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Lacunary Ideal Convergence of Interval Number Sequences Defined by Orlicz Function

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Abstract

An ideal I is a family of subsets of positive integers $\mathbb{N}$ which is closed under taking finite unions and subsets of its elements. In this paper we introduce some lacunary ideal convergent double interval valued number sequence spaces defined by Orlicz function and study different properties of these spaces like completeness, solidity, etc. We establish some inclusion relations among them. Also we introduce the concept of ideal double lacunary convergence for interval numbers and study some basic properties of this notion.

Keywords: *Completeness, θ-convergence, ideal-convergence, lacunary sequence, interval numbers, Orlicz function.*

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1 Background

The notion of I-convergence was initially introduced by Kostyrko, et. al [11] as a generalization of statistical convergence (see [8], [22]) which is based on the structure of the ideal I of subset of natural numbers $\mathbb{N}$. Kostyrko, et. al [12] gave some of basic properties of I-convergence and dealt with extremal I-limit points. Although an ideal is defined as a hereditary and additive family of subsets of a non-empty arbitrary set X, here in our study it suffices to take I as a family of subsets of $\mathbb{N}$, positive integers, i.e. $I \subset 2^{\mathbb{N}}$, such that $A \cup B \in I$ for each $A, B \in I$, and each subset of an element of I is an element of I.

A non-empty family of sets $F \subset 2^{\mathbb{N}}$ is a filter on $\mathbb{N}$ if and only if $\phi \notin F$, $A \cap B \in F$ for each $A, B \in F$, and any subset of an element of F is in F. An ideal I is called *non-trivial* if $I \neq \phi$ and $\mathbb{N} \notin I$. Clearly I is a non-trivial ideal if and only if $F = F(I) = \{\mathbb{N} - A : A \in I\}$ is a filter in $\mathbb{N}$, called the filter associated with the ideal I. A non-trivial ideal I is called *admissible* if and only if $\{\{n\} : n \in \mathbb{N}\} \subset I$. A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset. Further details on ideals can be found in Kostyrko, et.al (see [11]). Recall that a sequence $x = (x_k)$ of points in $\mathbb{R}$ is said to be I-convergent to a real number ℓ if $\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\} \in I$ for every $\varepsilon > 0$ ([11]). In this case we write $I - \lim x_k = \ell$. Further details on ideal convergence can be found in [21], [26]. The notion of I-convergence double sequence was initially introduced by Tripathy and Tripathy (see [25]).

Interval arithmetic was first suggested by Dwyer [2] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [15] in 1959 and Moore and Yang [16] in 1962. Further works on interval numbers can be found in Dwyer [3], Fischer [9], Markov [14]. Furthermore, Moore and Yang [17], have developed applications to differential equations.

Chiao in [1] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Şengönül and Eryilmaz in [23] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete metric space. Esi in [4], [5] and Esi and Esi [29], [30], Esi and Catalbas [31]introduced and studied strongly almost $\lambda-$convergence and statistically almost $\lambda-$convergence of interval numbers and lacunary sequence spaces of interval numbers, respectively. In [6], Esi and Hazarika introduced the difference classes of interval numbers. Recently Esi [7] has studied double sequences of interval numbers.

2 Preliminary Results

We denote the set of all real valued closed intervals by $\mathbb{IR}$. Any elements of $\mathbb{IR}$ is called closed interval and denoted by $\overline{x}$. That is $\overline{x} = \{x \in \mathbb{R} : \ a \leq x \leq b\}$. An interval number $\overline{x}$ is a closed subset of real numbers [1]. Let x_l and x_r be first and last points of $\overline{x}$ interval number, respectively. For $\overline{x}_1, \overline{x}_2 \in \mathrm{I}\mathbb{R}$, we have $\overline{x}_1 = \overline{x}_2 \Leftrightarrow x_{1_l}{=}x_{2_l}, x_{1_r}{=}x_{2_r}$. $\overline{x}_1 + \overline{x}_2 = \{x \in \mathbb{R} : x_{1_l} + x_{2_l} \leq x \leq x_{1_r} + x_{2_r}\}$, and if $\alpha \geq 0$, then $\alpha\overline{x} = \{x \in \mathbb{R} :\ \alpha x_{1_l} \leq x \leq \alpha x_{1_r}\}$ and if $\alpha < 0$, then $\alpha\overline{x} = \{x \in \mathbb{R} :\ \alpha x_{1_r} \leq x \leq \alpha x_{1_l}\}$,

$$\begin{aligned} \overline{x}_1.\overline{x}_2 = \{x \in \mathbb{R} : \min\{x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}\} \leq x \\ \leq \max\{x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}\}\}. \end{aligned}$$

In [15], Moore proved that the set of all interval numbers $\mathrm{I}\mathbb{R}$ is a complete metric space with a metric d defined by

$$d(\overline{x}_1, \overline{x}_2) = \max\{|x_{1_l} - x_{2_l}|, |x_{1_r} - x_{2_r}|\}.$$

In the special case $\overline{x}_1 = [a, a]$ and $\overline{x}_2 = [b, b]$, we obtain usual metric of $\mathbb{R}$.

The algebric properties of $\mathrm{I}\mathbb{R}$ can be found in [1].

Now we give the definition of convergence of interval numbers:

Definition 2.1 *([1]) A sequence $\overline{x} = (\overline{x}_k)$ of interval numbers is said to be convergent to the interval number $\overline{x}_o$ if for each $\varepsilon > 0$ there exists a positive integer k_o such that $d(\overline{x}_k, \overline{x}_o) < \varepsilon$ for all $k \geq k_o$. We write $\lim_k \overline{x}_k = \overline{x}_o$ to denote that $\overline{x}_o$ is the limit of a sequence $\overline{x} = (\overline{x}_k)$.*

Thus, $\lim_k \overline{x}_k = \overline{x}_o \Leftrightarrow \lim_k x_{k_l} = x_{o_l}$ and $\lim_k x_{k_r} = x_{o_r}$.

An interval valued double sequence $\overline{x} = (\overline{x}_{k,l})$ is said to be convergent in the Pringsheim's sense or P-convergent to an interval number $\overline{x}_o$, if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(\overline{x}_{k,l}, \overline{x}_o) < \varepsilon \text{ for } k, l > N$$

(see [19]) and we denote it by $P - \lim \overline{x}_{k,l} = \overline{x}_o$, where $d(\overline{x}_{k,l}, \overline{y}_{k,l})$ is the Hausdorff distance between $\overline{x} = (\overline{x}_{k,l})$ and $\overline{y} = (\overline{y}_{k,l})$ (see [28]). The interval number $\overline{x}_o$ is called the Pringsheim limit of $\overline{x} = (\overline{x}_{k,l})$. More exactly, we say that a double sequence of interval numbers $\overline{x} = (\overline{x}_{k,l})$ converges to a finite interval number $\overline{x}_o$ if $\overline{x}_{k,l}$ tends to $\overline{x}_o$ as both k and l tend to infinity independently of each another. We denote by $\overline{c}^2$ the set of all double convergent interval numbers of double interval numbers.

The interval number double sequence $\overline{x} = (\overline{x}_{k,l})$ is bounded if and only if there exists a positive number B such that $d(\overline{x}_{k,l}, \overline{0}) < B$ for all k and l. We shall denote all bounded interval number double sequences by $\overline{l}^2_\infty$. Let $\overline{w}^2$ denote the set of all double sequences of interval numbers.

Recall in [18], [13] that an Orlicz function M is a continuous, convex, nondecreasing function define for $x \geq 0$ such that $M(0) = 0$ and $M(x) > 0$ for

$x > 0$. If convexity of Orlicz function is replaced by $M(x+y) \le M(x)+M(y)$ then this function is called the modulus function and characterized by Ruckle [20]. An Orlicz function M is said to satisfy Δ_2-condition for all values u, if there exists $K > 0$ such that $M(2u) \le KM(u)$, $u \ge 0$. Subsequently, the notion of Orlicz function was used to defined sequence spaces by Tripathy et.al [24], Tripathyd and Hazarika [27] and many others.

Definition 2.2 *([10]) A* lacunary sequence *is an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$. The intervals determined by θ will be defined by $J_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be defined by ϕ_r.*

Freedman et.al., [10] defined the space N_θ in the following way. For any lacunary sequence $\theta = (k_r)$,

$$N_\theta = \left\{ (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in J_r} |x_k - L| = 0, \text{ for some } L \right\}.$$

The space N_θ is a *BK-space* with the norm

$$||(x_k)||_\theta = \sup_r \frac{1}{h_r} \sum_{k\in J_r} |x_k|.$$

N_θ^0 denote the subset of these sequences in N_θ for which $L = 0$, $(N_\theta^0, ||.||_\theta)$ is also a *BK-space.*

Let θ_1 and θ_2 be any two lacunary sequences and $J_r = \{k : k_{r-1} < k \le k_r\}$, $J_s = \{l : l_{s-1} < l \le l_s\}$. $h_{r,s} = h_r h_s$ and $k_0 = 0$, $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$; $l_0 = 0$, $h_s = l_s - l_{s-1} \to \infty$ as $s \to \infty$. Also we defined $\theta = (h_{r,s})$ and $J_{rs} = J_r \times J_s$.

Let $p = (p_{i,j})$ be a double sequence of positive real numbers. If $0 < p_{i,j} \le \sup_{i,j} p_{i,j} = H < \infty$ and $D = \max\left(1, 2^{H-1}\right)$, then for $a_{i,j}, b_{i,j} \in \mathbb{R}$ for all $i, j \in \mathbb{N}$, we have

$$|a_{i,j} + b_{i,j}|^{p_{i,j}} \le D\left(|a_{i,j}|^{p_{i,j}} + |b_{i,j}|^{p_{i,j}}\right).$$

3 Main Results

We define the generalized double lacunary mean by

$$t_{rs}(x) = \frac{1}{h_{r,s}} \sum_{(k,l)\in J_{rs}} x_{k,l}.$$

In this paper, we define new double sequence spaces for interval sequences as follows.

Let $\mathcal{I}$ be an admissible ideal of $\mathbb{N} \times \mathbb{N}$. Let M be an Orlicz function and $p = (p_{i,j})$ be a double sequence of strictly positive real numbers. We introduce the following sequence spaces:

$$ {}_2\overline{w}^{\mathcal{I}}\left(\overline{\theta}, M, p\right) = \left\{ \overline{x} = (\overline{x}_{i,j}) : \left\{ (r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s}} \sum_{(i,j)\in J_{rs}} \left[M\left(\frac{d\left(\overline{x}_{i,j}, \overline{x}_o\right)}{\rho} \right) \right]^{p_{i,j}} \right\} \in \mathcal{I}, \right. $$
$$ \left. \text{for some } \rho > 0, \text{and } \overline{x}_o \in \mathbf{IR} \right\}, $$

$$ {}_2\overline{w}^{\mathcal{I}}_o\left(\overline{\theta}, M, p\right) = \left\{ \overline{x} = (\overline{x}_{i,j}) : \left\{ (r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s}} \sum_{(i,j)\in J_{rs}} \left[M\left(\frac{d\left(\overline{x}_{i,j}, \overline{0}\right)}{\rho} \right) \right]^{p_{i,j}} \right\} \in \mathcal{I}, \right. $$
$$ \left. \text{for some } \rho > 0 \right\}, $$

$$ {}_2\overline{w}^{\mathcal{I}}_\infty\left(\overline{\theta}, M, p\right) = \left\{ \overline{x} = (\overline{x}_{i,j}) : \exists K > 0 \ s.t. \right. $$
$$ \left\{ (r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s}} \sum_{(i,j)\in J_{rs}} \left[M\left(\frac{d\left(\overline{x}_{i,j}, \overline{0}\right)}{\rho} \right) \right]^{p_{i,j}} \geq K \right\} \in \mathcal{I}, $$
$$ \left. \text{for some } \rho > 0 \right\} $$

and

$$ {}_2\overline{w}_\infty\left(\overline{\theta}, M, p\right) = \left\{ \ \overline{x} = (\overline{x}_{i,j}) : \ \sup_{r,s} \tfrac{1}{h_{r,s}} \textstyle\sum_{(i,j)\in J_{rs}} \left[M\left(\frac{d\left(\overline{x}_{i,j}, \overline{0}\right)}{\rho} \right) \right]^{p_{i,j}} < \infty, \right. $$
$$ \left. \text{for some } \rho > 0 \right\}, $$

where $d\left(\overline{x}_{i,j}, \overline{y}_{i,j}\right)$ is the Hausdorff distance between $\overline{x} = (\overline{x}_{i,j})$ and $\overline{y} = \left(\overline{y}_{i,j}\right)$.

Theorem 3.1 *Let $p = (p_{i,j})$ be bounded and $r, s \in N$. The double sequence spaces ${}_2\overline{w}^{\mathcal{I}}\left(\overline{\theta}, M, p\right)$, ${}_2\overline{w}^{\mathcal{I}}_o\left(\overline{\theta}, M, p\right)$, ${}_2\overline{w}^{\mathcal{I}}_\infty\left(\overline{\theta}, M, p\right)$ and ${}_2\overline{w}_\infty\left(\overline{\theta}, M, p\right)$ are complete metric spaces defined by*

$$ g(\overline{x}, \overline{y}) = \inf \left\{ \rho^{\frac{p_{r,s}}{H}} : \left(\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in J_{rs}} \left[M\left(\frac{d\left(\overline{x}_{i,j}, \overline{0}\right)}{\rho} \right) \right]^{p_{i,j}} \right)^{\frac{1}{H}} \leq 1, \text{ for some } \rho > 0 \right\} $$

where $H = \max\left\{1, \sup_{i,j} p_{i,j} < \infty\right\}$.

Proof. It can be easily verified that the classes are metric spaces. To prove completeness the class ${}_2\overline{w}_\infty\big(\overline{\theta},M,p\big)$, let $(\overline{x}^s_{i,j})$ be a Cauchy sequence in ${}_2\overline{w}_\infty\big(\overline{\theta},M,p\big)$. Then $g((\overline{x}^s_{i,j}) - (\overline{x}^t_{i,j})) \to 0$ as $s,t \to \infty$. For given $\varepsilon > 0$, choose $a > 0$ and $x_o > 0$ be such that $\frac{\varepsilon}{ax_o} > 0$ and $M\left(\frac{ax_o}{2}\right) \geq 1$. Now $g\left((\overline{x}^s_{i,j}) - (\overline{x}^t_{i,j})\right) \to 0$ as $s,t \to \infty$ implies that there exists $n_o \in \mathbb{N}$ such that $g\left((\overline{x}^s_{i,j}) - (\overline{x}^t_{i,j})\right) < \frac{\varepsilon}{ax_o}$ for all $s,t \geq n_o$. Then

$$\inf\left\{\rho^{\frac{p_{r,s}}{H}} : \left(\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in J_{rs}}\left[M\left(\frac{d\left(\overline{x}^s_{i,j}-\overline{x}^t_{i,j},\overline{0}\right)}{\rho}\right)\right]^{p_{i,j}}\right)^{\frac{1}{H}} \leq 1, \quad \text{for some } \rho > 0\right\} < \frac{\varepsilon}{ax_o}, \tag{3.1}$$

where H is the same as in Theorem 3.1. Now from (3.1), we have

$$M\left(\frac{d\left(\overline{x}^s_{i,j}-\overline{x}^t_{i,j},\overline{0}\right)}{\rho}\right) \leq 1 \leq M\left(\frac{ax_o}{2}\right).$$

Therefore

$$\frac{d\left(\overline{x}^s_{i,j}-\overline{x}^t_{i,j},\overline{0}\right)}{g\left((\overline{x}^s_{i,j})-(\overline{x}^t_{i,j})\right)} < \frac{ax_o}{2}.\frac{\varepsilon}{ax_o} = \frac{\varepsilon}{2}.$$

This implies that $(\overline{x}^s_{i,j})$ is a Cauchy sequence of interval numbers. So it is convergent, let $\lim_{s\to\infty}\overline{x}^s_{i,j} = \overline{x}_{i,j}$. Using continuity of M, we have

$$\lim_{t\to\infty}\left(\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in J_{rs}}\left[M\left(\frac{d\left(\overline{x}^s_{i,j}-\overline{x}^t_{i,j},\overline{0}\right)}{\rho}\right)\right]^{p_{i,j}}\right)^{\frac{1}{H}} \leq 1$$

and then

$$\left(\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in J_{rs}}\left[M\left(\frac{d\left(\overline{x}^s_{i,j}-\overline{x}_{i,j},\overline{0}\right)}{\rho}\right)\right]^{p_{i,j}}\right)^{\frac{1}{H}} \leq 1.$$

Let $s \geq n_o$, then taking infimum of such $\rho's$ we obtain $g\left(\overline{x}^s_{i,j}-\overline{x}_{i,j}\right) < \varepsilon$. Now using $g((\overline{x}_{i,j})-\overline{0}) \leq g((\overline{x}_{i,j})-\overline{x}^s_{i,j})+g((\overline{x}^s_{i,j})-\overline{0})$, we get $(\overline{x}_{i,j}) \in {}_2\overline{w}_\infty\left(\overline{\theta},M,p\right)$. Hence ${}_2\overline{w}_\infty\left(\overline{\theta},M,p\right)$ is complete. This completes the proof. ∎

Theorem 3.2 ***(a)*** ${}_2\overline{w}^{\mathcal{I}}\left(\overline{\theta},M,p\right) \subset {}_2\overline{w}_\infty\left(\overline{\theta},M,p\right)$, ***(b)*** ${}_2\overline{w}^{\mathcal{I}}_o\left(\overline{\theta},M,p\right) \subset {}_2\overline{w}_\infty\left(\overline{\theta},M,p\right)$.

Proof. It is easy, so omitted. ∎

Theorem 3.3 ***(a)*** *If* $0 < \inf_{i,j} p_{i,j} \leq p_{i,j} < 1$, *then* ${}_2\overline{w}^{\mathcal{I}}\left(\overline{\theta}, M, p\right) \subset {}_2\overline{w}^{\mathcal{I}}\left(\overline{\theta}, M\right)$, *(b) If* $1 < p_{i,j} < \sup_{i,j} p_{i,j} < \infty$, *then* ${}_2\overline{w}^{\mathcal{I}}\left(\overline{\theta}, M\right) \subset {}_2\overline{w}^{\mathcal{I}}\left(\overline{\theta}, M, p\right)$, *(c) If* $0 < p_{i,j} \leq q_{i,j} < \infty$ *and* $\left(\frac{q_{i,j}}{p_{i,j}}\right)$ *is bounded, then* ${}_2\overline{w}^{\mathcal{I}}\left(\overline{\theta}, M, p\right) \subset {}_2\overline{w}^{\mathcal{I}}\left(\overline{\theta}, M, q\right)$.

Proof. The first part of the result follows from the relation

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(i,j) \in J_{rs}} M\left(\frac{d\left(\overline{x}_{i,j}, \overline{x}_o\right)}{\rho}\right) \geq \varepsilon\right\}$$

$$\subseteq \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(i,j) \in J_{rs}} \left[M\left(\frac{d\left(\overline{x}_{i,j}, \overline{x}_o\right)}{\rho}\right)\right]^{p_{i,j}} \geq \varepsilon\right\}$$

and the second part of the result follows from the relation

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(i,j) \in J_{rs}} \left[M\left(\frac{d\left(\overline{x}_{i,j}, \overline{x}_o\right)}{\rho}\right)\right]^{p_{i,j}} \geq \varepsilon\right\}$$

$$\subseteq \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(i,j) \in J_{rs}} M\left(\frac{d\left(\overline{x}_{i,j}, \overline{x}_o\right)}{\rho}\right) \geq \varepsilon\right\}.$$

This completes the proof. The proof of the part three is easy, so omitted. ■

Theorem 3.4 ***(a)*** *If* $0 < \inf_{i,j} p_{i,j} \leq p_{i,j} < 1$, *then* ${}_2\overline{w}_0^{\mathcal{I}}\left(\overline{\theta}, M, p\right) \subset {}_2\overline{w}_0^{\mathcal{I}}\left(\overline{\theta}, M\right)$, *(b) If* $1 < p_{i,j} < \sup_{i,j} p_{i,j} < \infty$, *then* ${}_2\overline{w}_0^{\mathcal{I}}\left(\overline{\theta}, M\right) \subset {}_2\overline{w}_0^{\mathcal{I}}\left(\overline{\theta}, M, p\right)$, *(c) If* $0 < p_{i,j} \leq q_{i,j} < \infty$ *and* $\left(\frac{q_{i,j}}{p_{i,j}}\right)$ *is bounded, then* ${}_2\overline{w}_0^{\mathcal{I}}\left(\overline{\theta}, M, p\right) \subset {}_2\overline{w}_0^{\mathcal{I}}\left(\overline{\theta}, M, q\right)$.

Proof of the result follows from the Theorem 3.3.

Theorem 3.5 *Let* M_1 *and* M_2 *be two Orlicz functions. Then*

$${}_2\overline{w}^{\mathcal{I}}\left(\overline{\theta}, M_1, p\right) \cap {}_2\overline{w}^{\mathcal{I}}\left(\overline{\theta}, M_2, p\right) \subset {}_2\overline{w}^{\mathcal{I}}\left(\overline{\theta}, M_1 + M_2, p\right).$$

Proof. Let $(\overline{x}_{i,j}) \in {}_2\overline{w}^{\mathcal{I}}\left(\overline{\theta}, M_1, p\right) \cap {}_2\overline{w}^{\mathcal{I}}\left(\overline{\theta}, M_2, p\right)$. Then for every $\varepsilon > 0$ we have

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(i,j) \in J_{rs}} \left[M_1\left(\frac{d\left(\overline{x}_{i,j}, \overline{x}_o\right)}{\rho_1}\right)\right]^{p_{i,j}} \geq \varepsilon\right\} \in \mathcal{I}, \text{ for some } \rho_1 > 0$$

and

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(i,j) \in J_{rs}} \left[M_2\left(\frac{d\left(\overline{x}_{i,j}, \overline{x}_o\right)}{\rho_2}\right)\right]^{p_{i,j}} \geq \varepsilon\right\} \in \mathcal{I}, \text{ for some } \rho_2 > 0.$$

Let $\rho = \max\{\rho_1, \rho_2\}$. The result follows from the following inequality

$$\sum_{(i,j)\in J_{rs}} \left[(M_1 + M_2)\left(\frac{d\left(\overline{x}_{i,j}, \overline{x}_o\right)}{\rho}\right)\right]^{p_{i,j}}$$

$$\leq D\left(\sum_{(i,j)\in J_{rs}} \left[M_1\left(\frac{d\left(\overline{x}_{i,j}, \overline{x}_o\right)}{\rho_1}\right)\right]^{p_{i,j}} + \sum_{(i,j)\in J_{rs}} \left[M_2\left(\frac{d\left(\overline{x}_{i,j}, \overline{x}_o\right)}{\rho_2}\right)\right]^{p_{i,j}}\right).$$

This completes the proof. ■

Theorem 3.6 *Let M_1 and M_2 be two Orlicz functions. Then*

$${}_2\overline{w}^{\mathcal{I}}\left(\overline{\theta}, M_1, p\right) \subset {}_2\overline{w}^{\mathcal{I}}\left(\overline{\theta}, M_2 \circ M_1, p\right).$$

Proof. Let $\inf p_{i,j} = H_0$. For given $\varepsilon > 0$, we first choose $\varepsilon_0 > 0$ such that $\max\{\varepsilon_0^H, \varepsilon_0^{H_0}\} < \varepsilon$. Now using the continuity of M_2 choose $0 < \delta < 1$ such that $0 < t < \delta$ implies $M_2(t) < \varepsilon_0$. Let $(\overline{x}_{i,j}) \in {}_2\overline{w}^{\mathcal{I}}\left(\overline{\theta}, M_1, p\right)$. Now from the definition of ${}_2\overline{w}^{\mathcal{I}}\left(\overline{\theta}, M_1, p\right)$, for some $\rho > 0$

$$A(\delta) = \left\{(r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s}} \sum_{(i,j)\in J_{rs}} \left[M_1\left(\frac{d\left(\overline{x}_{i,j}, \overline{x}_o\right)}{\rho}\right)\right]^{p_{i,j}} \geq \delta^H\right\} \in \mathcal{I}.$$

Thus if $(r,s) \notin A(\delta)$ then we have

$$\frac{1}{h_{r,s}} \sum_{(i,j)\in J_{rs}} \left[M_1\left(\frac{d\left(\overline{x}_{i,j}, \overline{x}_o\right)}{\rho}\right)\right]^{p_{i,j}} < \delta^H$$

$$\Rightarrow \sum_{(i,j)\in J_{rs}} \left[M_1\left(\frac{d\left(\overline{x}_{i,j}, \overline{x}_o\right)}{\rho}\right)\right]^{p_{i,j}} < h_{r,s}\delta^H$$

$$\Rightarrow \left[M_1\left(\frac{d\left(\overline{x}_{i,j}, \overline{x}_o\right)}{\rho}\right)\right]^{p_{i,j}} < \delta^H, \text{ for all } i,j = 1,2,3,\ldots$$

$$\Rightarrow M_1\left(\frac{d\left(\overline{x}_{i,j}, \overline{x}_o\right)}{\rho}\right) < \delta, \text{ for all } i,j = 1,2,3,\ldots$$

Hence from above inequality and using continuity of M_2, we must have

$$M_2\left(M_1\left(\frac{d\left(\overline{x}_{i,j}, \overline{x}_o\right)}{\rho}\right)\right) < \varepsilon_0, \text{ for all } i,j = 1,2,3,\ldots$$

which consequently implies that

$$\sum_{(i,j)\in J_{rs}} \left[M_2\left(M_1\left(\frac{d\left(\overline{x}_{i,j}, \overline{x}_o\right)}{\rho}\right)\right)\right]^{p_{i,j}} < h_{r,s}\max\{\varepsilon_0^H, \varepsilon_0^{H_0}\} < h_{r,s}\varepsilon.$$

So,

$$\frac{1}{h_{rs}}\sum_{(i,j)\in J_{rs}}\left[M_2\left(M_1\left(\frac{d\left(\overline{x}_{i,j},\overline{x}_o\right)}{\rho}\right)\right)\right]^{p_{i,j}}<\varepsilon.$$

This shows that

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{r,s}}\sum_{(i,j)\in J_{rs}}\left[M_2\left(M_1\left(\frac{d\left(\overline{x}_{i,j},\overline{x}_o\right)}{\rho}\right)\right)\right]^{p_{i,j}}\geq\varepsilon\right\}\subset A(\delta)$$

and so belongs to $\mathcal{I}$. This completes the proof. ■

Theorem 3.7 *Let M_1 and M_2 be two Orlicz functions. Then (a) ${}_2\overline{w}_0^{\mathcal{I}}\left(\overline{\theta},M_1,p\right)\cap{}_2\overline{w}_0^{\mathcal{I}}\left(\overline{\theta},M_2,p\right)\subset{}_2\overline{w}_0^{\mathcal{I}}\left(\overline{\theta},M_1+M_2,p\right)$; (b) ${}_2\overline{w}_0^{\mathcal{I}}\left(\overline{\theta},M_1,p\right)\subset{}_2\overline{w}_0^{\mathcal{I}}\left(\overline{\theta},M_2\circ M_1,p\right)$.*

The proof of the theorem follows from the Theorems 3.5 and 3.6.

Theorem 3.8 *Let M_1 and M_2 be two Orlicz functions satisfying Δ_2-condition. If $\beta=\lim_{t\to\infty}\frac{M_2(t)}{t}\geq 1$, then (a) ${}_2\overline{w}_0^{\mathcal{I}}\left(\overline{\theta},M_1,p\right)={}_2\overline{w}_0^{\mathcal{I}}\left(\overline{\theta},M_2\circ M_1,p\right)$, (b) ${}_2\overline{w}^{\mathcal{I}}\left(\overline{\theta},M_1,p\right)={}_2\overline{w}^{\mathcal{I}}\left(\overline{\theta},M_2\circ M_1,p\right)$.*

Proof. It is easy, so omitted. ■

The following Theorem is a direct consequence of definition of equivalent mapping.

Theorem 3.9 *Let M_1 and M_2 be two Orlicz functions such that $M_1\approx M_2$. Then $Z(\overline{\theta},M_1,p)=Z(\overline{\theta},M_2,p)$, for $Z={}_2\overline{w}_0^{\mathcal{I}}$, ${}_2\overline{w}^{\mathcal{I}}$, ${}_2\overline{w}_\infty^{\mathcal{I}}$ and ${}_2\overline{w}_\infty$.*

Let w denotes be the all sequences real or complex numbers. A sequence space E is said to be *solid* (or *normal*) if $(x_k)\in E$ and for all sequence (α_k) of scalars with $|\alpha_k|\leq 1$, for all $k\in\mathbb{N}$ implies $(\alpha_k x_k)\in E$.

Let $K=\{k_1<k_2<\cdots\}\subseteq\mathbb{N}$ and E be a sequence space. A K-step space of E is a sequence space $\lambda_K^E=\{(x_{k_n})\in w:(k_n)\in E\}$.

A canonical preimage of a sequence $(x_{k_n})\in\lambda_K^E$ is a sequence $(y_n)\in w$ defined as

$$y_n=\begin{cases}x_n, & \text{if } n\in\mathbb{N};\\ 0, & \text{otherwise}\end{cases}$$

A canonical preimage of a step space λ_K^E is a set of canonical preimages of all elements in λ_K^E, i.e. y is in canonical preimage of λ_K^E if and only if y is canonical preimage of some $x\in\lambda_K^E$.

A sequence space E is said to be *monotone* if it contains the canonical preimages of its step spaces.

Lemma 3.10 *Every normal space is monotone.*

Theorem 3.11 *The double sequence space* ${}_2\overline{w}^{\mathcal{I}}\left(\overline{\theta}, M, p\right)$, ${}_2\overline{w}_o^{\mathcal{I}}\left(\overline{\theta}, M, p\right)$, ${}_2\overline{w}_\infty^{\mathcal{I}}\left(\overline{\theta}, M, p\right)$ *and* ${}_2\overline{w}_\infty\left(\overline{\theta}, M, p\right)$ *are solid as well as monotone.*

Proof. We give the proof for only ${}_2\overline{w}_o^{\mathcal{I}}\left(\overline{\theta}, M, p\right)$. The others can be proved similarly. Let $\overline{x} = (\overline{x}_{i,j}) \in {}_2\overline{w}_o^{\mathcal{I}}\left(\overline{\theta}, M, p\right)$ and $(\alpha_{i,j})$ be a scalar sequence such that $|\alpha_{i,j}| \leq 1$ for all $i, j \in \mathbb{N}$. Then for every $\varepsilon > 0$ we have

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(i,j) \in J_{rs}} \left[M\left(\frac{d\left(\alpha_{i,j}\overline{x}_{i,j}, \overline{0}\right)}{\rho}\right)\right]^{p_{i,j}} \geq \varepsilon\right\}$$

$$\subseteq \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{E}{h_{r,s}} \sum_{(i,j) \in J_{rs}} \left[M\left(\frac{d\left(\overline{x}_{i,j}, \overline{0}\right)}{\rho}\right)\right]^{p_{i,j}} \geq \varepsilon\right\} \in \mathcal{I},$$

where $E = \max\{1, |\alpha_{k,l}|^H\}$. Hence $(\alpha\overline{x}) \in {}_2\overline{w}_o^{\mathcal{I}}\left(\overline{\theta}, M, p\right)$. By Lemma 3.10, the double sequence space ${}_2\overline{w}_o^{\mathcal{I}}\left(\overline{\theta}, M, p\right)$ is monotone. This completes the proof. ■

4 $\mathcal{I}_{\overline{\theta}}$ -convergence of Interval Numbers

Definition 4.1 *A double sequence* $\overline{x} = (\overline{x}_{i,j})$ *of interval numbers is said to be* $\overline{\theta}$*-convergent to* $\overline{x}_0 \in \mathbf{IR}$ *if for every* $\varepsilon > 0$*, there exists a positive integer* N *such that*

$$d(t_{rs}(\overline{x}), \overline{x}_0) < \varepsilon \text{ for all } r, s > N.$$

Definition 4.2 *A double sequence* $\overline{x} = (\overline{x}_{i,j})$ *of interval numbers is said to be* $\mathcal{I}_{\overline{\theta}}$*-convergent to* $\overline{x}_0 \in \mathbf{IR}$ *if for every* $\varepsilon > 0$*, the set*

$$K_\varepsilon(\overline{\theta}) = \{(r,s) \in \mathbb{N} \times \mathbb{N} : d(t_{rs}(\overline{x}), \overline{x}_0) \geq \varepsilon\} \in \mathcal{I}.$$

or equivalently

$$\{(r,s) \in \mathbb{N} \times \mathbb{N} : d(t_{rs}(\overline{x}), \overline{x}_0) < \varepsilon\} \in F(\mathcal{I}).$$

In this case we write $\mathcal{I}_{\overline{\theta}} - \lim \overline{x} = \overline{x}_0$.

Theorem 4.3 *Let* $\overline{x} = (\overline{x}_{i,j})$ *be a double sequence in* $\mathbf{IR}$*. If* $\overline{\theta} - \lim \overline{x} = \overline{x}_0$*, then* $\mathcal{I}_{\overline{\theta}} - \lim \overline{x} = \overline{x}_0$*.*

Proof. Let $\overline{\theta} - \lim \overline{x} = \overline{x}_0$, then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(t_{rs}(\overline{x}), \overline{x}_0) < \varepsilon, \text{ for all } r, s > N.$$

Therefore the set

$$B = \{(r,s) \in \mathbb{N} : d(t_{rs}(\overline{x}), \overline{x}_0) \geq \varepsilon\} \subseteq \{(1,1), (2,2), \ldots, (N-1, N-1)\}.$$

But, $\mathcal{I}$ being admissible, we have $B \in \mathcal{I}$. Hence $\mathcal{I}_{\overline{\theta}} - \lim \overline{x} = \overline{x}_0$. ■

Theorem 4.4 *Sequential method $I_{\overline{\theta}}$ is regular.*

Proof. The proof follows from the fact that $\mathcal{I}$ is admissible and Theorem 4.3. ■

5 Open Problem

We introduce some lacunary ideal convergent double interval valued number sequence spaces defined by Orlicz function and study different properties of these spaces like completeness, solidity, some inclusion relations among them and also we introduced the concept of ideal double lacunary convergence for interval numbers and studied some basic properties of this notion. It is open problem that these classes have similar properties in three-dimensional space or not.

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