

Some Unavoidable Questions about Krasner Hypermodules

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Abstract

In this paper, we discuss some questions about Krasner hyperring and Krasner hypermodules. In this regard, we introduce some classes of Krasner hypermodules and obtain some results to give some approaches to answer the questions. Finally, we state some open problems as the enriched version of the questions.

Keywords: *Krasner hypermodule, Krasner hyperring, linearly pointed Krasner hypermodule, pointed Krasner hypermodule, zero-pinned hyperring.*

1 Introduction

The theory of hyperstructures was introduced by Marty in 1934. Hyperstructures have many applications to several sectors of both pure and applied mathematics (see [2, 6, 7, 20]). Many books and papers have been published related to the applications of hyperstructures in the fields of hypergraphs, binary relations, geometry, lattices, fuzzy sets and rough sets, automata, cryptography, combinatorics, codes, artificial intelligence, probabilistic, etc., for example see [3, 4, 5, 13, 14, 15, 19, 21]. Marty introduced hypergroups as a generalization of groups during the 8th Congress of the Scandinavian Mathematicians

in [10]. Some time later, the notion of hyperring was studied by Krasner [8] as a generalization of classical ring. Then some researchers followed him and studied this kind of hyperring named "Krasner hyperring". Naturally, one can define hypermodule to generalize the concept of module. Up to now, several kinds of hyperrings and hypermodules have been introduced and studied by many authors in [1, 9, 20]. In this paper, we fix a Krasner hyperring R and discuss an important type of hypermodule named "Krasner R -hypermodule" (see [16]). In fact, we consider some questions and suggest some helpful hints to find some answers to these questions.

2 Preliminaries

Here we state some basic definitions related to hyperstructures theory.

Let H be a non-empty set and $P(H)$ denote the family of all subsets of H . If $P^*(H) = P(H) \setminus \{\emptyset\}$, then every map

$$\circ: H \times H \longrightarrow P^*(H)$$

$$(a, b) \mapsto a \circ b$$

is called a *hyperproduct* or *hyperoperation* on H and the couple (H, \circ) is called a *hypergroupoid*. For convenience, we may use H instead of (H, \circ) . For any two subsets $A, B \subseteq H$, the set $A \circ B$ means $\bigcup_{(a,b) \in A \times B} a \circ b$. For simplicity, the notations $A \circ b$ and $a \circ B$ are used for $A \circ \{b\}$ and $\{a\} \circ B$, respectively. Generally, the singleton $\{a\}$ is identified with its member a . Also, we may use ab instead of $a \circ b$ for $a, b \in H$.

A hypergroupoid (H, \circ) is called a *semihypergroup* if $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in H$. The semihypergroup (H, \circ) is called a *hypergroup* if $x \circ H = H \circ x = H$ for every $x \in H$.

Let e be an element of semihypergroup $(H, +)$ (resp., (H, \cdot)) such that $e + x = x + e = x$ (resp., $e \cdot x = x \cdot e = x$) for all $x \in H$. Then e is called a scalar identity (resp., scalar unit). Every scalar identity (resp., scalar unit) in a semihypergroup is unique. We denote the scalar identity (resp., scalar unit) of $(H, +)$ (resp., (H, \cdot)) by 0_H (resp., 1_H). Let 0_H be the scalar identity of hypergroup $(H, +)$ and $x \in H$. An element $x' \in H$ is called an *additive inverse* of x in $(H, +)$ if $0_H \in x + x' \cap x' + x$.

Definition 2.1 ([11]) *A non-empty set H together with the hyperoperation $+$ is called a canonical hypergroup if the following axioms hold:*

1. $(H, +)$ is a semihypergroup (associativity);
2. $(H, +)$ is commutative (commutativity);

3. there is a scalar identity 0_H (existence of scalar identity);
4. for every $x \in H$, there is a unique (additive) inverse denoted by $-x$ such that $0_H \in x + (-x)$, which for simplicity we write $0_H \in x - x$ (existence of inverse);
5. $\forall x, y, z \in H : x \in y + z \implies y \in x - z$ (reversibility).

Definition 2.2 A non-empty set R together with the hyperoperation $+$ and the operation \cdot , denoted by $(R, +, \cdot)$, is called a Krasner hyperring if the following axioms hold:

1. $(R, +)$ is a canonical hypergroup;
2. (R, \cdot) is a semigroup including 0_R as a bilaterally absorbing element, that is $0_R \cdot x = x \cdot 0_R = 0_R$ for all $x \in R$;
3. $(y + z) \cdot x = (y \cdot x) + (z \cdot x)$ and $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

If there is no confusion, we may use R instead of $(R, +, \cdot)$. We say the Krasner hyperring R has the unit element 1_R if the semi(hyper)group (R, \cdot) has the (scalar) unit element 1_R , i.e., $1_R \cdot r = r \cdot 1_R = r$ for all $r \in R$.

Now we introduce the concept of Krasner hypermodule over a Krasner hyperring. First, we start with the following concept:

Definition 2.3 Let X and Y be two non-empty sets. A map $*$: $X \times Y \longrightarrow Y$ sending (x, y) to $x * y \in Y$ is called a scalar multiplication on Y and extended to all non-empty subsets $U \subseteq X$ and $V \subseteq Y$ as $x * V := \{x * v \mid v \in V\}$ and $U * y := \{u * y \mid u \in U\}$.

Definition 2.4 Let $(R, +, \cdot)$ be a Krasner hyperring. A canonical hypergroup $(A, +)$ together with the scalar multiplication $*$: $R \times A \longrightarrow A$ (on A) is called a Krasner hypermodule over R , for short a Krasner R -hypermodule, if for all $r_1, r_2 \in R$ and for all $a_1, a_2 \in A$ the following axioms hold:

1. $r_1 * (a_1 + a_2) = r_1 * a_1 + r_1 * a_2$;
2. $(r_1 + r_2) * a_1 = r_1 * a_1 + r_2 * a_1$;
3. $(r_1 \cdot r_2) * a_1 = r_1 * (r_2 * a_1)$;
4. $0_R * a_1 = 0_A$.

The name "Krasner" has been given to this kind of hypermodule by the authors in [16]. In Definition 2.4, the map $*$ is said to be trivial if $r * a = 0_A$ for all $(r, a) \in R \times A$.

A Krasner hypermodule A over a Krasner hyperring R with 1_R is called unitary if $1_R * a = a$ for all $a \in A$. In the sequel, for simplicity, we use ra instead of $r * a$. We say a Krasner hyperring R (resp., hypermodule) is *proper* if there exist $r, s \in R$ (resp., $x, y \in A$) such that $r + s$ (resp., $x + y$) is not a singleton. We denote every singleton Krasner hyperring or hypermodule by $\mathbf{0}$.

3 Questions and Results

In this section, we ask some questions and introduce some classes of Krasner hyperrings and hypermodules.

First, notice that the concept of canonical hypergroup in hyperstructures theory is a generalization of the concept of abelian group. As every abelian group $(A, +)$ can be considered as a \mathbb{Z} -module, one may ask the following question: Can we say that every canonical hypergroup $(A, +)$ is a unitary Krasner hypermodule over the ring \mathbb{Z} with the usual scalar multiplication? It is clear that the answer is negative since $m x \subseteq A$ and the scalar multiplication $*$ is a map into A . So this problem can be a motivation to introduce another kind of hypermodule like "general Krasner hypermodule" (see [17, 18]). Also, as another problem, assume there is a canonical hypergroup $(A, +)$ with $x \in A$ such that $x - x \subseteq A$ has more than one element. If A is a unitary Krasner hypermodule over the ring \mathbb{Z} , then $x - x = 1 * x - 1 * x = (1 - 1) * x = 0 * x = 0$ is a contradiction. So we state the following question:

Question 3.1 *Is it possible to define any hyperoperation and multiplication on \mathbb{Z} to be a proper Krasner hyperring so that every canonical hypergroup can be considered as a Krasner \mathbb{Z} -hypermodule with nonzero scalar multiplication?*

We continue with the following questions:

Question 3.2 *Are there any proper hyperring R and any Krasner R -hypermodule $A \neq \mathbf{0}$ such that $rx + y$ is a singleton for all $r \in R$ and all $x, y \in A$?*

Question 3.3 *Are there any proper hyperring R and any unitary Krasner R -hypermodule $A \neq \mathbf{0}$ such that $rx + y$ is a singleton for all $r \in R$ and all $x, y \in A$?*

Question 3.4 *Are there any proper hyperring R and any Krasner R -hypermodule $A \neq \mathbf{0}$ such that $(A, +)$ is an abelian group?*

Question 3.5 *Are there any proper hyperring R and any unitary Krasner R -hypermodule $A \neq \mathbf{0}$ such that $(A, +)$ is an abelian group?*

Definition 3.6 Let R be a Krasner hyperring. A Krasner R -hypermodule A is called

1. linearly pointed if $rx + y$ is a singleton for all $r \in R$ and all $x, y \in A$.
2. pointed if $(A, +)$ is an abelian group.

In the sequel, when we speak of a unitary Krasner R -hypermodule, automatically R is supposed to have the unit element 1_R .

Notation 3.7 For an arbitrary Krasner hyperring R , we denote the class of all (resp., unitary) linearly pointed Krasner R -hypermodules by ${}_R\text{LPKHM}$ (resp., ${}_R\text{LPKHM}^1$). Also, we denote the class of all (resp., unitary) pointed Krasner R -hypermodules by ${}_R\text{PKHM}$ (resp., ${}_R\text{PKHM}^1$). The notation ${}_R\text{M}$ (resp., ${}_R\text{M}^1$) is used for the class of all (resp., unitary) modules over a ring R .

Remark 3.8 Let A be a unitary Krasner R -hypermodule. Then clearly $rx + y$ is a singleton for all $r \in R$ and all $x, y \in A$ if and only if $(A, +)$ is an abelian group. So we have the following straightforward observations.

Proposition 3.9 Always

1. ${}_R\text{PKHM} \subseteq {}_R\text{LPKHM}$.
2. ${}_R\text{LPKHM}^1 = {}_R\text{PKHM}^1$.

Clearly, if the Krasner hyperring R is not proper, then R is indeed a ring. Thus the following result is straightforward and we omit its proof.

Proposition 3.10 Let R be a Krasner hyperring that is not proper.

1. If $R = \mathbf{0}$ or R does not have 1_R , then ${}_R\text{M} = {}_R\text{PKHM} \subseteq {}_R\text{LPKHM}$.
2. If $\mathbf{0} \neq R$ has 1_R , then ${}_R\text{M}^1 = {}_R\text{PKHM}^1 = {}_R\text{LPKHM}^1$.

Let R be a Krasner hyperring. So we have $R = \cup_{r \in R} r - r$ or $R \neq \cup_{r \in R} r - r$. In the first case, the family $\{r - r\}_{r \in R}$ forms a cover for R with the property $0_R \in \cap_{r \in R} r - r$. So we call the first kind of hyperring zero-pinned and naturally, the latter is called *non-zero-pinned*. Similarly, for an arbitrary Krasner hyperring R , a Krasner R -hypermodule A is called zero-pinned if $A = \cup_{a \in A} a - a$. Also, a Krasner R -hypermodule A is called non-zero-pinned if $A \neq \cup_{a \in A} a - a$.

Example 3.11 [12] Let (G, \cdot) be a group with $|G| \geq 4$, and define a hyper-addition and a multiplication on $R = G \cup \{0\}$, by:

$$a + 0 = 0 + a = a \quad \text{for all } a \in R;$$

$$\begin{aligned}
a + a &= \{a, 0\} \quad \text{for all } a \in G; \\
a + b &= b + a = G \setminus \{a, b\} \quad \text{for all } a, b \in G : a \neq b; \\
a \odot 0 &= 0 \odot a = 0 \quad \text{for all } a \in R; \\
a \odot b &= a \cdot b \quad \text{for all } a, b \in G.
\end{aligned}$$

Then it is easy to check that $(R, +, \odot)$ is a Krasner hyperring with the scalar identity 0. Since $0 \in r + r$ for every $r \in R$, r is the inverse of itself in $(R, +)$, i.e., $-r = r$. Clearly, $R = \cup_{r \in R} r + r = \cup_{r \in R} r - r$. Thus $(R, +, \odot)$ is zero-pinned.

Notation 3.12 Let R be a Krasner hyperring. The class of all zero-pinned Krasner R -hypermodules and non-zero-pinned Krasner R -hypermodules are denoted by ${}_R\text{ZPHM}$ and ${}_R\text{ZPHM}^c$, respectively.

The following result gives a partial answer or an assistance to find any example as answers for the Questions 3.2 to 3.5 if they exist.

Proposition 3.13 Let R be a zero-pinned proper Krasner hyperring. Then

1. the scalar multiplication of every element of ${}_R\text{LPKHM}$ or ${}_R\text{PKHM}$ is trivial.
2. there is no any unitary Krasner R -hypermodule $A \neq \mathbf{0}$. Consequently, ${}_R\text{LPKHM}^1 = {}_R\text{PKHM}^1 = \{\mathbf{0}\}$.

Proof. Suppose R is a zero-pinned proper Krasner hyperring.

1. Let A be a Krasner R -hypermodule in which $rx + y$ is a singleton for all $r \in R$ and all $x, y \in A$ or $(A, +)$ is an abelian group. By assumption, if $t \in R$, then $t \in r - r$ for some $r \in R$. On the other hand, $(r - r)a = ra + (-r)a$ for every $r \in R$ and $a \in A$. Let $t, t' \in r - r$. Since the right side of the latter equality is clearly a singleton, we have $ta = t'a$. Setting $t' = 0$ implies $ta = 0$. This means the map $*$ in Definition 2.4 is zero.
2. This follows by applying the proof of the first part, since the map $*$ cannot define a nonzero unitary Krasner R -hypermodule on every canonical group $A \neq \mathbf{0}$.

Thus when a Krasner hyperring R is zero-pinned, having trivial scalar multiplication is a necessary condition for every Krasner R -hypermodule in ${}_R\text{LPKHM}$ or ${}_R\text{PKHM}$. So if R is zero-pinned, then there is no any Krasner R -hypermodule with non-trivial scalar multiplication.

Hence, in order to find any example as an answer to Questions 3.2 to 3.5, we should either look for a zero-pinned proper Krasner hyperring R together with a nonzero Krasner R -hypermodule with trivial scalar multiplication or find a non-zero-pinned proper Krasner hyperring R together with a nonzero Krasner R -hypermodule having the mentioned properties.

4 Open Problems

According to the previous section, we state the following open problems:

Problem 4.1 *Is it possible to define any (non-trivial) hyperoperation on the set \mathbb{Z} so that it is a proper Krasner hyperring and any canonical hypergroup can be a Krasner \mathbb{Z} -hypermodule as every abelian group is a \mathbb{Z} -module?*

Problem 4.2 1. *Are there any zero-pinned proper Krasner hyperring R and any Krasner R -hypermodule $A \neq \mathbf{0}$ contained in ${}_R\text{LPKHM}$ or ${}_R\text{PKHM}$ (with trivial scalar multiplication)?*

2. *Are there any non-zero-pinned proper Krasner hyperring R and any Krasner R -hypermodule $A \neq \mathbf{0}$ contained in ${}_R\text{LPKHM}$ or ${}_R\text{PKHM}$?*

3. *Are there any non-zero-pinned proper Krasner hyperring R and any Krasner R -hypermodule $A \neq \mathbf{0}$ contained in ${}_R\text{LPKHM}^1 = {}_R\text{PKHM}^1$?*

Also, according to Propositions 3.9, we settle the following problem:

Problem 4.3 1. *Let R be a zero-pinned proper Krasner hyperring. Is there any Krasner R -hypermodule $A \neq \mathbf{0}$ (with trivial scalar multiplication) in ${}_R\text{LPKHM}$ which does not belong to ${}_R\text{PKHM}$?*

2. *Let R be a non-zero-pinned proper Krasner hyperring. Is there any non-unitary Krasner R -hypermodule $A \neq \mathbf{0}$ in ${}_R\text{LPKHM}$ which does not belong to ${}_R\text{PKHM}$?*

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