodies like metals, rocks polymers and so on, for which the rate of deformation depends on the internal variables. In the framework of the Mechanics of Cotinuum Medias, many interseting internal variables have been the literature some of them are the spatial display of dislocation, the work-hardening of materials, the damage and the absolute temperature, see for examples and details the references [3, 7, 9, 10, 14, 16, 17, 19].

The stress-strain behaviour has a direct relation with the temperature. Indeed, the work of internal forces generates a temperature and, inversely, variations of temperature may generate deformations in rigid materials. It has been proved experimently that mechanical properties change dramatically with temperature, going from glass-like brittle behaviour at low temperatures to a rubber-like behaviour at high temperatures. The phenomena of collision and interaction between particles constituting any material generate an energy dissipation, which can be written mathematically as the product of the stress tensor and the dissipative part of the strain rate tensor.

The object of this work is to study the dynamic evolution of linear thermo-viscoelastic materials taking into account the viscous dissipative. For this, we consider a rate-type constitutive law of the form

$$\frac{\partial \boldsymbol{\sigma}}{\partial t} = \mathcal{A}\left(\boldsymbol{\varepsilon}\left(\frac{\partial \mathbf{u}}{\partial t}\right)\right) + \mu(\theta)\left(\mathcal{G}_1(\boldsymbol{\sigma}) + \mathcal{G}_2(\varepsilon(\mathbf{u}))\right), \tag{1.1}$$

in which $\mathbf{u}$, $\boldsymbol{\sigma}$ represent, respectively, the displacement field and stress field, θ represents the absolute temperature, $\mathcal{A}$, $\mathcal{G}_1$ and $\mathcal{G}_2$ are real tensors and μ is a real function. The paper is organized as follows. In Section 2 we present the mechanical problem of the dynamic evolution of thermo-viscoelastic materials, we introduce some notations and preliminaries and we derive the variational formulation of the problem. We demonstrate in Section 3 an existence results and we apply the obtained results to the thermo-viscoelastic Maxwell model.

2 Problem Statement and Preliminaries

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be a bounded domain with a Lipschitz boundary Γ, partitioned into two disjoint measurable parts Γ_1 and Γ_2 such that $|\Gamma_1| > 0$ and let $Q = \Omega \times (0, T)$. We denote by $\mathbb{S}_n$ the space of symmetric tensors on $\mathbb{R}^n$. We define the inner product and the Euclidean norm on $\mathbb{R}^n$ and $\mathbb{S}_n$, respectively, by

$$\mathbf{u} \cdot \mathbf{v} = u_i v_i \quad \forall\, \mathbf{u},\ \mathbf{v} \in \mathbb{R}^n \ \text{ and } \ \boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij} \tau_{ij} \quad \forall\, \boldsymbol{\sigma},\ \boldsymbol{\tau} \in \mathbb{S}_n.$$

$$|\mathbf{u}| = (\mathbf{u} \cdot \mathbf{u})^{\frac{1}{2}} \quad \forall \mathbf{u} \in \mathbb{R}^n \ \text{ and } \ |\boldsymbol{\sigma}| = (\boldsymbol{\sigma} \cdot \boldsymbol{\sigma})^{\frac{1}{2}} \quad \forall \boldsymbol{\sigma} \in \mathbb{S}_n.$$

Here and below, the indices i and j run from 1 to n and the summation convention over repeated indices is used.

For the rest of this article, we will denote by c possibly different positive constants depending only on the data of the problem. Denote by p' and q' the conjugates of p and q, where $1 \leq p < \dfrac{n}{n-1}$, $q \geq 1$ and let $s \geq n$. We define the function spaces

$$H = L^2(\Omega)^n = \left\{ \mathbf{u} = \{u_i\} \mid u_i \in L^2(\Omega) \right\},$$
$$\mathcal{H} = \left\{ \boldsymbol{\sigma} = \{\sigma_{ij}\} \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \right\},$$
$$H_1 = \{ \mathbf{u} \in H \mid \boldsymbol{\varepsilon}(\mathbf{u}) \in \mathcal{H} \},$$
$$\mathcal{H}_1 = \{ \boldsymbol{\sigma} \in \mathcal{H} \mid \operatorname{div} \boldsymbol{\sigma} \in H \},$$
$$\mathcal{X} = \left\{ \varphi \in L^2\left(0,T; H^1(\Omega)\right) \cap \mathcal{C}^0\left([0,T], L^2(\Omega)\right),\ \frac{\partial \varphi}{\partial t} \in L^2\left(0,T; H^1(\Omega)'\right) \right\},$$
$$\mathcal{Y}_{p,q} = \left\{ \varphi \in L^q\left(0,T; W^{1,p}(\Omega)\right),\ \frac{\partial \varphi}{\partial t} \in L^q\left(0,T; W^{-1,p}(\Omega)\right) \right\},$$
$$\mathcal{Z}_{p,q} = \left\{ \varphi \in W^{1,q}\left(0,T; W^{1,p}(\Omega)\right),\ \varphi(x,T) = 0 \text{ in } \Omega \right\}.$$

Here $\boldsymbol{\varepsilon} : H_1 \longrightarrow \mathcal{H}$ and $\operatorname{div} : \mathcal{H}_1 \longrightarrow H$ are the deformation (small strain) and the divergence (stress) operators, respectively, defined by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}),$$
$$\operatorname{div} \boldsymbol{\sigma} = (\sigma_{ij,j}).$$

H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$\langle \mathbf{u}, \mathbf{v} \rangle_H = \int_\Omega u_i v_i dx, \quad \langle \boldsymbol{\sigma}, \boldsymbol{\tau} \rangle_{\mathcal{H}} = \int_\Omega \sigma_{ij} \tau_{ij} dx,$$
$$\langle \mathbf{u}, \mathbf{v} \rangle_{H_1} = \langle \mathbf{u}, \mathbf{v} \rangle_H + \langle \boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}) \rangle_{\mathcal{H}},$$
$$\langle \boldsymbol{\sigma}, \boldsymbol{\tau} \rangle_{\mathcal{H}_1} = \langle \sigma, \tau \rangle_{\mathcal{H}} + \langle \operatorname{div} \sigma, \operatorname{div} \tau \rangle_H.$$

The associated norms on the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $\|.\|_H$, $\|.\|_{\mathcal{H}}$, $\|.\|_{H_1}$ and $\|.\|_{\mathcal{H}_1}$, respectively. In addition, $\mathcal{X}$, $\mathcal{Y}_{p,q}$ and $\mathcal{Z}_{p,q}$ are Banach spaces equipped, respectively, with the norms

$$\|\zeta\|_{\mathcal{X}} = \|\zeta\|_{L^2(0,T;H^1(\Omega))} + \|\zeta\|_{\mathcal{C}^0([0,T],L^2(\Omega))} + \left\| \frac{\partial \zeta}{\partial t} \right\|_{L^2\left(0,T;H^1(\Omega)'\right)} \quad \forall \zeta \in \mathcal{X},$$
$$\|\zeta\|_{\mathcal{Y}_{p,q}} = \|\zeta\|_{L^q(0,T;W^{1,p}(\Omega))} + \left\| \frac{\partial \zeta}{\partial t} \right\|_{L^q(0,T;W^{-1,p}(\Omega))} \quad \forall \zeta \in \mathcal{Y}_{p,q},$$
$$\|\zeta\|_{\mathcal{Z}_{p,q}} = \|\zeta\|_{L^q(0,T;W^{1,p}(\Omega))} + \left\| \frac{\partial \zeta}{\partial t} \right\|_{L^q(0,T;W^{1,p}(\Omega))} \quad \forall \zeta \in \mathcal{Z}_{p,q}.$$

Since the boundary Γ is Lipschitz continuous, the unit outward normal vector field $\mathbf{n}$ on the boundary is defined a.e. Let $H_\Gamma = \left(H^{\frac{1}{2}}(\Gamma)\right)^n$ and $\gamma : H_1 \longrightarrow H_{\boldsymbol{\Gamma}}$ be the trace map.

We also denote by $\mathcal{V}$ the closed subspace of H_1 defined by

$$\mathcal{V} = \{\mathbf{v} \in H_1 \mid \gamma\mathbf{v} = 0 \quad \text{on} \quad \Gamma_1\}.$$

Since $|\Gamma_1| > 0$, Korn's inequality holds in $\mathcal{V}$ and thus there exists a positive constant C_0 depending only on Ω, Γ such that

$$\|\boldsymbol{\varepsilon}(\mathbf{v})\|_{\mathcal{H}} \geq C_0 \|\mathbf{v}\|_{H_1} \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}.$$

Moreover, the dual of H_Γ is denoted $H'_{\mathbf{\Gamma}}$. If $\boldsymbol{\sigma} \in \mathcal{H}_1$ there exists $\boldsymbol{\sigma}\nu \in H'_{\mathbf{\Gamma}}$ such that the following Green formula holds :

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\mathbf{v})\, dx + \int_\Omega \operatorname{div} \boldsymbol{\sigma} \cdot \mathbf{v} dx = (\boldsymbol{\sigma}\nu, \gamma\mathbf{v})_{H'_{\mathbf{\Gamma}} \times H_{\mathbf{\Gamma}}} \quad \forall \mathbf{v} \in H_1.$$

In addition, if $\boldsymbol{\sigma}$ is sufficiently regular (say $\mathcal{C}^1$), then

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\mathbf{v})\, dx + \int_\Omega \operatorname{div} \boldsymbol{\sigma} \cdot \mathbf{v} dx = \int_\Gamma \boldsymbol{\sigma}\nu \cdot \gamma\mathbf{v} d\gamma \quad \forall \mathbf{v} \in H_1.$$

where $d\gamma$ represents the surface element.

The physical setting is the following. A linear thermo-viscoelastic body occupies the domain Ω. We assume that the body is clamped on $\Gamma_1 \times (0, T)$ and therefore the displacement field vanishes there. Surface tractions of density $\mathbf{f}_0$ act on $\Gamma_2 \times (0, T)$ and a volume forces of density $\mathbf{f}$ is applied in Q. In addition, we admit a possible external heat source applied in Q, given by the function r. Moreover, we take into account the effect of internal forces that generate a non linear viscous dissipative function.

The mechanical problem may be formulated as follows.

Problem P1. Find the displacement field $\mathbf{u} : Q \longrightarrow \mathbb{R}^n$, the stress field $\boldsymbol{\sigma} : Q \longrightarrow \mathbb{S}_n$ and the temperature $\theta : Q \longrightarrow \mathbb{R}$ such that

$$\frac{\partial \boldsymbol{\sigma}}{\partial t} = \mathcal{A}\left(\boldsymbol{\varepsilon}\left(\frac{\partial \mathbf{u}}{\partial t}\right)\right) + \mu(\theta)\left(\mathcal{G}_1(\boldsymbol{\sigma}) + \mathcal{G}_2(\boldsymbol{\varepsilon}(\mathbf{u}))\right) \text{ in } Q, \tag{2.1}$$

$$\rho\ddot{\mathbf{u}} = \operatorname{div}(\boldsymbol{\sigma}) + \mathbf{f} \text{ in } Q, \tag{2.2}$$

$$\frac{\partial \theta}{\partial t} - \operatorname{div}(\mathcal{K}(\nabla\theta)) = \boldsymbol{\varepsilon}^d\left(\frac{\partial \mathbf{u}}{\partial t}\right) \cdot \boldsymbol{\sigma} + r \text{ in } Q, \tag{2.3}$$

$$\mathbf{u} = 0 \text{ on } \Gamma_1 \times (0, T), \tag{2.4}$$

$$\boldsymbol{\sigma}\nu = \mathbf{f}_0 \text{ on } \Gamma_2 \times (0, T), \tag{2.5}$$

$$\frac{\partial \theta}{\partial \mathbf{n}} = 0 \text{ on } \Gamma \times (0, T), \tag{2.6}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \dot{\mathbf{u}}(0) = \mathbf{u}_1, \quad \sigma(0) = \sigma_0 \text{ and } \theta(0) = \theta_0 \text{ in } \Omega. \tag{2.7}$$

This problem represents the dynamic evolution of a linear thermo-viscoelastic material with viscous dissipation. Relation (2.1) is the thermo-viscoelastic constitutive law where $\mathcal{A}$ is a real tensor describing the elastic properties of the material, μ is a real function which represents the viscosity of the material, corresponding to the thermal properties and $\mathcal{G}_1$, $\mathcal{G}_2$ are real tensors describing the viscoelastic behaviour of the material. This constitutive law is also called "Standard linear solid model", see [9]. (2.2) represents the equation of motion in which the dot above denotes the derivative with respect to the time variable and ρ is the density of mass. Equation (2.3) represents the energy conservation where $\mathcal{K}$ is an isotropic real tensor called the thermal conductivity tensor and the term $\boldsymbol{\varepsilon}^d\left(\frac{\partial \mathbf{u}}{\partial t}\right)\cdot\boldsymbol{\sigma}$ is the viscous dissipative function, generated by the work of internal forces, in which $\boldsymbol{\varepsilon}^d$ denotes the dissipative part in the strain rate tensor and r is a given volume heat source.
Equalities (2.4) and (2.5) are the displacement-traction boundary conditions, respectively. (2.6) is an homogeneous Neumann boundary conditions on $\Gamma\times(0,T)$ for the temperature. Finally, the functions $\mathbf{u}_0$, $\mathbf{u}_1$, $\boldsymbol{\sigma}_0$ and θ_0 in (2.7) represent the initial data.

In the study of the mechanical problem (P1) we consider the following hypotheses:

$$\left\{\begin{array}{l}\mathcal{A}:\Omega\times\mathbb{S}_n\longrightarrow\mathbb{S}_n \text{ is a tensor verifying :}\\ \text{(a) } \mathcal{A}_{ijkh}\in L^\infty(\Omega)\quad\forall i,j,k,h=1,...,n;\\ \text{(b) } \mathcal{A}(x)\boldsymbol{\sigma}\cdot\boldsymbol{\tau}=\sigma\cdot\mathcal{A}(x)\cdot\boldsymbol{\tau}\quad\forall\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{S}_n,\quad\text{a.e. in }\Omega;\\ \text{(c) There exists an } \alpha_{\mathcal{A}}>0 \text{ such that}\\ \mathcal{A}(x)\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\geq\alpha_{\mathcal{A}}\left|\boldsymbol{\sigma}\right|^2\quad\forall\boldsymbol{\sigma}\in\mathbb{S}_n,\quad\text{a.e. in }\Omega.\end{array}\right. \tag{2.8}$$

$$\left\{\begin{array}{l}\mathcal{G}_e:\Omega\times\mathbb{S}_n\longrightarrow\mathbb{S}_n,\ (e=1,2), \text{ is a tensor verifying :}\\ \text{(a) } (\mathcal{G}_e)_{ijkh}\in L^\infty(\Omega)\quad\forall i,j,k,h=1,...,n;\\ \text{(b) } \mathcal{G}_e(x)\boldsymbol{\sigma}\cdot\boldsymbol{\tau}=\sigma\cdot\mathcal{G}_e(x)\boldsymbol{\tau}\quad\forall\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{S}_n,\quad\text{a.e. in }\Omega.\end{array}\right. \tag{2.9}$$

$$\left\{\begin{array}{l}\mathcal{K}:\Omega\times\mathbb{R}^n\longrightarrow\mathbb{R}^n \text{ is a tensor verifying :}\\ \text{(a) } \mathcal{K}_{ij}(x)=\mathcal{K}_{ji}(x)\in L^\infty(\Omega)\quad\forall i,j=1,...,n,\quad\text{a.e. in }\Omega;\\ \text{(b) There exists an } \alpha_{\mathcal{K}}>0 \text{ such that}\\ \mathcal{K}(x)\mathbf{v}\cdot\mathbf{v}\geq\alpha_{\mathcal{K}}\left|\mathbf{v}\right|^2\quad\forall\mathbf{v}\in\mathbb{R}^n,\quad\text{a.e. in }\Omega;\end{array}\right. \tag{2.10}$$

$$\left\{\begin{array}{l}\mu:\mathbb{R}\longrightarrow\mathbb{R} \text{ is a real function verifying :}\\ \text{(a) } \mu\in\mathcal{C}^0(\mathbb{R});\\ \text{(b) There exists } \mu_*,\mu^*>0 \text{ such that}\\ \mu_*\leq\mu(s)\leq\mu^*\quad\forall s\in\mathbb{R}\end{array}\right. \tag{2.11}$$

We also suppose that the mass density satisfies

$$\rho\in L^\infty(\Omega),\quad \rho\geq\rho^*>0.$$

$$\left\{\begin{array}{c}\mathbf{f}\in W^{1,+\infty}(0,T;H),\quad \mathbf{f}_0\in W^{1,+\infty}\left(0,T;L^2(\Gamma_2)^n\right),\\ r\in L^1(Q).\end{array}\right. \tag{2.12}$$

$$\mathbf{u}_0 \in \mathcal{V}, \mathbf{u}_1 \in H, \quad \sigma_0 \in \mathcal{H}_1 \text{ and } \theta_0 \in L^1(\Omega). \tag{2.13}$$

We denote by $\mathbf{F} \in \mathcal{V}'$ the following element:

$$\langle \mathbf{F}(t), \mathbf{v} \rangle_{\mathcal{V}' \times \mathcal{V}} = \int_\Omega \mathbf{f}(t) \cdot \mathbf{v} dx + \int_{\Gamma_2} \mathbf{f}_0(t) \cdot \gamma \mathbf{v} d\gamma \quad \forall \mathbf{v} \in \mathcal{V}. \tag{2.14}$$

The use of (2.12) permits us to verify that

$$\mathbf{F} \in W^{1,+\infty}(0, T; \mathcal{V}'). \tag{2.15}$$

Moreover, we remark that hypotheses (2.8), (2.9) and (2.10) imply the existence of positive constants $m_{\mathcal{A}}$, m_e, $(e = 1, 2)$ and $m_{\mathcal{K}}$ such that

$$\|\mathcal{A}\boldsymbol{\sigma}\|_{\mathcal{H}} \leq m_{\mathcal{A}} \|\boldsymbol{\sigma}\|_{\mathcal{H}} \quad \forall \boldsymbol{\sigma} \in \mathcal{H}, \tag{2.16}$$

$$\|\mathcal{G}_e\boldsymbol{\sigma}\|_{\mathcal{H}} \leq m_e \|\boldsymbol{\sigma}\|_{\mathcal{H}}, \quad (e = 1, 2), \quad \forall \boldsymbol{\sigma} \in \mathcal{H}, \tag{2.17}$$

$$\|\mathcal{K}\mathbf{v}\|_H \leq m_{\mathcal{K}} \|\mathbf{v}\|_H \quad \forall \mathbf{v} \in H. \tag{2.18}$$

Furthermore, one can check that under the hypothesis (2.8), the constitutive law (2.1) can be rewritten

$$\boldsymbol{\varepsilon}\left(\frac{\partial \mathbf{u}}{\partial t}\right) = \mathcal{A}^{-1}\left(\frac{\partial \boldsymbol{\sigma}}{\partial t}\right) - \mu(\theta)\mathcal{A}^{-1}\left(\mathcal{G}_1(\boldsymbol{\sigma}) + \mathcal{G}_2(\varepsilon(\mathbf{u}))\right) \quad \text{in} \quad \Omega \times (0, T),$$

where $\mathcal{A}^{-1}$ denotes the inverse tensor of $\mathcal{A}$.
Consequently the non dissipative and the dissipative parts of the strain rate tensor are, respectively

$$\boldsymbol{\varepsilon}^{nd}\left(\frac{\partial \mathbf{u}}{\partial t}\right) = \mathcal{A}^{-1}\left(\frac{\partial \boldsymbol{\sigma}}{\partial t}\right), \tag{2.19}$$

$$\boldsymbol{\varepsilon}^{d}\left(\frac{\partial \mathbf{u}}{\partial t}\right) = -\mu(\theta)\mathcal{A}^{-1}\left(\mathcal{G}_1(\boldsymbol{\sigma}) + \mathcal{G}_2(\boldsymbol{\varepsilon}(\mathbf{u}))\right). \tag{2.20}$$

Using the above notations and Green's formula, we can easily derive the following variational formulation of the mechanical problem (P1).
Problem (P2). Find the displacement field $\mathbf{u} : Q \longrightarrow \mathbb{R}^n$, the stress field $\boldsymbol{\sigma} : Q \longrightarrow \mathbb{S}_n$ and the temperature $\theta : Q \longrightarrow \mathbb{R}$ such that

$$\frac{\partial \boldsymbol{\sigma}}{\partial t} = \mathcal{A}\left(\boldsymbol{\varepsilon}\left(\frac{\partial \mathbf{u}}{\partial t}\right)\right) + \mu(\theta(t))\left(\mathcal{G}_1(\boldsymbol{\sigma}(t)) + \mathcal{G}_2(\boldsymbol{\varepsilon}(\mathbf{u}(t)))\right) \quad \text{a.e. } t \in (0, T), \tag{2.21}$$

$$\int_\Omega \boldsymbol{\sigma}(t) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) dx = \int_\Omega \mathbf{F}(t) \cdot \mathbf{v} dx \quad \forall \mathbf{v} \in \mathcal{V}, \quad \text{a.e. } t \in (0, T), \tag{2.22}$$

$$\int_{\Omega}\frac{\partial\theta}{\partial t}\varphi dx+\int_{\Omega}\mathcal{K}\left(\nabla\theta\left(t\right)\right)\cdot\nabla\varphi dx$$
$$=-\int_{\Omega}\mu\left(\theta\left(t\right)\right)\mathcal{A}^{-1}\left(\mathcal{G}_1\left(\boldsymbol{\sigma}\right)+\mathcal{G}_2\left(\varepsilon\left(\mathbf{u}\right)\right)\right)\cdot\boldsymbol{\sigma}\left(t\right)\varphi dx+\int_{\Omega}r\left(t\right)\varphi dx$$
$$\forall\varphi\in W^{1,p'}\left(\Omega\right),\quad \text{a.e. } t\in\left(0,T\right), \tag{2.23}$$

$$\mathbf{u}\left(0\right)=\mathbf{u}_0,\dot{\mathbf{u}}(0)=\mathbf{u}_1,\quad \boldsymbol{\sigma}\left(0\right)=\sigma_0 \text{ and } \theta\left(0\right)=\theta_0 \text{ in }\ \Omega. \tag{2.24}$$

If $(\mathbf{u}\left(t\right),\boldsymbol{\sigma}\left(t\right))$ has at least the regularity $\mathcal{V}\times\mathcal{H}$ a.e. $t\in\left(0,T\right)$, then in equation (2.23), the terms on the right hand side have sense, since the injection $W^{1,p'}\left(\Omega\right)\subset\mathcal{C}^0\left(\overline{\Omega}\right)$ is continuous for $p'>n$, that is, $p<\dfrac{n}{n-1}$.

We will say that a function $\theta\in\mathcal{Y}_{p,q}$ is a weak solution of the variational equation (2.23) if

$$-\int_{Q}\theta\frac{\partial\varphi}{\partial t}dxdt+\int_{Q}\mathcal{K}\left(\nabla\theta\right)\cdot\nabla\varphi dxdt$$
$$=-\int_{Q}\mu\left(\theta\right)\mathcal{A}^{-1}\left(\mathcal{G}_1\left(\boldsymbol{\sigma}\right)+\mathcal{G}_2\left(\varepsilon\left(\mathbf{u}\right)\right)\right)\cdot\boldsymbol{\sigma}\varphi dxdt+\int_{Q}r\varphi dxdt$$
$$+\int_{\Omega}\theta_0\varphi\left(0\right)dx\quad\forall\varphi\in\mathcal{Z}_{p',q'}.$$

We can then reformulate the variational problem (P2) as follows.

Problem (P3). Find the displacement field $\mathbf{u}\left(t\right)\in\mathcal{V},\dot{\mathbf{u}}(t)\in H,\ddot{\mathbf{u}}(t)\in\mathcal{V}'$ the stress field $\boldsymbol{\sigma}\left(t\right)\in\mathcal{H}$,$\dot{\boldsymbol{\sigma}}\left(t\right)\in\mathcal{H}_1'$and the temperature $\theta\left(t\right)\in\mathcal{Y}_{p,q}$ such that

$$\frac{\partial\boldsymbol{\sigma}}{\partial t}=\mathcal{A}\left(\boldsymbol{\varepsilon}\left(\frac{\partial\mathbf{u}}{\partial t}\right)\right)+\mu\left(\theta\left(t\right)\right)\left(\mathcal{G}_1\left(\boldsymbol{\sigma}\left(t\right)\right)+\mathcal{G}_2\left(\boldsymbol{\varepsilon}\left(\mathbf{u}\left(t\right)\right)\right)\right)\quad \text{a.e. } t\in\left(0,T\right), \tag{2.25}$$

$$\int_{\Omega}\boldsymbol{\sigma}\left(t\right)\cdot\boldsymbol{\varepsilon}\left(\mathbf{v}\right)dx=\int_{\Omega}\mathbf{F}\left(t\right)\cdot\mathbf{v}dx\quad\forall\mathbf{v}\in\mathcal{V},\quad \text{a.e. } t\in\left(0,T\right), \tag{2.26}$$

$$-\int_{Q}\theta\frac{\partial\varphi}{\partial t}dxdt+\int_{Q}\mathcal{K}\left(\nabla\theta\right)\cdot\nabla\varphi dxdt$$
$$=-\int_{Q}\mu\left(\theta\right)\mathcal{A}^{-1}\left(\mathcal{G}_1\left(\boldsymbol{\sigma}\right)+\mathcal{G}_2\left(\varepsilon\left(\mathbf{u}\right)\right)\right)\cdot\boldsymbol{\sigma}\varphi dxdt+\int_{Q}r\varphi dxdt$$
$$+\int_{\Omega}\theta_0\varphi\left(0\right)dx\quad\forall\varphi\in\mathcal{Z}_{p',q'}. \tag{2.27}$$

3 Existence Results

The main result of this section is stated by the following existence theorem.

Theorem 3.1 *Under the assumptions (2.8)-(2.13)), there exists at least one solution* $(\mathbf{u}, \boldsymbol{\sigma}, \theta)$ *to problem (P3). Moreover, the solution has the regularity*

$$\begin{gathered}\mathbf{u} \in L^{\infty}(0,T;\mathcal{V})\\ \dot{\mathbf{u}} \in L^{\infty}(0,T;H),\\ \ddot{\mathbf{u}} \in L^{\infty}(0,T;\mathcal{V}'),\\ \boldsymbol{\sigma} \in L^{\infty}(0,T;\mathcal{H}),\\ \dot{\boldsymbol{\sigma}} \in L^{\infty}(0,T;\mathcal{H}_1'),\\ \theta \in \mathcal{Y}_{p,q}.\end{gathered}$$

Where q *is such that*

$$\left\{\begin{array}{c} 1 \le q < 2,\\ \frac{2}{q} + \frac{n}{p} > n+1. \end{array}\right. \tag{3.1}$$

The proof will be carried up by several steps. Based on classical results of functional analysis concerning evolution problems, Banach and Kakutani-Glicksberg fixed point theorems, see [6, 12, 11], as well as some monotonicity and compactness arguments, using two auxiliary existence results.

Let $\lambda \in \mathcal{Y}_{p,q}$ and consider the following auxiliary problem.

Problem($\mathrm{P}_{1\lambda}$). Find the displacement field $\mathbf{u}_\lambda : Q \longrightarrow \mathbb{R}^n$, the stress field $\boldsymbol{\sigma}_\lambda : Q \longrightarrow \mathbb{S}_n$ such that

$$\frac{\partial \boldsymbol{\sigma}_\lambda}{\partial t} = \mathcal{A}(\varepsilon(\frac{\partial u_\lambda}{\partial t})) + \mu(\lambda(t))(\mathcal{G}_1(\boldsymbol{\sigma}_\lambda(t)) + \mathcal{G}_2(\varepsilon(\mathbf{u}_\lambda(t))) \quad \text{a.e. } t \in (0,T), \tag{3.2}$$

$$\int_\Omega \boldsymbol{\sigma}(t) \cdot \boldsymbol{\varepsilon}(\mathbf{v})\, dx = \int_\Omega \mathbf{F}(t) \cdot \mathbf{v} dx \quad \forall \mathbf{v} \in \mathcal{V}, \quad \text{a.e. } t \in (0,T). \tag{3.3}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \dot{\mathbf{u}}(0) = \mathbf{u}_1, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \ \text{ in } \ \Omega. \tag{3.4}$$

Lemma 3.2 *For all* $\lambda \in \mathcal{Y}_{p,q}$ *there exists a unique solution* $(\mathbf{u}_\lambda, \dot{\mathbf{u}}_\lambda, \ddot{\mathbf{u}}_\lambda, \boldsymbol{\sigma}_\lambda, \dot{\boldsymbol{\sigma}}_\lambda) \in L^\infty(0,T;\mathcal{V}) \times L^\infty(0,T;H) \times L^\infty(0,T;\mathcal{V}) \times L^\infty(0,T;\mathcal{H}) \times L^\infty(0,T;\mathcal{H}_1')$ *to the auxiliary problem* $(P_{1\lambda})$.

Proof. Take an arbitrary

$$\boldsymbol{\eta} \in L^\infty(0,T;\mathcal{H}) \tag{3.5}$$

and let Z_η be the function

$$Z_\eta = \int_0^t \boldsymbol{\eta}(s) ds + \boldsymbol{\sigma}_\mathbf{0} - \mathcal{A}(\varepsilon(\mathbf{u}_0)). \tag{3.6}$$

Considering the following auxiliary problem.
Problem $(\mathrm{P}^{\eta}_{1\lambda})$. Find the displacement field $\mathbf{u}(\mathbf{t}) \in L^{\infty}(0,T;\mathcal{V})$, such that

$$\boldsymbol{\sigma}_{\lambda\eta}(t) = \mathcal{A}(\varepsilon(\mathbf{u}_{\lambda\eta}(t)) + Z_{\eta}(t) \quad \text{a.e. } t \in (0,T), \tag{3.7}$$

$$\int_{\Omega} \rho \ddot{\mathbf{u}}_{\lambda\eta}(t)\mathbf{v}dx + \int_{\Omega} \boldsymbol{\sigma}_{\lambda\eta}(t) \cdot \boldsymbol{\varepsilon}(\mathbf{v})\, dx = \int_{\Omega} \mathbf{F}(t) \cdot \mathbf{v}dx \quad \forall \mathbf{v} \in \mathcal{V}, \quad \text{a.e. } t \in (0,T), \tag{3.8}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{u_1}, \quad \theta(0) = \theta_0, \quad \sigma(0) = \sigma_0 \quad \text{in } \Omega. \tag{3.9}$$

We start by proving that the problem $(P^{\eta}_{1\lambda})$ has a unique solution $(\mathbf{u}_{\lambda\eta}, \dot{\mathbf{u}}_{\lambda\eta}, \ddot{\mathbf{u}}_{\lambda\eta}, \boldsymbol{\sigma}_{\lambda\eta}, \dot{\boldsymbol{\sigma}}_{\lambda\eta}) \in L^{\infty}(0,T;\mathcal{V}) \times L^{\infty}(0,T;H) \times L^{\infty}(0,T;\mathcal{V}'^{\infty}(0,T;\mathcal{H}) \times L^{\infty}(0,T;\mathcal{H}'_1)$. To do this, we find making use (3.7) and (3.8)

$$\int_{\Omega} \rho \ddot{\mathbf{u}}_{\lambda\eta}(t)\mathbf{v}dx + \int_{\Omega} \mathcal{A}(\varepsilon(\mathbf{u}_{\lambda\eta}(t))\varepsilon(\mathbf{v}))dx = \int_{\Omega} \mathbf{F}(t)\mathbf{v}dx - \int_{\Omega} Z_{\eta}(t)\varepsilon(\mathbf{v})dx$$

$$\forall \mathbf{v} \in \mathcal{V}, \ \text{a.e. } t \in (0,T),$$

It is will know that this hyperbolic equation has a unique solution $\mathbf{u}_{\lambda\eta} \in L^{\infty}(0,T;\mathcal{V})$,
$\dot{\mathbf{u}}_{\lambda\eta} \in L^{\infty}(0,T;H), \ddot{\mathbf{u}}_{\lambda\eta} \in L^{\infty}(0,T;\mathcal{V}')$, for more detail see [12].
Elsewhere, the existence of the stress field $\boldsymbol{\sigma}_{\lambda\eta}, \dot{\boldsymbol{\sigma}}_{\lambda\eta} \in L^{\infty}(0,T;\mathcal{H}) \times L^{\infty}(0,T;\mathcal{H}'_1)$ is an immediate consequence of relation (3.7).

Which implies under the hypothesis that $\boldsymbol{\sigma}_{\lambda\eta}, \dot{\boldsymbol{\sigma}}_{\lambda\eta} \in L^{\infty}(0,T;\mathcal{H}) \times L^{\infty}(0,T;\mathcal{H}'_1)$ satisfies the estimate

$$\|\boldsymbol{\sigma}_{\lambda\eta}(t)\|_{\mathcal{H}} \leq m_{\mathcal{A}} \|\mathbf{u}_{\lambda\eta}(t)\|_{\mathcal{V}} + \int_0^t \|\boldsymbol{\eta}(s)\|_{\mathcal{H}} ds + \|\boldsymbol{\sigma}_0\|_{\mathcal{H}} + \|\mathbf{u}_0\|_{\mathcal{V}}$$

$$\text{a.e. } t \in (0,T),$$

by differntial of (3.7) with respect to the time variable $t \in (0,T)$, we get

$$\|\frac{\partial \boldsymbol{\sigma}_{\lambda\eta}(t)}{\partial t}\|_{\mathcal{H}'_1} \leq m_{\mathcal{A}} \|\varepsilon(\frac{\partial \mathbf{u}_{\lambda\eta}(t)}{\partial t})\|_{\mathcal{H}'_1} + \|\boldsymbol{\eta}(t)\|_{\mathcal{H}} \quad \text{a.e. } t \in (0,T) \tag{3.12}$$

Introducing now the operator $\Lambda : L^{\infty}(0,T;\mathcal{H}) \to L^{\infty}(0,T;\mathcal{H})$

$$\Lambda(\eta(t)) = \mu(\lambda(t))(\mathcal{G}_1(\boldsymbol{\sigma}_{\lambda\eta}(t)) + \mathcal{G}_2(\varepsilon(\mathbf{u}_{\lambda\eta}(t))) \tag{3.13}$$

Let $t \in (0,T)$ and $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in L^{\infty}(0,T;\mathcal{H})$, it is easy to check that for a.e $t \in (0,T)$

$$\|\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)\|_{\mathcal{H}} \leq \mu^* \max(m_1, m_2)(\|\mathbf{u}_{\lambda\eta_1(t)} - \mathbf{u}_{\lambda\eta_2(t)}\|_{\mathcal{V}} + \|\boldsymbol{\sigma}_{\lambda\eta_1(t)} - \boldsymbol{\sigma}_{\lambda\eta_2(t)}\|_{\mathcal{H}})$$

Taking $\boldsymbol{\eta} = \boldsymbol{\eta}_1, \boldsymbol{\eta} = \boldsymbol{\eta}_2$ respectively, in equation (3.8), subtracting the two obtained equations, we can infer by choosing $\mathbf{v} = \dot{\mathbf{u}}_{\lambda\eta_1} - \dot{\mathbf{u}}_{\lambda\eta_2}$ as test function that

$$\int_\Omega \rho(\ddot{\mathbf{u}}_{\lambda\eta_1} - \ddot{\mathbf{u}}_{\lambda\eta_2})(\dot{\mathbf{u}}_{\lambda\eta_1} - \dot{\mathbf{u}}_{\lambda\eta_2})dx + \int_\Omega \mathcal{A}(\varepsilon(\mathbf{u}_{\lambda\eta_1}(t) - \mathbf{u}_{\lambda\eta_2}(t))\varepsilon(\dot{\mathbf{u}}_{\lambda\eta_1}(t) - \dot{\mathbf{u}}_{\lambda\eta_2}(t))dx$$

$$= -\int_\Omega (Z_{\lambda\eta_1} - Z_{\lambda\eta_2})\varepsilon(\dot{\mathbf{u}}_{\lambda\eta_1}(t) - \dot{\mathbf{u}}_{\lambda\eta_2}(t))dx$$

this equation becomes

$$\frac{(\rho^*)^2}{2}\frac{d}{dt}\|\dot{\mathbf{u}}_{\lambda\eta_1}(t) - \dot{\mathbf{u}}_{\lambda\eta_2}(t)\|_H^2 + \frac{C_0^2}{2}\frac{d}{dt}\|\mathbf{u}_{\lambda\eta_1}(t) - \mathbf{u}_{\lambda\eta_2}(t)\|_{\mathcal{V}}^2$$

$$\begin{aligned} = & \int_\Omega (\dot{Z}_{\lambda\eta_1}(t) - \dot{Z}_{\lambda\eta_2}(t))\varepsilon(\mathbf{u}_{\lambda\eta_1}(t) - \mathbf{u}_{\lambda\eta_2}(t))dx \\ & + \frac{d}{dt}\int_\Omega (Z_{\lambda\eta_1} - Z_{\lambda\eta_2})\varepsilon(\mathbf{u}_{\lambda\eta_1}(t) - \mathbf{u}_{\lambda\eta_2}(t))dx \end{aligned}$$

Integrating this inequality over the interval time variable $(0, t)$, Young inequality leads to

$$\frac{C_0^2}{2}\|\mathbf{u}_{\lambda\eta_1}(t) - \mathbf{u}_{\lambda\eta_2}(t)\|_{\mathcal{V}}^2$$

$$\leq \frac{1}{2C_0^2}\int_0^t \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{\mathcal{H}}^2 ds + \frac{C_0^2}{2}\int_0^t \|\mathbf{u}_{\lambda\eta_1}(t) - \mathbf{u}_{\lambda\eta_2}(t)\|_{\mathcal{V}}^2 ds + \frac{C_0^2}{4}\|\mathbf{u}_{\lambda\eta_1}(t) - \mathbf{u}_{\lambda\eta_2}(t)\|_{\mathcal{V}}$$

$$+ \frac{T}{C_0^2}\int_0^t \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{\mathcal{H}}^2 ds$$

using Gronwall's lemma

$$\|\mathbf{u}_{\lambda\eta_1}(t) - \mathbf{u}_{\lambda\eta_2}(t)\|_{\mathcal{V}}^2 \leq C_1 \int_0^t \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{\mathcal{H}}^2 ds$$

Then

$$\|\mathbf{u}_{\lambda\eta_1}(t) - \mathbf{u}_{\lambda\eta_2}(t)\|_{\mathcal{V}} \leq C_2\|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{L^\infty(0,T;\,\mathcal{H})} \tag{3.15}$$

Moreover, Taking into account (3.15), equation (3.7) gives for a.e. $t \in (0, T)$,

$$\left\|\sigma_{\lambda\eta_1}(t) - \sigma_{\lambda\eta_2}(t)\right\|_{\mathcal{H}} \leq (m_{\mathcal{A}}C_2 + T)\int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{\mathcal{H}}\, ds. \tag{3.16}$$

We conclude from (**??**)-(3.16) that

$$\|\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)\|_{\mathcal{H}} \leq C\,\|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{L^\infty(0,T;\,\mathcal{H})}\,.$$

where

$$C = \mu^* \max(m_1, m_2) \max(C_2, (m_{\mathcal{A}} C_2 + T)).$$

This implies that

$$\|\Lambda \boldsymbol{\eta}_1 - \Lambda \boldsymbol{\eta}_2\|_{L^\infty(0,T;\,\mathcal{H})} \leq C \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{L^\infty(0,T;\,\mathcal{H})}.$$

Applying Λ another time. Similarly, we get

$$\left\|\Lambda^2 \boldsymbol{\eta}_1 - \Lambda^2 \boldsymbol{\eta}_2\right\|_{L^\infty(0,T;\,\mathcal{H})} \leq \frac{C^2}{2!} \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{L^\infty(0,T;\,\mathcal{H})}.$$

We generalize this procedure by recurrence on n, we obtain the following formula, see for more details the reference [14]

$$\|\Lambda^n \boldsymbol{\eta}_1 - \Lambda^n \boldsymbol{\eta}_2\|_{L^\infty(0,T;\,\mathcal{H})} \leq \frac{C^n}{n!} \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{L^\infty(0,T;\,\mathcal{H})}. \tag{3.17}$$

The sequence $\left(\frac{C^n}{n!}\right)_n$ converges to 0. Thus, for n sufficiently large $\frac{C^n}{n!} < 1$. It means that a large power n of the operator Λ is a contraction on $L^\infty(0,T;\mathcal{H})$. Then, Banach fixed point theorem asserts that Λ^n admits a unique fixed point

$$\tilde{\boldsymbol{\eta}} \in L^\infty(0,T;\mathcal{H}) : \Lambda^n \tilde{\boldsymbol{\eta}} = \tilde{\boldsymbol{\eta}}.$$

By applying the operator Λ another time we get

$$\Lambda^{n+1} \tilde{\boldsymbol{\eta}} = \Lambda \tilde{\boldsymbol{\eta}}.$$

Which means that

$$\Lambda^n (\Lambda \tilde{\boldsymbol{\eta}}) = \Lambda \tilde{\boldsymbol{\eta}}.$$

Then, $\Lambda \tilde{\boldsymbol{\eta}}$ is another fixed point of Λ^n. Thus, the uniqueness of fixed point of Λ^n leads to

$$\Lambda \tilde{\boldsymbol{\eta}} = \tilde{\boldsymbol{\eta}}.$$

Which proves that the problem $(\mathrm{P}_{1\lambda}^{\eta})$ has a unique solution

$$\begin{aligned}(\mathbf{u}_{\lambda\eta}, \dot{\mathbf{u}}_{\lambda\eta}, \ddot{\mathbf{u}}_{\lambda\eta}, \boldsymbol{\sigma}_{\lambda\eta}, \dot{\boldsymbol{\sigma}}_{\lambda\eta}) \in\ & L^\infty(0,T;\mathcal{V}) \times L^\infty(0,T;H) \times L^\infty(0,T;\mathcal{V}) \times \\ & L^\infty(0,T;\mathcal{H}) \times L^\infty(0,T;\mathcal{H}_1').\end{aligned}$$

We are ready now to prove the existence of solution to problem $(\mathrm{P}_{1\lambda})$. To this aim, by definition of Λ we get

$$\mu(\lambda(t)) (\mathcal{G}_1(\boldsymbol{\sigma}_{\lambda\tilde{\eta}}(t)) + \mathcal{G}_2(\boldsymbol{\varepsilon}(\mathbf{u}_{\lambda\tilde{\eta}}(t)))) = \tilde{\boldsymbol{\eta}}(t) \quad \text{a.e. } t \in (0,T),$$

and so equation (3.7) becomes

$$\boldsymbol{\sigma}_{\lambda\tilde{\eta}}(t) = \mathcal{A}(\boldsymbol{\varepsilon}(\mathbf{u}_{\lambda\tilde{\eta}}(t))) + Z_{\tilde{\eta}}(t) \text{ a.e. } t \in (0,T).$$

By taking the derivative of the above equation with respect to time variable t we deduce for a.e. $t \in (0,T)$

$$\frac{\partial \boldsymbol{\sigma}_{\lambda\tilde{\eta}}}{\partial t} = \mathcal{A}\left(\boldsymbol{\varepsilon}\left(\frac{\partial \mathbf{u}_{\lambda\tilde{\eta}}}{\partial t}\right)\right) + \mu(\lambda(t))(\mathcal{G}_1(\boldsymbol{\sigma}_{\lambda\tilde{\eta}}(t)) + \mathcal{G}_2(\boldsymbol{\varepsilon}(\mathbf{u}_{\lambda\tilde{\eta}}(t)))).$$

Then,

$$\begin{aligned}(\mathbf{u}_{\lambda\tilde{\eta}}, \dot{\mathbf{u}}_{\lambda\tilde{\eta}}, \ddot{\mathbf{u}}_{\lambda\tilde{\eta}}, \boldsymbol{\sigma}_{\lambda\tilde{\eta}}, \dot{\boldsymbol{\sigma}}_{\lambda\tilde{\eta}}) \in \; & L^\infty(0,T;\mathcal{V}) \times L^\infty(0,T;H) \times L^\infty(0,T;\mathcal{V}') \times \\ & (0,T;\mathcal{H}) \times L^\infty(0,T;\mathcal{H}'_1)\end{aligned}$$

represents the unique solution of the auxiliary problem ($\mathrm{P}_{1\lambda}$). This permits us to conclude the proof of Lemma 3.2.

Considering now the following auxiliary problem.

Problem ($\mathrm{P}_{2\lambda}$). Find the temperature $\theta_\lambda \in \mathcal{Y}_{p,q}$ solution of the variational equation

$$\begin{aligned} & -\int_Q \theta_\lambda \frac{\partial \varphi}{\partial t} dxdt + \int_Q \mathcal{K}(\nabla\theta_\lambda) \cdot \nabla\varphi dxdt \\ = & -\int_Q \mu(\lambda) \mathcal{A}^{-1}(\mathcal{G}_1(\boldsymbol{\sigma}_\lambda) + \mathcal{G}_2(\boldsymbol{\varepsilon}(\mathbf{u}_\lambda))) \cdot \boldsymbol{\sigma}_\lambda \varphi dxdt + \\ & \int_Q r\varphi dxdt + \int_\Omega \theta_0 \varphi(0) dx \quad \forall \varphi \in \mathcal{Z}_{p',q'}, \end{aligned} \tag{3.18}$$

$$\theta_\lambda(0) = \theta_0 \text{ in } \Omega. \tag{3.19}$$

Lemma 3.3 *Let*

$$\begin{aligned}(\mathbf{u}_{\lambda\eta}, \dot{\mathbf{u}}_{\lambda\eta}, \ddot{\mathbf{u}}_{\lambda\eta}, \boldsymbol{\sigma}_{\lambda\eta}, \dot{\boldsymbol{\sigma}}_{\lambda\eta}) \in \; & L^\infty(0,T;\mathcal{V}) \times L^\infty(0,T;H) \times L^\infty(0,T;\mathcal{V}') \times \\ & L^\infty(0,T;\mathcal{H}) \times L^\infty(0,T;\mathcal{H}'_1)\end{aligned}$$

be the solution of problem ($P_{1\lambda}$) given by Lemma 3.2. Then, there exists $\theta_\lambda \in \mathcal{Y}_{p,q}$, q given by the relation (3.1), solution to the auxiliary weak problem ($P_{2\lambda}$).

The proof of this Lemma is based on classical arguments of functional analysis concerning parabolic equation, see for more details [3, 12].

Proof of Theorem 3.1. In order to apply the Kakutani-Glicksberg fixed point theorem, see [14, 11], we consider the closed convex ball

$$K = \left\{\lambda \in \mathcal{Y}_{p,q} : \|\lambda\|_{\mathcal{Y}_{p,q}} \leq d\right\}. \tag{3.20}$$

The ball K is compact when the topological vector space is provided by the weak topology of the space $\mathcal{Y}_{p,q}$. Let us built the mapping $\mathcal{L} : K \longrightarrow 2^K$, as follows

$$\lambda \longmapsto \mathcal{L}(\lambda) \subset K. \tag{3.21}$$

whither $\mathcal{L}(\lambda)$ represents the set of solutions θ_λ of the auxiliary problem $(\mathrm{P}_{2\lambda})$. For every $\lambda \in K$, problem $(\mathrm{P}_{2\lambda})$ is linear with respect to the function θ_λ. Moreover, $(\mathbf{u}_{\lambda\eta}, \dot{\mathbf{u}}_{\lambda\eta}, \ddot{\mathbf{u}}_{\lambda\eta}, \boldsymbol{\sigma}_{\lambda\eta}, \dot{\boldsymbol{\sigma}}_{\lambda\eta}) \in L^\infty(0,T;\mathcal{V}) \times L^\infty(0,T;H) \times L^\infty(0,T;\mathcal{V}'^\infty(0,T;\mathcal{H}) \times L^\infty(0,T;\mathcal{H}_1')$ represents the unique solution of problem $(\mathrm{P}_{1\lambda})$. Consequently the set $\mathcal{L}(\lambda)$ is convex. To conclude the proof it remains to verify the closeness in $K \times K$ of the graph set

$$G(\mathcal{L}) = \{(\lambda, \theta_\lambda) \in K \times K \mid \theta_\lambda \in \mathcal{L}(\lambda)\}. \tag{3.22}$$

To do so, we consider a sequence $(\lambda_m) \in K$, such that

$$\lambda_m \longrightarrow \lambda \text{ in } \mathcal{Y}_{p,q} \text{ weakly,} \tag{3.23}$$

and let $\theta_{m\lambda} \in \mathcal{L}(\lambda)$.
Remembering that $\theta_{m\lambda}$ is solution of the following equation

$$\begin{aligned} &-\int_Q \theta_{m\lambda} \frac{\partial \varphi}{\partial t} dxdt + \int_Q \mathcal{K}(\nabla \theta_{m\lambda}) \cdot \nabla \varphi dxdt \\ &= -\int_Q \mu(\lambda_m) \mathcal{A}^{-1}(\mathcal{G}_1(\boldsymbol{\sigma}_{m\lambda}) + \mathcal{G}_2(\boldsymbol{\varepsilon}(\mathbf{u}_{m\lambda}))) \cdot \boldsymbol{\sigma}_{m\lambda} \varphi dxdt + \\ &\int_Q r\varphi dxdt + \int_\Omega \theta_0 \varphi(0)\, dx \quad \forall \varphi \in \mathcal{Z}_{p',q'}, \end{aligned} \tag{3.24}$$

$$\theta_\lambda(0) = \theta_0 \text{ in } \Omega. \tag{3.25}$$

where

$$\begin{aligned} (\mathbf{u}_{\lambda\eta}, \dot{\mathbf{u}}_{\lambda\eta}, \ddot{\mathbf{u}}_{\lambda\eta}, \boldsymbol{\sigma}_{\lambda\eta}, \dot{\boldsymbol{\sigma}}_{\lambda\eta}) \in\ & L^\infty(0,T;\mathcal{V}) \times L^\infty(0,T;H) \times L^\infty(0,T;\mathcal{V}') \times \\ & (0,T;\mathcal{H}) \times L^\infty(0,T;\mathcal{H}_1') \end{aligned}$$

is the unique solution of system

$$\frac{\partial \boldsymbol{\sigma}_{m\lambda}}{\partial t} = \mathcal{A}\left(\boldsymbol{\varepsilon}\left(\frac{\partial \mathbf{u}_{m\lambda}}{\partial t}\right)\right) + \mu(\lambda_m(t))(\mathcal{G}_1(\boldsymbol{\sigma}_{m\lambda}) + \mathcal{G}_2(\boldsymbol{\varepsilon}(\mathbf{u}_{m\lambda}))) \quad \text{a.e. } t \in (0,T), \tag{3.26}$$

$$\int_\Omega \boldsymbol{\sigma}_{m\lambda}(t) \cdot \boldsymbol{\varepsilon}(\mathbf{v})\, dx = \int_\Omega \mathbf{F}(t) \cdot \mathbf{v} dx \quad \forall \mathbf{v} \in \mathcal{V}, \quad \text{a.e. } t \in (0,T), \tag{3.27}$$

$$\mathbf{u}_{m\lambda}(0) = \mathbf{u}_0, \quad \boldsymbol{\sigma}_{m\lambda}(0) = \boldsymbol{\sigma}_0 \text{ in } \Omega. \tag{3.28}$$

Then, by virtue of Lemmas 3.1 and 3.2 we can extract a subsequences, still denoted $\mathbf{u}_{m\lambda}$, $\boldsymbol{\sigma}_{m\lambda}$ and $\theta_{m\lambda}$ such that

$$\mathbf{u}_{m\lambda} \longrightarrow \mathbf{u}_{\lambda} \text{ in } L^{\infty}(0,T;\mathcal{V}) \text{ weakly*}, \tag{3.29}$$

$$\frac{\partial \mathbf{u}_{m\lambda}}{\partial t} \longrightarrow \frac{\partial \mathbf{u}_{\lambda}}{\partial t} \text{ in } L^{\infty}(0,T;H) \text{ weakly*}, \tag{3.30}$$

$$\frac{\partial^2 \mathbf{u}_{m\lambda}}{\partial t^2} \longrightarrow \frac{\partial^2 \mathbf{u}_{\lambda}}{\partial t^2} \text{ in } L^{\infty}(0,T;\mathcal{V}') \text{ weakly*}, \tag{3.31}$$

$$\boldsymbol{\sigma}_{m\lambda} \longrightarrow \boldsymbol{\sigma}_{\lambda} \text{ in } L^{\infty}(0,T;\mathcal{H}) \text{ weakly*}, \tag{3.32}$$

$$\frac{\partial \boldsymbol{\sigma}_{m\lambda}}{\partial t} \longrightarrow \frac{\partial \boldsymbol{\sigma}_{\lambda}}{\partial t} \text{ in } L^{\infty}(0,T;\mathcal{H}_1') \text{ weakly*}, \tag{3.33}$$

$$\theta_{m\lambda} \longrightarrow \theta_{\lambda} \text{ in } L^{q}\left(0,T;W^{1,p}(\Omega)\right) \text{ weakly}, \tag{3.34}$$

$$\frac{\partial \theta_{m\lambda}}{\partial t} \longrightarrow \frac{\partial \theta_{\lambda}}{\partial t} \text{ in } L^{q}\left(0,T;W^{-1,p}(\Omega)\right) \text{ weakly}, \tag{3.35}$$

These allow us, via Aubin-Simon's compactness theorems to extract subsequences, still denoted λ_m, $\mathbf{u}_{m\lambda}$ and $\theta_{m\lambda}$ such that

$$\lambda_m \longrightarrow \lambda \text{ in } L^{q}(0,T;L^{p}(\Omega)) \text{ strongly and a.e. in } Q. \tag{3.36}$$

$$\mathbf{u}_{m\lambda} \longrightarrow \mathbf{u}_{\lambda} \text{ in } L^{\infty}(0,T;H) \text{ strongly and a.e. in } Q, \tag{3.37}$$

$$\theta_{m\lambda} \longrightarrow \theta_{\lambda} \text{ in } L^{q}(0,T;L^{p}(\Omega)) \text{ strongly and a.e. in } Q. \tag{3.38}$$

Since the system (3.26)-(3.28) is linear , we can easily pass to the limit, using the convergence results (3.29)-(3.33), the fact that $\mu \in \mathcal{C}^0(\mathbb{R})$ and (3.36), to obtain

$$\frac{\partial \boldsymbol{\sigma}_{\lambda}}{\partial t} = \mathcal{A}\left(\boldsymbol{\varepsilon}\left(\frac{\partial \mathbf{u}_{\lambda}}{\partial t}\right)\right) + \mu(\lambda(t))\left(\mathcal{G}_1(\boldsymbol{\sigma}_{\lambda}(t)) + \mathcal{G}_2(\boldsymbol{\varepsilon}(\mathbf{u}_{\lambda}(t)))\right) \quad \text{a.e. } t \in (0,T), \tag{3.39}$$

$$\int_{\Omega} \boldsymbol{\sigma}_{\lambda}(t) \cdot \boldsymbol{\varepsilon}(\mathbf{v})\, dx = \int_{\Omega} \mathbf{F}(t) \cdot \mathbf{v} dx \quad \forall \mathbf{v} \in \mathcal{V}, \quad \text{a.e. } t \in (0,T), \tag{3.40}$$

$$\mathbf{u}_{\lambda}(0) = \mathbf{u}_0, \quad \sigma_{\lambda}(0) = \sigma_0 \text{ in } \Omega. \tag{3.41}$$

Our goal now is to prove that

$$\mathbf{u}_{m\lambda} \longrightarrow \mathbf{u}_{\lambda} \text{ in } L^{\infty}(0,T;\mathcal{V}) \text{ strongly}, \tag{3.42}$$

$$\boldsymbol{\sigma}_{m\lambda} \longrightarrow \boldsymbol{\sigma}_{\lambda} \text{ in } L^{\infty}(0,T;\mathcal{H}) \text{ strongly}. \tag{3.43}$$

To this aim, we proceed as follows. Subtracting equations (3.26) and (3.39) and integrating over the variable $t \in (0, T)$ we find

$$\|\boldsymbol{\sigma}_{m\lambda}(t) - \boldsymbol{\sigma}_{\lambda}(t)\|_{\mathcal{H}} \leq m_{\mathcal{A}} \|\mathbf{u}_{m\lambda}(t) - \mathbf{u}_{\lambda}(t)\|_{\mathcal{V}} +$$
$$\int_0^t \|(\mu(\lambda_m(s)) - \mu(\lambda(s)))(\mathcal{G}_1(\boldsymbol{\sigma}_{\lambda}(s)) + \mathcal{G}_2(\boldsymbol{\varepsilon}(\mathbf{u}_{\lambda}(s))))\|_{\mathcal{H}} ds +$$
$$\mu^* \int_0^t (m_1 \|\boldsymbol{\sigma}_{m\lambda}(s) - \boldsymbol{\sigma}_{\lambda}(s)\|_{\mathcal{H}} + m_2 \|\mathbf{u}_{m\lambda}(s) - \mathbf{u}_{\lambda}(s)\|_{\mathcal{V}}) ds$$
$$\text{a.e. } t \in (0, T). \tag{3.44}$$

Consequently,

$$\|\boldsymbol{\sigma}_{m\lambda}(t) - \boldsymbol{\sigma}_{\lambda}(t)\|_{\mathcal{H}}^2 \leq 4m_{\mathcal{A}}^2 \|\mathbf{u}_{m\lambda}(t) - \mathbf{u}_{\lambda}(t)\|_{\mathcal{V}}^2 +$$
$$\int_0^t \|(\mu(\lambda_m(s)) - \mu(\lambda(s)))(\mathcal{G}_1(\boldsymbol{\sigma}_{\lambda}(s)) + \mathcal{G}_2(\boldsymbol{\varepsilon}(\mathbf{u}_{\lambda}(s))))\|_{\mathcal{H}}^2 ds +$$
$$8T(\mu^*)^2 \int_0^t \left(m_1^2 \|\boldsymbol{\sigma}_{m\lambda}(s) - \boldsymbol{\sigma}_{\lambda}(s)\|_{\mathcal{H}}^2 + m_2^2 \|\mathbf{u}_{m\lambda}(s) - \mathbf{u}_{\lambda}(s)\|_{\mathcal{V}}^2\right) ds$$
$$\text{a.e. } t \in (0, T). \tag{3.45}$$

Furthermore, we get by subtracting equations (3.27) and (3.40), making use again equations (3.26) and (3.39) and setting $\mathbf{v} = \dot{\mathbf{u}}_{m\lambda} - \dot{\mathbf{u}}_{\lambda}$ as test function in the obtained equation

$$\frac{(\rho)^2}{2} \frac{d}{dt} \|\dot{\mathbf{u}}_{m\lambda} - \dot{\mathbf{u}}_{\lambda}\|_H^2 + \frac{C_0^2}{2} \frac{d}{dt} \|\mathbf{u}_{m\lambda} - \mathbf{u}_{\lambda}\|_{\mathcal{V}}^2 =$$
$$-\int_{\Omega} \Big(\int_0^t (\mu(\lambda_m(s)) - \mu(\lambda(s)))(G_1(\boldsymbol{\sigma}_{\lambda}(s)) + G_2(\varepsilon(\mathbf{u}_{\lambda}(s)) +$$
$$\mu(\lambda(s))(G_1(\boldsymbol{\sigma}_{m\lambda}(s) - \boldsymbol{\sigma}_{\lambda}(s))) + G_2(\varepsilon(\mathbf{u}_{m\lambda}(s) - \mathbf{u}_{\lambda}(s))))ds \Big) \varepsilon(\dot{\mathbf{u}}_{m\lambda}(s) - \dot{\mathbf{u}}_{\lambda}(s)) dx$$

this equation becomes

$$\frac{(\rho^*)^2}{2} \frac{d}{dt} \|\dot{\mathbf{u}}_{m\lambda} - \dot{\mathbf{u}}_{\lambda}\|_H^2 + \frac{C_0^2}{2} \frac{d}{dt} \|\mathbf{u}_{m\lambda} - \mathbf{u}_{\lambda}\|_{\mathcal{V}}^2$$
$$\leq \int_{\Omega} (\mu(\lambda_m(s)) - \mu(\lambda(s)))(G_1(\boldsymbol{\sigma}_{\lambda}(s)) + G_2(\varepsilon(\mathbf{u}_{\lambda}(s)) +$$
$$\mu(\lambda(s))(G_1(\boldsymbol{\sigma}_{m\lambda}(s) - \boldsymbol{\sigma}_{\lambda}(s))) + G_2(\varepsilon(\mathbf{u}_{m\lambda}(s) - \mathbf{u}_{\lambda}(s))))ds) \varepsilon(\mathbf{u}_{m\lambda}(s) - \mathbf{u}_{\lambda}(s)) dx$$
$$-\frac{d}{dt} \int_{\Omega} \Big(\int_0^t (\mu(\lambda_m(s)) - \mu(\lambda(s)))(G_1(\boldsymbol{\sigma}_{\lambda}(s)) + G_2(\varepsilon(\mathbf{u}_{\lambda}(s)) +$$
$$\mu(\lambda(s))(G_1(\boldsymbol{\sigma}_{m\lambda}(s) - \boldsymbol{\sigma}_{\lambda}(s))) + G_2(\varepsilon(\mathbf{u}_{m\lambda}(s) - \mathbf{u}_{\lambda}(s)\lambda)))ds \Big) \Big) \varepsilon(\mathbf{u}_{m\lambda}(s) - \mathbf{u}_{\lambda}(s)) dx$$

Integrating this inequality over the interval time variable $[0,t]$, Young inequality leads to

$$\frac{C_0^2}{2}\|\mathbf{u}_{m\lambda}-\mathbf{u}_\lambda\|_{\mathcal{V}}^2$$
$$\leq \frac{1}{2C_0^2}\int_0^t \|(\mu(\lambda_m(s))-\mu(\lambda(s)))(G_1(\boldsymbol{\sigma}_\lambda(s))+G_2(\varepsilon(\mathbf{u}_\lambda(s))+$$
$$\mu(\lambda(s))(G_1(\boldsymbol{\sigma}_{m\lambda}(s)-\boldsymbol{\sigma}_\lambda(s)))+G_2(\varepsilon(\mathbf{u}_{m\lambda}(s)-\mathbf{u}_\lambda(s)))\|_{\mathcal{H}}^2 ds+$$
$$\frac{C_0^2}{2}\int_0^t \|\mathbf{u}_{m\lambda}-\mathbf{u}_\lambda\|^2\, ds+\frac{T}{C_0^2}\int_0^t \|(\mu(\lambda_m(s))-\mu(\lambda(s)))(G_1(\boldsymbol{\sigma}_\lambda(s))+G_2(\varepsilon(\mathbf{u}_\lambda(s))+$$
$$\mu(\lambda(s))(G_1(\boldsymbol{\sigma}_{m\lambda}(s)-\boldsymbol{\sigma}_\lambda(s)))+G_2(\varepsilon(\mathbf{u}_{m\lambda}(s)-\mathbf{u}_\lambda(s)))\|_{\mathcal{H}}^2 ds+$$
$$\frac{C_0^2}{4}\|\mathbf{u}_{m\lambda}-\mathbf{u}_\lambda\|_{\mathcal{V}}^2$$

Applying Gronwall's lemma, we obtain

$$\|\mathbf{u}_{m\lambda}-\mathbf{u}_\lambda\|_{\mathcal{V}}^2 \leq C\int_0^t \|(\mu(\lambda_m(s))-\mu(\lambda(s)))(G_1(\boldsymbol{\sigma}_\lambda(s))+G_2(\varepsilon(\mathbf{u}_\lambda(s))\|_{\mathcal{H}}^2 ds+$$
$$(\mu^*)^2 C\int_0^t \left(m_1^2\|\boldsymbol{\sigma}_{m\lambda}(s)-\boldsymbol{\sigma}_\lambda(s)\|_{\mathcal{H}}^2+m_2^2\|\mathbf{u}_{m\lambda}(s)-\mathbf{u}_\lambda(s)\|_{\mathcal{V}}^2\right) ds$$
$$\text{a.e. } t\in(0,T). \tag{3.46}$$

From (3.45), (3.46) and exploiting Gronwall's lemma, we obtain for a.e. $t \in (0,T)$

$$\|\boldsymbol{\sigma}_{m\lambda}(t)-\boldsymbol{\sigma}_\lambda(t)\|_{\mathcal{H}}^2+\|\mathbf{u}_{m\lambda}(t)-\mathbf{u}_\lambda(t)\|_{\mathcal{V}}^2$$
$$\leq c\int_0^t \|(\mu(\lambda_m(s))-\mu(\lambda(s)))(\mathcal{G}_1(\boldsymbol{\sigma}_\lambda(s))+\mathcal{G}_2(\boldsymbol{\varepsilon}(\mathbf{u}_\lambda(s))))\|_{\mathcal{H}}^2\, ds. \tag{3.47}$$

On the other hand, since $\lambda_m \in K$, we extract a subsequence still denoted λ_m such that

$$\lambda_m \longrightarrow \lambda \text{ a.e. in } Q. \tag{3.48}$$

So, since $\mu \in \mathcal{C}^0(\mathbb{R})\cap L^\infty(\mathbb{R})$, we find

$$\begin{cases} \mu(\lambda_m) \longrightarrow \mu(\lambda) \text{ a.e. in } Q, \\ \mu(\lambda_m) \longrightarrow \mu(\lambda) \text{ in } L^\infty(Q) \text{ weakly*, (after a new extraction).} \end{cases} \tag{3.49}$$

Hence, it follows using (3.47) and (3.49) as well as the fact that the term $\mathcal{G}_1(\boldsymbol{\sigma}_\lambda)+\mathcal{G}_2(\boldsymbol{\varepsilon}(\mathbf{u}_\lambda))$ belongs to a bounded of $L^\infty(Q)\subset L^1(Q)$

$$\lim\left(\|\boldsymbol{\sigma}_{m\lambda}(t)-\boldsymbol{\sigma}_\lambda(t)\|_{\mathcal{H}}+\|\mathbf{u}_{m\lambda}(t)-\mathbf{u}_\lambda(t)\|_{\mathcal{V}}\right)=0 \text{ a.e. } t\in(0,T),$$

which allows us to validate (3.41)-(3.42).

Elsewhere, to pass to the limit in problem (3.24)-(3.25) it is enough to check the right hand side. To this end, it is well known that after a possible modification on a set of measure zero, $\varphi \in \mathcal{Z}_{p',q'}$ is continuous from $[0,T]$ into $W^{1,p'}(\Omega) \subset \mathcal{C}\left(\overline{\Omega}\right)$. Thus, we can write

$$\left|\int_Q \mu(\lambda_m)\,\mathcal{A}^{-1}\left(\mathcal{G}_1(\boldsymbol{\sigma}_{m\lambda}) + \mathcal{G}_2(\boldsymbol{\varepsilon}(\mathbf{u}_{m\lambda}))\right)\cdot \boldsymbol{\sigma}_{m\lambda}\varphi dxdt - \right.$$
$$\left.\int_Q \mu(\lambda)\,\mathcal{A}^{-1}\left(\mathcal{G}_1(\boldsymbol{\sigma}_{\lambda}) + \mathcal{G}_2(\boldsymbol{\varepsilon}(\mathbf{u}_{\lambda}))\right)\cdot \boldsymbol{\sigma}_{\lambda}\varphi dxdt\right|$$
$$\leq c\operatorname*{ess\,sup}_Q |\varphi(x,t)| \left[\left\|\left(\mu(\lambda_m) - \mu(\lambda)\right)\mathcal{A}^{-1}\left(\mathcal{G}_1(\boldsymbol{\sigma}_{\lambda}) + \mathcal{A}^{-1}\mathcal{G}_2(\boldsymbol{\varepsilon}(\mathbf{u}_{\lambda}))\right)\right\|_{L^2(0,T;\mathcal{H})}\right.$$
$$\left. + \left\|\boldsymbol{\sigma}_{m\lambda} - \boldsymbol{\sigma}_{\lambda}\right\|_{L^\infty(0,T;\mathcal{H})} + \left\|\mathbf{u}_{m\lambda} - \mathbf{u}_{\lambda}\right\|_{L^\infty(0,T;\mathcal{V})}\right]\left\|\boldsymbol{\sigma}_{\lambda}\right\|_{L^\infty(0,T;\mathcal{H})} \quad \forall\varphi \in \mathcal{Z}_{p',q'}.$$

Hence, by the foregoing

$$\int_Q \mu(\lambda_m)\left(\mathcal{G}_1(\boldsymbol{\sigma}_{m\lambda}) + \mathcal{G}_2(\boldsymbol{\varepsilon}(\mathbf{u}_{m\lambda}))\right)\cdot \boldsymbol{\sigma}_{m\lambda}\varphi dxdt \longrightarrow$$
$$\int_Q \mu(\lambda)\left(\mathcal{G}_1(\boldsymbol{\sigma}_{\lambda}) + \mathcal{G}_2(\boldsymbol{\varepsilon}(\mathbf{u}_{\lambda}))\right)\cdot \boldsymbol{\sigma}_{\lambda}\varphi dxdt \quad \forall\varphi \in \mathcal{Z}_{p',q'}. \tag{3.50}$$

Moreover, remarking from definition of the space function $\mathcal{Y}_{p,q}$, that $\mathcal{Y}_{p,q} \subset \mathcal{C}^0([0,T]; L^p(\Omega))$. Consequently, the condition $\theta_m(0) = \theta_{0m}$ has sense in the space $L^1(\Omega)$ and we have

$$\theta_{0m} \longrightarrow \theta_0 \text{ in } L^1(\Omega) \text{ weakly.} \tag{3.51}$$

On the other hand, the condition $\varphi \in \mathcal{Z}_{q',r'}$ implies

$$\varphi(0) \in \mathcal{C}^0\left(\bar{\Omega}\right). \tag{3.52}$$

Thus, it follows by (3.34), (3.38), (3.50), (3.51) and (3.52) that θ_λ solves the limit problem

$$-\int_Q \theta_\lambda \frac{\partial\varphi}{\partial t} dxdt + \int_Q \mathcal{K}(\nabla\theta_\lambda)\cdot\nabla\varphi dxdt$$
$$= -\int_Q \mu(\lambda)\,\mathcal{A}^{-1}\left(\mathcal{G}_1(\boldsymbol{\sigma}_{\lambda}(t)) + \mathcal{G}_2(\boldsymbol{\varepsilon}(\mathbf{u}_{\lambda}(t)))\right)\cdot \boldsymbol{\sigma}_{\lambda}\varphi dxdt +$$
$$\int_Q \theta_0\varphi(0)\,dx \quad \forall\varphi \in \mathcal{Z}_{p',q'}, \tag{3.53}$$
$$\theta_\lambda(0) = \theta_0 \text{ in } \Omega. \tag{3.54}$$

By virtue of Kakutani-Glicksberg's fixed point theorem, we conclude that the mapping $\mathcal{L}$ admits at least one fixed point $\theta_{\tilde{\lambda}} \in \mathcal{L}\left(\theta_{\tilde{\lambda}}\right)$, which permits us to achieve the proof.

We describe in the following example the concrete constitutive law of Maxwell's thermo-viscoelatic model which may be cast in the abstract form (1.1) and for which our main results apply.

Example. The Maxwell model is a linear thermo-viscoelastic constitutive law of the form, see [9]

$$\frac{\partial \boldsymbol{\sigma}}{\partial t} = \mathcal{A}\left(\boldsymbol{\varepsilon}\left(\frac{\partial \mathbf{u}}{\partial t}\right)\right) - \mu(\theta)\,\mathcal{G}(\boldsymbol{\sigma}) \quad \text{in} \quad Q, \tag{3.55}$$

where $\mathcal{A}$, $\mathcal{G}$, μ verify, respectively, the hypotheses (2.8), (2.9) and (2.11). This model can be used to describe soft solids: thermoplastic polymers in the vicinity of their melting temperature, fresh concrete (neglecting its aging), numerous metals at a temperature close to their melting point. The model (3.55) is a particular case of (1.1).

4 Open Problem

In our case the problem of uniqueness remaines unsolved. It is also interested to use numerical techniques to approximate and simulate the problem.

In addition, the case when the considered material is thermo-viscoelastic with non linear constitutive law is an open problem. Moreover, it is of interest to investigate setting with more general constitutive laws (thermo-viscoplastic and elasto-thermo-viscoplastic).

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