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Complete monotonicity of a function involving the (p, k)-digamma function

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Abstract

We establish a completely monotonic theorem involving the generalized digamma function with two parameters, and pose two open problems.

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1 Introduction

The Euler's gamma function is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, x > 0. \tag{1}$$

The digamma or psi function $\psi(x)$ which is defined as the logarithmic derivative of the gamma function:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{n(n+x)} \tag{2}$$

where γ is the Euler-Mascheroni constant.

In 2007, Diaz and Pariguan[3] defined the Γ_k function for $k > 0$, by

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, x \in \mathbf{C} \setminus k\mathbf{Z}^-, \tag{3}$$

where $(x)_{n,k} = x(x+k)(x+2k)...(x+(n-1)k)$ and $\lim_{k\to 1} \Gamma_k(x) = \Gamma(x)$. Completely similar, we also may define the generalized digamma function $\psi_k(x)$. The reader may see the references [3][9][10][11]. Very recently, K. Nantomah, E. Prempeh and S. B. Twum[11] introduced a new definition of gamma function with two parameters as follows:

$$\Gamma_{p,k}(x) = \frac{(p+1)!k^{p+1}(pk)^{\frac{x}{k}-1}}{(x)_{p,k}}, x > 0 \tag{4}$$

where $(x)_{p,k} = x(x+k)(x+2k)...(x+pk)$ and $\lim_{p\to\infty} \Gamma_{p,k}(x) = \Gamma_k(x)$. Furthermore, we naturally define the (p,k)- analogue of the digamma and polygamma functions as follows: $\psi_{p,k}(x) = \frac{\Gamma'_{p,k}(x)}{\Gamma_{p,k}(x)}$ and $\psi^{(m)}_{p,k}(x) = \frac{d^m}{dx^m}\psi_{p,k}(x)$. The functions $\psi_{p,k}(x)$ and $\psi^{(m)}_{p,k}(x)$ satisfy the following series and integral representations.

$$\begin{aligned} \psi_{p,k}(x) &= \tfrac{1}{k}\ln(pk) - \sum_{n=0}^{p} \tfrac{1}{nk+x} \\ &= \tfrac{1}{k}\ln(pk) - \int_0^\infty \tfrac{1-e^{-k(p+1)t}}{1-e^{-kt}} e^{-xt} dt \end{aligned} \tag{5}$$

and

$$\begin{aligned} \psi^{(m)}_{p,k}(x) &= (-1)^m m! \sum_{n=0}^{p} \tfrac{1}{(nk+x)^{m+1}} \\ &= (-1)^{m+1} \int_0^\infty \tfrac{1-e^{-k(p+1)t}}{1-e^{-kt}} t^m e^{-xt} dt \end{aligned} \tag{6}$$

where $m \in \mathbf{N}$.

A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and satisfies $(-1)^n f^n(x) \geq 0$ for $x \in I$ and $n \geq 0$. For the background and application, the reader may see [13].

In [2], Alzer proved that the function $x^\alpha[\ln x - \psi(x)]$ is completely monotonic on $(0,\infty)$ if and only if $\alpha \leq 1$. In [4], Guo and Qi gave two new proofs to the above result of Alzer. Recently, Krasniqi and Qi [7] generalized this result to generalized digamma function $\psi_k(x)$ with single parameter. It is natural that how can generalize result of Alzer to generalized digamma function $\psi_{p,k}(x)$ with two parameters. It is the main purpose of this paper to turn out that our result generalizes the theorem given by Alzer. In the final section, we also present two open problems concerning Theorem 3.1 and Corollary 3.2.

2 Lemmas

Lemma 2.1 *[1, 5.1.32] For real numbers $a, b > 0$, we have*

$$\ln\frac{b}{a} = \int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt.$$

Lemma 2.2 *[7] The function* $\theta(t)=\frac{1}{1-e^{-t}}-\frac{1}{t}$ *is increasing on* $t>0$ *with* $\lim_{t\to\infty}\theta(t)=1$ *and* $\lim_{t\to 0}\theta(t)=\frac{1}{2}$.

Lemma 2.3 *[1, 6.1.1] For real numbers* $x,\lambda>0$, *we have*

$$\frac{1}{x^{\lambda}}=\frac{1}{\Gamma(\lambda)}\int_0^{\infty}t^{\lambda-1}e^{-xt}dt.$$

3 Main results

Theorem 3.1 *For* $p\in\mathbf{N}, k>0$ *and* $\alpha\le 1$, *the function*

$$\delta_{p,k,\alpha}(x)=x^{\alpha}\left[\frac{1}{k}\ln\frac{pkx}{x+k(p+1)}-\psi_{p,k}(x)\right]$$

is complete monotonic on $(0,\infty)$.

Proof. Using Lemma 2.1 and the formula (5), we have

$$\begin{aligned}\delta_{p,k,1}(x)&=x\left[\tfrac{1}{k}\ln\tfrac{pkx}{x+k(p+1)}-\psi_{p,k}(x)\right]\\&=x\left[\tfrac{1}{k}\ln\tfrac{x}{x+k(p+1)}+\int_0^{\infty}\tfrac{1-e^{-k(p+1)t}}{1-e^{-kt}}e^{-xt}dt\right]\\&=x\int_0^{\infty}(1-e^{-k(p+1)t})\theta_k(t)e^{-xt}dt\end{aligned}$$

where $\theta_k(t)=\frac{1}{1-e^{-kt}}-\frac{1}{kt}$. So, we easily obtain that the function $\theta_k(t)$ is increasing and positive on $k,t>0$ by using Lemma 2.2.

Leibniz formula and direct computation yield

$$\begin{aligned}&(-1)^n\delta^{(n)}_{p,k,1}(x)=x(-1)^n\tfrac{d^n}{dx^n}\int_0^{\infty}\left(1-e^{-k(p+1)t}\right)\theta_k(t)e^{-xt}dt\\&-n(-1)^{n-1}\tfrac{d^{n-1}}{dx^{n-1}}\int_0^{\infty}\left(1-e^{-k(p+1)t}\right)\theta_k(t)e^{-xt}dt\\&=x\int_0^{\infty}\left(1-e^{-k(p+1)t}\right)\theta_k(t)e^{-xt}t^n dt-n\int_0^{\infty}\left(1-e^{-k(p+1)t}\right)\theta_k(t)e^{-xt}t^{n-1}dt\\&=\int_0^{n/x}\left(1-e^{-k(p+1)t}\right)\theta_k(t)e^{-xt}t^{n-1}(xt-n)dt+\\&\int_{n/x}^{\infty}\left(1-e^{-k(p+1)t}\right)\theta_k(t)e^{-xt}t^{n-1}(xt-n)dt\\&\ge\int_0^{\infty}\left(1-e^{-k(p+1)t}\right)\theta_k\left(\tfrac{n}{x}\right)e^{-xt}t^{n-1}(xt-n)dt\\&=\theta_k\left(\tfrac{n}{x}\right)\int_0^{\infty}\left(1-e^{-k(p+1)t}\right)e^{-xt}t^{n-1}(xt-n)dt.\end{aligned}$$

Applying Lemma 2.3, we get

$$\begin{aligned}&(-1)^n\delta^{(n)}_{p,k,1}(x)\ge\theta_k\left(\tfrac{n}{x}\right)\int_0^{\infty}\left(1-e^{-k(p+1)t}\right)e^{-xt}t^{n-1}(xt-n)dt\\&=\theta_k\left(\tfrac{n}{x}\right)\left[x\tfrac{\Gamma(n+1)}{x^{n+1}}-x\tfrac{\Gamma(n+1)}{(x+k(p+1))^{n+1}}-n\tfrac{\Gamma(n)}{x^n}+n\tfrac{\Gamma(n)}{(x+k(p+1))^{n+1}}\right]\\&=0.\end{aligned}$$

This implies that the function $\delta_{p,k,1}(x)$ is complete monotonic on $(0,\infty)$. Since the product of any two completely monotonic function is also completely monotonic on their domain, and the function $x^{\alpha-1}$ for $\alpha<1$ is clearly completely monotonic on $(0,\infty)$, so we obtain that the function

$$\delta_{p,k,\alpha}(x)=x^{\alpha-1}\delta_{p,k,1}(x)$$

is complete monotonic on $(0,\infty)$. This completes the proof.

Corollary 3.2 *If the function*

$$\delta_{p,k,\alpha}(x)=x^{\alpha}\left[\frac{1}{k}\ln\frac{pkx}{x+k(p+1)}-\psi_{p,k}(x)\right]$$

is complete monotonic on $(0,\infty)$*, then* $\alpha\le 2$*.*

Proof. Since the function

$$\delta_{p,k,\alpha}(x)=x^{\alpha}\left[\frac{1}{k}\ln\frac{pkx}{x+k(p+1)}-\psi_{p,k}(x)\right]$$

is complete monotonic on $(0,\infty)$, so we have $\delta_{p,k,\alpha}(x)\ge 0, \delta'_{p,k,1}(x)\le 0$. That is

$$\delta'_{p,k,1}(x)=x^{\alpha-1}\left[\alpha\left(\frac{1}{k}\ln\frac{pkx}{x+k(p+1)}-\psi_{p,k}(x)\right)+\frac{p+1}{x+k(p+1)}-x\psi'_{p,k}(x)\right]\le 0.$$

So, we get

$$\alpha\le\frac{x{\psi'}_{p,k}(x)-\frac{p+1}{x+k(p+1)}}{\frac{1}{k}\ln\frac{pkx}{x+k(p+1)}-\psi_{p,k}(x)}.$$

Applying L,Hôspitai rule and the formula (6), we have

$$\begin{aligned}&\lim_{x\to\infty}\frac{x{\psi'}_{p,k}(x)-\frac{p+1}{x+k(p+1)}}{\frac{1}{k}\ln\frac{pkx}{x+k(p+1)}-\psi_{p,k}(x)}\\&=\frac{\sum\limits_{n=0}^{p}[4nk-2k(p+1)]}{\sum\limits_{n=0}^{p}[2nk-k(p+1)]}\\&=2.\end{aligned}$$

This completes the proof.

4 Further Comments and Open Problems

Based on $\delta_{p,k,\alpha}(x) \geq 0$, we easily obtain the following estimation:

$$\psi_{p,k}(x) \leq \frac{1}{k} \ln \frac{pkx}{x + k(p+1)}, k > 0, p \in \mathbf{N}.$$

Finally, we propose two open problems:

Open problem 4.1 If the function

$$\delta_{p,k,\alpha}(x) = x^{\alpha} \left[\frac{1}{k} \ln \frac{pkx}{x + k(p+1)} - \psi_{p,k}(x) \right]$$

is complete monotonic on $(0, \infty)$, then $\alpha \leq 1$?

Open problem 4.2 We also hope to suggest some ideas for further study. It would be natural to generalize the completely monotonic and logarithmically completely monotonic properties of classical gamma, digamma and polygamma functions to the (p, k)-analogue of these functions. Further research in this topic has begun. Please see the recent reference [10]

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References

[1] M. Abramowitz, I. Stegun, eds., *Handbook of mathematical functions with formulas, graphs and mathematical tables.* National Bureau of Standards, Dover, New York, 1965.

[2] H. Alzer, *On some inequalities for the gamma and psi function,* Math. Comp. 66 (1997) 373-389.

[3] R. Diaz and E. Pariguan, *On hypergeometric functions and pochhammer k-symbol,* Divulg. Math. 15, No. 2, 2007, 179-192.

[4] B.-N. Guo and F. Qi, *Two new proofs of the complete monotonicity of a function involving the psi function,* Bull. Korean Math. Soc. 47 (2010), no. 1, 103-111.

[5] F. Krasniqi and S. Guo, *Logarithmically Completely monotonic functions involving generalized gamma and q-gamma functions,* J. Inequal. Spec. Funct., 1 (2010), 8-16.

[6] F. Krasniqi and F. Merovci, *Some completely monotonic properties for the (p, q)-gamma function,* Mathematica Balkanica (N.S.), V. 26, 2012, Fasc. No. 1-2.

[7] F. Krasniqi and F. Qi, *Complete monotonicity of a function involving the p -psi function and alternative proofs,* Global Journal of Mathematical Analysis 2 (2014), No. 3, 204-208.

[8] F. Krasniqi and A. Shabani, *Convexity properties and inequalities for a generalized gamma functions,* Appl. Math. E-Notes, 10(2010), 27-35.

[9] K. Nantomah, *Convexity Properties and Inequalities Concerning the (p, k)-Gamma function,* Commun. Fac. Sci. Univ. Ank. Series A1, 66, No. 2,(2017), 130-140.

[10] K. Nantomah, F. Merovci and S. Nasiru, *Some monotonic properties and inequalities for the (p, q)-gamma function,* Kragujevac Journal of Mathematics, 42, No. 2, (2018), 287-297.

[11] K. Nantomah, E. Prempeh and S. B. Twum, *On a (p,k)-analogue of the Gamma function and some associated Inequalities,* Moroccan Journal of Pure and Applied Analysis, 2, No. 2, (2016), 79-90.

[12] F. Qi and Ch.-P. Chen, *A complete monotonicity property of the Gamma function,* J. Math. Anal. Appl. 296 (2004), 603-607.

[13] R. L. Schilling, R. Song, and Z. Vondrăcek, *Bernstein Functions,* de Gruyter Studies in Mathematics 37, De Gruyter, Berlin, Germany, 2010.