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$(\alpha, \beta)-$Reverse Derivations On Prime and Semiprime Rings

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Abstract

In this paper we investigate some properties of $(\alpha, \beta)-$***reverse derivations on prime and semiprime rings. The first of the main results is that if*** R ***is a prime ring of characteristic not*** 2***,*** $(\alpha, \beta)-$***reverse derivation and generalized*** $(\alpha, \beta)-$***reverse derivation are*** $(\alpha, \beta)-$***derivations and generalized*** $(\alpha, \beta)-$***derivations of*** R***, respectively. The next main result is that if*** R ***is a*** $2-$***torsion free semiprime ring, following properties are obtained. Let*** α, β ***be two homomorphisms of*** R***,*** $a, b, c \in R$***,*** D***,*** F ***and*** G ***be three*** $(\alpha, \beta)-$***reverse derivations of*** R. (i) ***Let*** α ***be automorphisms of*** R ***and*** β ***be epimorphisms of*** R. D ***is a*** $(\alpha, \beta)-$***reverse derivations of*** R ***if and only if*** D ***is a*** $(\beta, \alpha)-$***derivations of*** R. (ii) ***If*** $D(x) = c\alpha(x) + \beta(x)c$ ***then*** $D = 0$ ***and*** $c = 0$. (iii) ***If*** $D(x) = a\alpha(x) + \beta(x)b$ ***then*** D ***is inner*** $(\alpha, \beta)-$***derivation of*** R ***which is determined by*** a. (iv) ***If*** $F(x)\alpha(y) + \beta(y)G(x) = 0$ ***for all*** $x, y \in R$ ***and*** α, β ***are automorphisms of*** R ***then*** $F(y)\alpha([z, x]) = \beta([z, x])G(y) = 0$ ***for all*** $x, y, z \in R$; ***in particular,*** F ***and*** G ***map*** R ***into*** $Z(R)$.

Keywords: *semiprime ring, Reverse Derivation, generalized reverse derivation, Generalized* $(\alpha, \beta)-$*Reverse Derivations*
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1 Introduction

Let R be an associative ring with center $Z(R)$ and $\alpha, \beta : R \to R$ be two mappings. R is said to be $2-$*torsion free* if $2x = 0$ then $x = 0$. A ring R is called a *semiprime* if $a \in R$ and $aRa = (0)$ implies that $a = 0$. R is called a *prime ring* if $a, b \in R$ and $aRb = (0)$ implies that $a = 0$ or $b = 0$. For $x, y \in R$, $xy-yx$ is denoted by $[x, y]$ and $x\alpha(y) - \beta(y)x$ is denoted by $[x, y]_{\alpha,\beta}$. An additive mapping $d : R \to R$ is called a *derivation* if $d(xy) = d(x)y+xd(y)$ for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a : R \to R$ is given by $I_a(x) = [a, x]$ which is called an *inner derivation* determinated by a. In [1], generalized derivation in rings is defined as: An additive mapping $f : R \to R$ is called a *generalized derivation* if there exists a derivation d of R such that $f(xy) = f(x)y + xd(y)$ for all $x, y \in R$. Set $C_{\alpha,\beta} = \{c \in R \mid c\alpha(r) = \beta(r)c \text{ for all } r \in R\}$ and it is known as $(\alpha, \beta)-$center of R. In particular, $C_{1,1} = Z(R)$ where $1 : R \to R$ is identity map. An additive mapping $d : R \to R$ is called an $(\alpha, \beta)-$derivation if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$. An additive mapping $D : R \to R$ is said to be a *generalized* $(\alpha, \beta)-$*derivation* associated with d such that d is a $(\alpha, \beta)-$derivation of R if

$$D(xy) = D(x)\alpha(y) + \beta(x)d(y) \text{ for all } x, y \in R.$$

For a fixed $a \in R$, the mapping $I_a : R \to R$ is given by $I_a(x) = [a, x]_{\alpha,\beta}$ which is called $(\alpha, \beta)-$inner derivation determined by a.

The notion of reverse derivation arose in the previous paper of Herstein [2]. The notion of reverse derivation has relations with some generalizations of derivations. A *reverse derivation* is an additive mapping D from a ring R into itself satisfying $D(xy) = D(y)x+yD(x)$ for all $x, y \in R$. In [2], Herstein showed that if R is a prime ring and D is a nonzero reverse derivation of R, then R is a commutative integral domain and D is a derivation. Latter Samman and Alyamani extended the result of Herstein to semiprime rings in [5].

In this paper we extend the notion of reverse derivation to that of $(\alpha, \beta)-$reverse derivation and generalized $(\alpha, \beta)-$reverse derivation. An additive mapping $D : R \to R$ is said to be an $(\alpha, \beta)-$*reverse derivation* of R if

$$D(xy) = D(y)\alpha(x) + \beta(y)D(x) \text{ for all } x, y \in R.$$

Let d be a $(\alpha, \beta)-$reverse derivation. An additive mapping $D : R \to R$ is said to be a *generalized* $(\alpha, \beta)-$*reverse derivation* associated with d if

$$D(xy) = D(y)\alpha(x) + \beta(y)d(x) \text{ for all } x, y \in R.$$

One of the main purpose of this paper is to show that if R is a prime ring, D is a nonzero $(\alpha, \beta)-$reverse derivation of R, then D is a $(\alpha, \beta)-$derivation of R and if D is a nonzero generalized $(\alpha, \beta)-$reverse derivation of R, then D is a generalized $(\alpha, \beta)-$derivation of R.

We will show that while the notions of (α,β) −derivation and (α,β) −reverse derivation do not coincide, the set of all (α,β) −derivations and the set of all (α,β) −reverse derivations on a ring R are not disjoint.

One of the other main aim is to show that for a semiprime ring R, any (α,β) −rever-se derivation is in fact a (α,β) −derivation mapping R into the center.

Throughout this paper, R is a ring, $Z(R)$ is the center of R, α,β are homomorphisms of R and $C_{\alpha,\beta}=\{c\in R \mid c\alpha(x)=\beta(x)c,\ \forall x\in R\}$. We use the basic commutator identities:

- $[x,y]=-[y,x]=[-y,x]=[y,-x]=[-x,-y]$
- $(x,yz)=y(x,z)+[x,y]\,z$
- $[x,yz]_{\alpha,\beta}=[x,y]_{\alpha,\beta}\,\alpha(z)+\beta(y)\,[x,z]_{\alpha,\beta}$

The material in this work is a part of first author's Master's Thesis which is supervised by Prof. Dr. Neşet Aydın.

2 $(\alpha,\beta)-$Reverse Derivation on Prime Rings

The following theorem generalize [2, Theorem 2.1]

Theorem 2.1. *Let R be a prime ring and β be a automorphisms of R. A mapping D on R is a nonzero $(\alpha,\beta)-$reverse derivation of R if and only if R is commutative and D is an ordinary $(\alpha,\beta)-$derivation of R.*

Proof. Let $x,y,z\in R$. Thus

$$\begin{aligned} D(x(yz)) &= D(yz)\alpha(x)+\beta(yz)D(x) \\ &= D(z)\alpha(y)\alpha(x)+\beta(z)D(y)\alpha(x)+\beta(y)\beta(z)D(x) \end{aligned}$$

and so we have

$$D(x(yz)) = D(z)\alpha(y)\alpha(x)+\beta(z)D(y)\alpha(x)+\beta(y)\beta(z)D(x) \quad (1)$$

Additionally

$$D((xy)z) = D(z)\alpha(x)\alpha(y)+\beta(z)D(y)\alpha(x)+\beta(z)\beta(y)D(x) \quad (2)$$

Note that $(xy)z=x(yz)$, the results of (1) and (2) must be equal. Thus, by comparing them, we obtain

$$D(z)\alpha\left([x,y]\right)+\beta([z,y])D(x))=0 \quad (3)$$

If in (3), we substitute $y = x$ we see that

$$\beta([z,x])\,D(x) = 0, \text{ for all } x, z \in R. \tag{4}$$

Replacing z by zy, $y \in R$ in (4) and using (4)

$$\beta([z,x])\,\beta(y)D(x) = 0, \text{ for all } x, y, z \in R$$

is obtained. Since R is prime ring, it gives us

$$x \in Z(R) \text{ or } D(x) = 0 \text{ for all } x \in R.$$

Define $A = \{x \in R \mid x \in Z(R)\}$ and $B = \{x \in R \mid D(x) = 0\}$. It is clear that A and B are additive subgroups of R such that $R = A \cup B$. But a group can not be union two of its proper subgroups and therefore $R = A$ or $R = B$. If $R = B$ then $D = 0$. This is a contradiction. So $R = A$. For this reason R is commutative. So

$$D(xy) = D(yx) = D(x)\alpha(y) + \beta(x)D(y)$$

Therefore D is an ordinary $(\alpha,\beta)-$derivation. □

Furthermore, the following theorem generalizes Theorem 2.1 and [2, Theorem 2.1]

Theorem 2.2. *Let R be a prime ring and β be a automorphisms of R. A mapping D is a nonzero generalized $(\alpha,\beta)-$reverse derivation with $(\alpha,\beta)-$reverse derivation d of R if and only if R is commutative and D is an ordinary generalized $(\alpha,\beta)-$derivation with a $(\alpha,\beta)-$derivation d of R.*

Proof. Since d is a $(\alpha,\beta)-$reverse derivation, by Lemma 2.1 R is commutative and d is a $(\alpha,\beta)-$derivation of R. Since R is commutative, we get

$$D(xy) = D(yx) = D(x)\alpha(y) + \beta(x)d(y) \text{ for all } y \in R.$$

Therefore D is an ordinary generalized $(\alpha,\beta)-$derivation of R with $(\alpha,\beta)-$derivation d of R. □

3 $(\alpha,\beta)-$Reverse Derivation on Semiprime Rings

The following theorem generalizes [5, Proposition 3.1].

Theorem 3.1. *Let R be a semiprime ring, α be automorphisms of R, β be epimorphisms of R. A mapping D is a $(\alpha,\beta)-$reverse derivation of R if and only if D is central $(\beta,\alpha)-$derivation of R.*

Proof. It is clear that if D is central $(\beta,\alpha)-$derivation of R then $D(xy) = D(x)\beta(y) + \alpha(x)D(y) = D(y)\alpha(x) + \beta(y)D(x)$ and so D is a $(\alpha,\beta)-$reverse derivation of R. So let us suppose that D is a $(\alpha,\beta)-$reverse derivation of R. Then

$$D(x(yz)) = D(z)\alpha(y)\alpha(x) + \beta(z)D(y)\alpha(x) + \beta(y)\beta(z)D(x)$$

and also

$$D((xy)z)) = D(z)\alpha(x)\alpha(y) + \beta(z)D(y)\alpha(x) + \beta(z)\beta(y)D(x)$$

Together with the last two equations we get

$$\beta\left([y,z]\right)D(x) = D(z)\alpha\left([x,y]\right) \text{ for all } x,y,z \in R. \tag{5}$$

Replacing z by y in (5), we have

$$D(y)\alpha\left([x,y]\right) = 0 \text{ for all } x,y \in R. \tag{6}$$

By writing x by zx in equation (6) and using (6), we obtain

$$D(y)\alpha(z)\alpha\left([x,y]\right) = 0 \text{ for all } x,y,z \in R. \tag{7}$$

Linearizing (6) (in y) and using (6), we get

$$\begin{aligned} 0 &= D(u+y)\alpha\left([u+y,x]\right) \\ &= \left(D(u)+D(y)\right)\left(\alpha\left([u,x]\right)+\alpha\left([y,x]\right)\right) \\ &= D(u)\alpha\left([u,x]\right) + D(u)\alpha\left([y,x]\right) + D(y)\alpha\left([u,x]\right) + D(y)\alpha\left([y,x]\right) \end{aligned}$$

and so

$$D(u)\alpha\left([y,x]\right) + D(y)\alpha\left([u,x]\right) = 0 \text{ for all } x,y,u \in R.$$

That is,

$$D(y)\alpha\left([u,x]\right) = D(u)\alpha\left([x,y]\right) \text{ for all } x,y,u \in R. \tag{8}$$

By taking $[u,x]\,z\alpha^{-1}\left(D(u)\right)$ instead of z in the equation (7) and using (8) then we get

$$\begin{aligned} 0 &= D(y)\alpha([u,x]\,z\alpha^{-1}\left(D(u)\right))\alpha\left([x,y]\right) \\ &= D(y)\alpha\left([u,x]\right)\alpha\left(z\right)D(u)\alpha\left([x,y]\right) \\ &= \left(D(y)\alpha\left([u,x]\right)\alpha\left(z\right)D(y)\alpha\left([u,x]\right)\right) \end{aligned}$$

Since R is semiprime and α is automorphisms we have

$$D(y)\alpha\left([u,x]\right) = 0 \text{ for all } x,y,u \in R. \tag{9}$$

Replacing u by $\alpha^{-1}(D(y))$ in (9) we get

$$D(y)\left[D(y),\alpha(x)\right] = 0 \text{ for all } x,y \in R.$$

Hence, by [3, Lemma 1.1.4], $D(y) \in Z(R)$, for all $y \in R$, .i.e. $D(R) \subset Z(R)$. This implies that $D(xy) = D(y)\alpha(x) + \beta(y)D(x) = D(x)\beta(y) + \alpha(x)D(y)$. That is D is $(\beta,\alpha)-$derivation of R. □

Lemma 3.2. *Let R be a $2-$torsion free semiprime ring, $c \in R$, α, β be epimorphisms of R and $D : R \to R$ such that $D(x) = c\alpha(x) + \beta(x)c$. If D is a $(\alpha,\beta)-$reverse derivation of R then $D = 0$ and $c = 0$.*

Proof. For any $x, y \in R$, using the definition of D and the hypothesis we get

$$D(xy) = c\alpha(xy) + \beta(xy)c$$

on the other hand, using the D is a $(\alpha,\beta)-$reverse derivation of R

$$\begin{aligned} D(xy) &= D(y)\alpha(x) + \beta(y)D(x) \\ &= (c\alpha(y) + \beta(y)c)\,\alpha(x) + \beta(y)\,(c\alpha(x) + \beta(x)c) \\ &= c\alpha\,(y)\alpha(x) + \beta(y)c\alpha(x) + \beta(y)c\alpha(x) + \beta(y)\beta(x)c \\ &= c\alpha(yx) + 2\beta(y)c\alpha(x) + \beta\,(yx)\,c \\ &= D(yx) + 2\beta(y)c\alpha(x) \end{aligned}$$

Therefore

$$D\,([x,y]) = 2\beta(y)c\alpha(x) \text{ for all } x, y \in R. \tag{10}$$

Similarly

$$D\,([y,x]) = 2\beta(x)c\alpha(y) \text{ for all } x, y \in R$$

is obtained. Since $D([x,y]) + D([y,x]) = 0$ and R is $2-$torsion free we get

$$\beta(x)c\alpha(y) + \beta(y)c\alpha(x) = 0 \text{ for all } x, y \in R. \tag{11}$$

Replacing x by xz in (11) and using (11) we have

$$\beta(x)\beta(z)c\alpha(y) - \beta(x)c\alpha(y)\alpha(z) = 0 \text{ for all } x, y, z \in R$$

that is

$$\beta(x)\,(\beta(z)c\alpha(y) - c\alpha(y)\alpha(z)) = 0 \text{ for all } x, y, z \in R$$

Since β is epimorphism and R is semiprime ring, this implies that

$$\beta(z)c\alpha(y) - c\alpha(y)\alpha(z) = 0 \text{ for all } y, z \in R \tag{12}$$

From (11) and (12), we get

$$(\beta(y)c + c\alpha(y))\,\alpha(z) = 0 \text{ for all } y, z \in R$$

Since α is epimorphisms and R is semiprime ring, the last equation implies that

$$c\alpha(y) + \beta(y)c = 0 \text{ for all } y \in R \tag{13}$$

That is $D = 0$. From the equation (10), we have $2\beta(y)c\alpha(x) = 0$ for all $x, y \in R$. Since R is $2-$torsion free semiprime ring, $c = 0$ is obtained. □

Example 3.3. *Consider the ring* $R = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \mid x, y, z \in S \right\}$, *where* S *is a commutative ring with* 1 *and define* $\alpha\left(\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\right) = \begin{bmatrix} z & y \\ 0 & x \end{bmatrix}$, $\beta\left(\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\right) = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$. *Let* $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ *be fixed elements of* R *and define* $D : R \to R$ *by* $D(X) = A\alpha(X) + \beta(X)B$. *As a result it is easy to check that* D *is* $(\alpha, \beta)-$*reverse derivation of* R *but not* $(\alpha, \beta)-$*derivation of* R.

Lemma 3.4. *Let* R *be a ring,* $a, b \in R$, α, β *be mappings of* R *and* $D(x) = a\alpha(x) + \beta(x)b$. *If* D *is a* $(\alpha, \beta)-$*reverse derivation of* R *then the equality*

$$a\left(\alpha(xy) - \alpha(y)\alpha(x)\right) + \left(\beta(xy) - \beta(y)\beta(x)\right)b = \beta(y)(b + a)\alpha(x)$$

is satisfied.

Proof. For any $x, y \in R$,

$$D(xy) = a\alpha(xy) + \beta(xy)b \tag{14}$$

On the other hand, since D is also a $(\alpha, \beta)-$reverse derivation

$$\begin{aligned} D(xy) &= D(y)\alpha(x) + \beta(y)D(x) \\ &= \left(a\alpha(y) + \beta(y)b\right)\alpha(x) + \beta(y)\left(a\alpha(x) + \beta(x)b\right) \\ &= a\alpha\,(y)\alpha(x) + \beta(y)b\alpha(x) + \beta(y)a\alpha(x) + \beta(y)\beta(x)b \end{aligned}$$

Now using (14) and the last equation

$$a\alpha\,(y)\alpha(x) + \beta(y)b\alpha(x) + \beta(y)a\alpha(x) + \beta(y)\beta(x)b = a\alpha(xy) + \beta(xy)b$$

hence we get

$$a\left(\alpha(xy) - \alpha(y)\alpha(x)\right) + \left(\beta(xy) - \beta(y)\beta(x)\right)b = \beta(y)(b + a)\alpha(x) \text{ for all } x, y \in R \tag{15}$$

is obtained. □

The following theorem generalizes [5, Proposition 3.4].

Theorem 3.5. *Let* R *be a* $2-$*torsion semiprime ring,* $a, b \in R$, α *be epimorphisms of* R, β *be automorphisms of* R *and* $D : R \to R$ *such that* $D(x) = a\alpha(x) + \beta(x)b$. *If* D *is a nonzero* $(\alpha, \beta)-$*reverse derivation of* R *then* D *is ordinary inner* $(\alpha, \beta)-$*derivation of* R *which is determined by* a.

Proof. Using Lemma 3.4, we have

$$D\left([x, y]\right) = \beta(y)(b + a)\alpha(x), \text{ for all } x, y \in R$$

Similarly,

$$D([y,x]) = \beta(x)(b+a)\alpha(y) \text{ for all } x, y \in R$$

is obtained. Since $D([x,y]) + D([y,x]) = 0$, we get

$$\beta(x)(b+a)\alpha(y) + \beta(y)(b+a)\alpha(x) = 0 \text{ for all } x, y \in R \qquad (16)$$

Replacing x by xz in (16) and using (16) it gives

$$\begin{aligned} 0 &= \beta(x)\beta(z)(b+a)\alpha(y) + \beta(y)(b+a)\alpha(x)\alpha(z) \\ &= \beta(x)\beta(z)(b+a)\alpha(y) - \beta(x)(b+a)\alpha(y)\alpha(z) \end{aligned}$$

that is $\beta(x)\left(\beta(z)(b+a)\alpha(y) - (b+a)\alpha(y)\alpha(z)\right) = 0,$ for all $x, y, z \in R$. Since β is epimorphism and R is semiprime ring,

$$\beta(z)(b+a)\alpha(y) - (b+a)\alpha(y)\alpha(z) = 0 \text{ for all } x, y, z \in R$$

is obtained. Applying (16) the last equation gives

$$\left(\beta(y)(b+a) + (b+a)\alpha(y)\right)\alpha(z) = 0 \text{ for all } x, y, z \in R.$$

Since α is epimorphism and R is semiprime ring, we have

$$\beta(y)(b+a) + (b+a)\alpha(y) = 0 \text{ for all } x, y \in R \qquad (17)$$

Considering (17) together with (16) we get

$$\begin{aligned} 0 &= \beta(x)(b+a)\alpha(y) + \beta(y)(b+a)\alpha(x) \\ &= -\beta(x)\beta(y)(b+a) - \beta(y)\beta(x)(b+a) \end{aligned}$$

that is,

$$\beta(xy+yx)(b+a) = 0 \text{ for all } x, y \in R \qquad (18)$$

is obtained. In (18) replace y by yz and use the equation $x(yz) + (yz)x = y(xz+zx) + [x,y]\,z$ we have

$$\beta(y(xz+zx) + [x,y]\,z)(b+a) = 0 \text{ for all } x, y, z \in R$$

Using β is homomorphism and applying (18) it implies

$$\beta([x,y])\,\beta(z)\,(b+a) = 0 \text{ for all } x, y, z \in R$$

Putting $z = \beta^{-1}(b+a)z\,[x,y]$ in the last equation and by using β is automorphism, we have

$$\beta([x,y])\,(b+a)\,\beta(z)\beta([x,y])\,(b+a) = 0$$

Since R is semiprime we get

$$\beta\left([x,y]\right)(b+a)=0 \text{ for all } x,y\in R$$

that is

$$\beta(xy-yx)\,(b+a)=0 \text{ for all } x,y\in R \tag{19}$$

From equation (18) and (19) we get $2\beta(xy)\,(b+a)=0$ for all $x,y\in R$. Since R is $2-$torsion free semiprime ring we have $b+a=0$ that is $b=-a$. Now using the definition of D we get

$$D(x)=a\alpha(x)-\beta(x)a=[a,x]_{\alpha,\beta}$$

Therefore D is ordinary inner $(\alpha,\beta)-$derivation of R determined by a. □

This establishes the following corollary. Moreover the following corollary generalizes [5, Lemma 3.5]

Corollary 3.6. *Let R be a $2-$torsion semiprime ring, $a,b\in R$, α be epimorphisms of R,β be automorphisms of R and $D:R\to R$ such that $D(x)=a\alpha(x)+\beta(x)b$. If D is a nonzero $(\alpha,\beta)-$reverse derivation of R such that $D(R)=0$ then $a=-b\in C_{\alpha,\beta}$.*

The following theorem is another generalization of [5, Proposition 3.4]

Theorem 3.7. *Let R be a $2-$torsion free semiprime ring, $a,b\in R$, α be anti-epimorphisms of R,β be anti-automorphisms of R and $D(x)=a\alpha(x)+\beta(x)b$. If D is a nonzero $(\alpha,\beta)-$reverse derivation of R then D is ordinary inner $(\alpha,\beta)-$derivation of R which is determined by a.*

Proof. By using (15), and α,β are anti-homomorphisms, we have

$$\beta(y)(b+a)\alpha(x)=0 \text{ for all } x,y\in R$$

Since R is semiprime ring we have $b+a=0$, that is, $b=-a$. Now using the definition of D we get

$$D(x)=a\alpha(x)-\beta(x)a=[a,x]_{\alpha,\beta}$$

Therefore D is ordinary inner $(\alpha,\beta)-$derivation of R determined by a. □

A generalization of [4, Theorem 2.2] is presented in the following theorem.

Theorem 3.8. *Let R be a semiprime ring, α,β be a automorphisms and F,G be $(\alpha,\beta)-$reverse derivations of R such that*

$$F(x)\alpha(y)+\beta(y)G(x)=0 \textit{ for all } x,y\in R. \tag{20}$$

Then $F(y)\,\alpha\left([z,x]\right)=\beta\left([z,x]\right)G(y)=0$ for all $x,y,z\in R$; in particular, F and G map R into $Z(R)$.

Proof. Replacing x by xy in (20), we get

$$\begin{aligned}0 &= F(xy)\alpha(y) + \beta(y)G(xy)\\ &= F(y)\alpha(x)\alpha(y) + \beta(y)F(x)\alpha(y) + \beta(y)G(y)\alpha(x) + \beta(y)\beta(y)G(x)\\ &= F(y)\alpha(xy) + \beta(y)G(y)\alpha(x) + \beta(y)\left(F(x)\alpha(y) + \beta(y)G(x)\right)\\ &= F(y)\alpha(xy) + \beta(y)G(y)\alpha(x)\end{aligned}$$

that is,

$$F(y)\alpha(xy) + \beta(y)G(y)\alpha(x) = 0 \text{ for all } x, y \in R. \tag{21}$$

By (20), $\beta(y)G(y) = -F(y)\alpha(y)$. So from (21), we get

$$F(y)\alpha(xy) + \beta(y)G(y)\alpha(x) = F(y)\alpha(xy) - F(y)\alpha(y)\alpha(x) = 0.$$

That is

$$F(y)\alpha([x, y]) = 0 \text{ for all } x, y \in R. \tag{22}$$

Let $z \in R$. Replacing x by $\alpha^{-1}(z)\,x$ in (22) and using (22), we have

$$F(y)z\alpha([x, y]) = 0 \text{ for all } x, y, z \in R. \tag{23}$$

Linearizing (22) (in y) and using (22),

$$\begin{aligned}0 &= F(y+z)\alpha([x, y+z]) = \left(F(y) + F(z)\right)\alpha\left([x, y] + [x, z]\right)\\ &= F(y)\alpha\left([x, y]\right) + F(y)\alpha\left([x, z]\right) + F(z)\alpha\left([x, y]\right) + F(z)\alpha\left([x, z]\right)\\ &= F(y)\alpha\left([x, z]\right) + F(z)\alpha\left([x, y]\right)\end{aligned}$$

is obtained. That is, $F(y)\alpha\left([x, z]\right) = -F(z)\alpha\left([x, y]\right)$. So

$$F(z)\alpha\left([x, y]\right) = F(y)\alpha\left([z, x]\right) \text{ for all } x, y, z \in R. \tag{24}$$

We want to prove that $F(y)\alpha\left([z, x]\right) = 0$. For this purpose, let $v \in R$ and consider $F(y)\alpha\left([z, x]\right) vF(y)\alpha\left([z, x]\right)$. Then by (24), we have

$$F(y)\alpha\left([z, x]\right) vF(y)\alpha\left([z, x]\right) = F(y)\alpha\left([z, x]\right) vF(z)\alpha\left([x, y]\right) \tag{25}$$

Putting $w = \alpha\left([z, x]\right) vF(z)$, by (23) and (25), we get $F(y)\alpha\left([z, x]\right) vF(y)\alpha\left([z, x]\right) = F(y)w\alpha\left([x, y]\right) = 0$ for all $x, y, w \in R$. Hence by the semiprimeness of R we get

$$F(y)\alpha\left([z, x]\right) = 0 \text{ for all } x, y, z \in R. \tag{26}$$

This proves the first identity.

We now show that $F(R) \subseteq Z(R)$. Replacing z by $z\alpha^{-1}\left(F(y)\right)$ in (26) and using (26), we get

$$F(y)\alpha(z)\left[F(y), \alpha(x)\right] = 0 \text{ for all } x, y, z, u \in R. \tag{27}$$

Replacing z by xz in (27) we have

$$F(y)\alpha(x)\alpha(z)\left[F(y),\alpha(x)\right]=0 \text{ for all } x,y,z,u\in R. \tag{28}$$

Multiplying (27) by $-\alpha(x)$ on the left, we get

$$-\alpha(x)F(y)\alpha(z)\left[F(y),\alpha(x)\right]=0 \text{ for all } x,y,z,u\in R. \tag{29}$$

Adding (28) and (29), it gives

$$\left[F(y),\alpha(x)\right]\alpha(z)\left[F(y),\alpha(x)\right]=0 \text{ for all } x,z,u\in R.$$

Hence by the semiprimeness of R we get

$$\left[F(y),\alpha(x)\right]=0 \text{ for all } x,y\in R. \tag{30}$$

This shows that $F(R)\subseteq Z(R)$. Moreover, this proves another assertion about F.

We now show that $\beta\left([x,y]\right)G(z)=0$ for all $x,y,z\in R$. For this purpose, replacing y by $[x,y]$ in (20) and using (22), we obtain

$$\beta([x,y])G(x)=0 \text{ for all } x,y\in R. \tag{31}$$

Replacing y by $y\beta^{-1}(z)$ in (31) and using (31), we get

$$\beta([x,y])zG(x)=0 \text{ for all } x,y\in R. \tag{32}$$

Linearizing (31) (in x) and using (31) we get

$$\begin{aligned}0&=\left(\beta([x,y])+\beta([z,y])\right)\left(G(x)+G(z)\right)\\&=\beta([x,y])G(x)+\beta([z,y])G(x)+\beta([x,y])G(z)+\beta([z,y])G(z)\\&=\beta([z,y])G(x)+\beta([x,y])G(z)\end{aligned}$$

that is,

$$\beta([x,y])G(z)=\beta([y,z])G(x) \text{ for all } x,y,z\in R. \tag{33}$$

Let $v\in R$ and consider $\beta([x,y])G(z)v\beta([x,y])G(z)$. Then by (33), we have

$$\beta([x,y])G(z)v\beta([x,y])G(z)=\beta([x,y])G(z)v\beta([y,z])G(x) \tag{34}$$

Putting $w=G(z)v\beta([y,z])$, by (32) and (34), we get $\beta([x,y])G(z)v\beta([x,y])G(z)=\beta([x,y])wG(x)=0$ for all $x,y,w\in R$. Hence by the semiprimeness of R we get

$$\beta([x,y])G(z)=0 \text{ for all } x,y,z\in R. \tag{35}$$

This gives the second identity.

We now show that $G(R) \subseteq Z(R)$. Replacing y by $\beta^{-1}(G(y))\,w$ in (35) and using (35), we get

$$[\beta(x), G(y)])\beta(w)G(z) = 0 \text{ for all } x, y, w, z \in R. \tag{36}$$

Replacing w by wx in (36) we have

$$[\beta(x), G(y)])\beta(w)\beta(x)G(z) = 0 \text{ for all } x, y, w, z \in R. \tag{37}$$

Multiplying (36) by $\beta(x)$ on the right, we get

$$-[\beta(x), G(y)])\beta(w)G(z)\beta(x) = 0 \text{ for all } x, y, w, z \in R. \tag{38}$$

Adding (37) and (38), it gives

$$[\beta(x), G(y)]\,\beta(w)\,[\beta(x), G(z)] = 0 \text{ for all } x, y, w, z \in R.$$

in particular

$$[\beta(x), G(y)]\,\beta(w)\,[\beta(x), G(y)] = 0 \text{ for all } x, y, w \in R.$$

Hence by the semiprimeness of R we get

$$[\beta(x), G(y)] = 0 \text{ for all } x, y \in R.$$

This shows that $G(R) \subseteq Z(R)$. This proves another assertion about G. □

4 Open Problem

How to generalize these theorems and results for a nonzero ideal or Lie ideal of semiprime rings?

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