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On the poset of classes of isomorphic subgroups of a finite group

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Abstract

In this short note we determine the finite groups whose poset of classes of isomorphic subgroups has breaking points. A generalization of this property is also studied.

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1 Introduction

Let G be a finite group and $L(G)$ be the subgroup lattice of G. The starting point for our discussion is given by [2], where the proper nontrivial subgroups H of G satisfying the condition

(1) for every $X \in L(G)$ we have either $X \leq H$ or $H \leq X$

have been studied. Such a subgroup is called a *breaking point* for the lattice $L(G)$, and a group G whose subgroup lattice possesses breaking points is called a *BP-group*. Clearly, all cyclic p-groups of order at least p^2 and all generalized quaternion 2-groups $Q_{2^n} = \langle a, b \mid a^{2^{n-2}} = b^2, a^{2^{n-1}} = 1, b^{-1}ab = a^{-1}\rangle$, $n \geq 3$, are BP-groups. Note that a complete classification of BP-groups can be found in [2]. Also, we observe that the condition (1) is equivalent to

(2) $$L(G) = [1, H] \cup [H, G],$$

where for $X, Y \in L(G)$ with $X \subseteq Y$ we denote by $[X, Y]$ the interval in $L(G)$ between X and Y. A natural generalization of (2) has been suggested by Roland Schmidt, namely

$$L(G) = [1, M] \cup [N, G] \text{ with } 1 < M, N < G, \tag{3}$$

and the abelian groups G satisfying (3) have been determined in [1].

The above concepts can be naturally extended to other remarkable posets of subgroups of G, such us the posets $\overline{L}(G)$, $C(G)$, and $\overline{C}(G)$ of conjugacy classes of subgroups, of cyclic subgroups, and of conjugacy classes of cyclic subgroups of G, respectively. We recall here that the generalized quaternion 2-groups Q_{2^n}, $n \geq 3$, can be characterized as being the unique finite non-cyclic groups G for which $\overline{L}(G)$, $C(G)$, and $\overline{C}(G)$ have breaking points (see [10], [7], and [3], respectively).

In the current note, we will focus on the set of equivalence classes of subgroups of G. It is defined by

$$\text{Iso}(G) = \{[H] \mid H \in L(G)\},$$

where

$$[H] = \{K \in L(G) \mid K \cong H\}, \forall\, H \in L(G).$$

Under the ordering relation

$$[H_1] \leq [H_2] \text{ if and only if } K_1 \subseteq K_2 \text{ for some } K_1 \in [H_1] \text{ and } K_2 \in [H_2]$$

this is a poset with the least element $[1] = \{1\}$ and the greatest element $[G] = \{G\}$ (see [9]). We will prove that the finite p-groups are the unique finite groups G for which $\text{Iso}(G)$ has breaking points. The more general problem of finding the finite groups G such that $\text{Iso}(G)$ is a union of two proper intervals will be also addressed.

Most of our notation is standard and will usually not be repeated here. Elementary concepts and results on group theory can be found in [4, 6]. For subgroup lattice notions we refer the reader to [5, 8] .

2 Main results

Our first theorem deals with the breaking points of $\text{Iso}(G)$.

Theorem 1. *Let G be a finite group and* $\text{Iso}(G)$ *be the poset of classes of isomorphic subgroups of G. Then* $\text{Iso}(G)$ *possesses breaking points if and only if G is a p-group for some prime p.*

Proof. If G is a p-group for some prime p, then all its subgroups of order p are isomorphic and consequently they form a unique class in $\text{Iso}(G)$. Clearly, this is a breaking point for $\text{Iso}(G)$.

Conversely, let $[H]$ be a breaking point of $\text{Iso}(G)$. Assume that the order of G has at least two distinct prime divisors. Then H cannot be a p-subgroup for any $p \in \pi(G)$. Indeed, if H is a p-subgroup for some $p \in \pi(G)$, then by choosing a non-trivial q-subgroup K of G with $q \neq p$, one obtains $[K] \leq [H]$ or $[H] \leq [K]$. This shows that $|K| \mid |H|$ or $|H| \mid |K|$, which is impossible. Therefore for any Sylow subgroup S of G we must have $[S] \leq [H]$, implying that $|S| \mid |H|$. This leads to $|H| = |G|$, i.e. $H = G$, a contradiction. □

Remark that the poset $\text{Iso}(G)$ associated to a p-group G can have more than one breaking point: for example, by taking $G = Q_8$ or $G = \mathbb{Z}_{p^n}$, where p is a prime and $n \geq 2$, we have that all classes $[H]$ with $1 < H < G$ are breaking points for $\text{Iso}(G)$.

Next we will denote by $\mathcal{C}$ the class of finite groups G satisfying

$$\text{Iso}(G) = [[1], [M]] \cup [[N], [G]] \text{ with } 1 < M, N < G. \quad (4)$$

By Theorem 1 we know that all finite p-groups belong to $\mathcal{C}$. This shows that we can restrict our study to finite groups which are not p-groups. Moreover, in (4) we can assume that M is a maximal subgroup of G and N is a minimal subgroup of G. Let $|N| = p$, where p is a prime. Then (4) can be seen as follows: *p divides the order of any subgroup of G all of whose isomorphic copies are not contained in isomorphic copies of M.*

At a first look, we observe that many groups of small order satisfy (4). More precisely, we have difficulties in finding finite groups which are not contained in $\mathcal{C}$. This is due to the following result:

Theorem 2. *Any finite solvable group belongs to $\mathcal{C}$.*

Proof. Let G be a finite solvable group and $p \in \pi(G)$. Pick a subgroup $N \leq G$ of order p and a p-complement M in G of order m. Then $p \nmid m$, any two subgroups of order m of G are conjugate, and any subgroup of G of order dividing m is contained in a subgroup of order m. Let $H \leq G$. If $|H| \mid m$, then H is contained in a conjugate of M. In other words, H is contained in an isomorphic copy of M, proving that $[H] \leq [M]$. If $|H| \nmid m$, then $p \mid |H|$ and so H contains a subgroup of order p of G. Thus $[N] \leq [H]$, completing the proof. □

The above theorem shows that the order of a finite group not contained in $\mathcal{C}$ is ≥ 60. By inspecting the subgroup structure of groups of order 60 we find such an example.

Example 3. *A_5 does not belong to $\mathcal{C}$.*

Now, it is naturally to look to other finite simple non-abelian groups. We will focus on $PSL(3,2)$. We easily infer that $\text{Iso}(PSL(3,2))$ consists of the following 12 elements:

- all classes $[H] \leq [S_4]$ (note that there are 9 such classes: $[1]$, $[\mathbb{Z}_2]$, $[\mathbb{Z}_3]$, $[\mathbb{Z}_4]$, $[\mathbb{Z}_2 \times \mathbb{Z}_2]$, $[D_8]$, $[A_4]$, $[S_3]$, and $[S_4]$),
- $[\mathbb{Z}_7]$,
- $[\mathbb{Z}_7 \rtimes \mathbb{Z}_3]$,
- $[PSL(3,2)]$.

Then

$$\text{Iso}(PSL(3,2)) = [[1],[S_4]] \cup [[\mathbb{Z}_7],[PSL(3,2)]],$$

proving that:

Example 4. *$PSL(3,2)$ belongs to $\mathcal{C}$.*

Clearly, a natural open problem is the following.

3 Open Problem

Determine the finite non-solvable groups which are contained in the class $\mathcal{C}$.

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