

Transcendence and approximation measure of continued fractions

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Abstract

In this article, we give sufficient conditions on the elements of continued fractions A and B which will assure us that the continued fractions A , B , $A \pm B$, AB and A/B are all transcendental numbers. The used method also permits us to calculate an approximate measure of a continued fraction A .

Keywords: *Continued fraction, transcendental number, approximation measure.*

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1 Introduction

In 1844, J. Liouville showed that a real number admitting very good rational approximation can not be algebraic, then it gives the first examples of transcendental numbers. So it is possible to prove by an elementary way the transcendence of $\sum_{n=0}^{+\infty} (1/10^{-n!})$ or generally $\sum_{n=0}^{+\infty} (1/b^{u_n})$, where b is an integer number ≥ 2 and $(u_n)_{n \geq 0}$ is an extremely lacunary sequence.

From this point of view, the transcendence of the continued fractions having partial quotients that increase rapidly have been studied by several authors such as P. Bundschuh [1], A. Durand [2], W. Lianxiang [4], G. Nettle [5], T. Okano [6]. Let A and B be two continued fractions which are defined by

$$A = a_0 + \frac{1}{|a_1|} + \frac{1}{|a_2|} + \dots$$

and

$$B = b_0 + \frac{1}{|b_1|} + \frac{1}{|b_2|} + \dots$$

where $a_i > 0$, $b_i > 0$ are integers for any $i \geq 1$.

In 1981, Nettler [5] proved that if $a_n > b_n > a_{n-1}^{(n-1)^2}$ for all n sufficiently large, then A , B , $A \pm B$, and $AB^{\pm 1}$ are all transcendental numbers.

Similarly, in 1987, T. Okano [6] proved the same result, if we have $a_n > b_n > a_{n-1}^{\gamma \times (n-1)}$ where γ is a real constant > 16 and in 1993, A. Kacha [3] proved the transcendence of the six numbers if $a_n > b_n > a_{n-1}^\alpha$ where α is a real constant > 7 .

In the present paper, we improve Okano's Theorem by using a direct approach and by studying the link between partial quotients and the convergents denominators of A and B . It is proven in this article that A , B , $A \pm B$, and $AB^{\pm 1}$ are transcendental numbers if $a_n > b_n > a_{n-1}^{3+\epsilon}$ for any $\epsilon > 0$ and for any sufficiently large n .

The second main result of this article is to calculate an approximate measure of a continued fraction A .

2 Preliminaries and methods

In this section, we provide the description of the used method. To prove the transcendence of continued fractions, we will use the Roth's Theorem which gives a sufficient condition of transcendence of real numbers. In order to calculate an approximate measure in subsection 3.2 of the continued fraction A , we need the theorem of Liouville.

Theorem of Roth [7]. *Let ξ be a real number, δ a real number > 2 , if there exist an infinity rational numbers $\frac{p}{q}$ with $\gcd(p, q) = 1$ such that*

$$\left| \xi - \frac{p}{q} \right| < \frac{1}{q^\delta},$$

then ξ is a transcendental number.

Theorem of Liouville [8]. *Let ξ be a real algebraic number of degree $d \geq 1$ and H is its height, then there exists a positive constant $C = C(\xi) = \left(d^2 (1 + |\xi|)^{d-1} H \right)^{-1}$ such that the following inequality is verified for every rational number p/q , ($p/q \neq \xi$):*

$$\left| \xi - \frac{p}{q} \right| > \frac{C}{q^d}.$$

3 Main results

3.1 Transcendence

With the same notations as above for A and B , we put

$$A_n = a_0 + \frac{1}{|a_1|} + \frac{1}{|a_2|} + \cdots + \frac{1}{|a_n|} = \frac{{}^a p_n}{{}^a q_n},$$

$$B_n = b_0 + \frac{1}{|b_1|} + \frac{1}{|b_2|} + \cdots + \frac{1}{|b_n|} = \frac{{}^b p_n}{{}^b q_n}.$$

The principal result of transcendence is given in the following theorem.

Theorem 1. 1) *If there exists a real constant $\alpha > 1$ and an integer $n_0 \geq 1$ such that for every $n \geq n_0, a_n > a_{n-1}^\alpha$ then A is a transcendental number.*

2) *If there exists a real constant $\alpha > 3$ and an integer $n_1 \geq 1$ such that $a_n > b_n > a_{n-1}^\alpha$ for all $n \geq n_1$, then $A, B, A + B, A - B, AB$ and A/B are all transcendental numbers.*

The proof of this theorem is based on some lemmas that express the link between partial quotients of the mentioned reals in Theorem 1 and the denominators of their convergents.

Lemma 1. *Let $\alpha > 1$ be a real number. If $a_n > a_{n-1}^\alpha$ for any sufficiently large n , then for any $\varepsilon > 0$ we have*

$${}^a q_n < a_n^{\frac{\alpha}{\alpha-1} + \varepsilon}$$

for any sufficiently large n .

We recall that Lemma 1 is an improvement of Lemma 3 of Okano [6].

Proof of Lemma 1. We have

$$\begin{aligned} {}^a q_n &= a_n {}^a q_{n-1} + {}^a q_{n-2} < (a_n + 1) {}^a q_{n-1} \\ &< {}^a q_{n_1-1} \prod_{k=n_1}^n (a_k + 1). \end{aligned}$$

This yields that

$${}^a q_n < {}^a q_{n_1-1} \prod_{k=n_1}^n \left(1 + \frac{1}{a_k}\right) \prod_{k=1}^n a_k.$$

However, for all $k \geq n_1$ we get

$$a_{k+1} > a_k^\alpha \text{ and } a_{n_1} \geq 2 \text{ then } a_k > 2^{\alpha^{k-n_1}}.$$

Hence, there exists a positive real constant $C(\alpha)$ such that

$$\prod_{k=n_1}^n \left(1 + \frac{1}{a_k}\right) < \prod_{k=n_1}^n \left(1 + \frac{1}{2^{\alpha^{k-n_1}}}\right) < C(\alpha).$$

Finally, we obtain

$$\begin{aligned} {}^a q_n &< C(\alpha) {}^a q_{n_1-1} \prod_{k=n_1}^n a_k \\ &< C(\alpha) {}^a q_{n_1-1} a_n^{1 + \frac{1}{\alpha} + \frac{1}{\alpha^2} + \dots + \frac{1}{\alpha^{n-n_1}}}. \end{aligned}$$

Which gives

$${}^a q_n < C(\alpha) {}^a q_{n_1-1} a_n^{\frac{1}{1-\frac{1}{\alpha}}} < a_n^{\frac{\alpha}{\alpha-1} + \varepsilon}.$$

Lemma 2. *Let $\alpha > 1$ be a real number. If*

$$a_n > b_n > a_{n-1}^\alpha$$

for any sufficiently large n , then for any $\varepsilon > 0$

$${}^a q_n^{1+\varepsilon} > {}^b q_n > {}^a q_{n-1}^{\alpha-\varepsilon}$$

for any sufficiently large n .

Remark. i) This part of Lemma 2 is an improvement of Lemma 2.2 of Nettler and it is also an improvement of Lemma 1.3 of [3].

ii) The improvement comes from the fact that we do not use the most difficult and long formulas of Nettler which are used by Okano to express $A_n + B_n$, $A_n - B_n$, $A_n B_n$ or A_n/B_n .

Proof. One can easily show that

$$\begin{aligned} {}^b q_n &= b_n {}^b q_{n-1} + {}^b q_{n-2} \\ &> b_n {}^b q_{n-1} \geq a_{n-1}^\alpha {}^b q_{n-1} \\ &> \prod_{i=1}^{n-1} a_i^\alpha \quad \text{because } {}^b q_1 = b_1 \geq 1. \end{aligned}$$

By the same arguments as in the proof of Lemma 1 we also have

$${}^a q_n < \prod_{i=1}^n \left(1 + \frac{1}{a_i}\right) \prod_{i=1}^n a_i$$

for any sufficiently large n . Therefore

$${}^b q_{n+1} > {}^a q_n^\alpha \prod_{i=1}^n \left(1 + \frac{1}{a_i}\right)^{-1} > {}^a q_n^{\alpha-\varepsilon}$$

for any sufficiently large n .

The left side of the inequality of Lemma 2 can be proved in a similar way, we observe that $b_n > b_{n-1}^\alpha$, so

$${}^b q_n < \prod_{i=1}^n \left(1 + \frac{1}{b_i}\right) \prod_{i=1}^n b_i,$$

and

$${}^a q_n > \prod_{i=1}^n a_i > \prod_{i=1}^n b_i,$$

whence

$${}^b q_n < {}^a q_n \prod_{i=1}^n \left(1 + \frac{1}{b_i}\right) < {}^a q_n^{1+\varepsilon}$$

for any sufficiently large n .

Proof of Theorem 1.

i) We first prove the transcendence of A . We have

$$\left| A - \frac{{}^a p_n}{{}^a q_n} \right| < \frac{1}{{}^a q_n {}^a q_{n+1}} < \frac{1}{{}^{a_{n+1}} a q_n^2}.$$

Since $a_{n+1} > a_n^\alpha$ for any sufficiently large n , by virtue of Lemma 1 for any $\varepsilon > 0$ we have

$$a_{n+1} > \frac{{}^a q_n^{\alpha-1}}{a_n^{\varepsilon(\alpha-1)}},$$

for any sufficiently large n . So, for any $\varepsilon > 0$ the inequality below becomes

$$\left| A - \frac{{}^a p_n}{{}^a q_n} \right| < \frac{{}^a n^{\varepsilon(\alpha-1)}}{{}^a q_n^{\alpha+1}}.$$

Now by choosing $\varepsilon \rightarrow 0$, we get

$$\left| A - \frac{{}^a p_n}{{}^a q_n} \right| < \frac{1}{{}^a q_n^{\alpha+1}}.$$

Since $\alpha + 1 > 2$, by Roth's Theorem, we conclude that A is a transcendental number.

From $b_n > a_n > b_{n-1}^\alpha$ it follows, again by Lemma 1, that B is transcendental as well.

ii) To prove that $A + B$ is transcendental, let

$$\left(\frac{{}^c p_n}{{}^c q_n} \right) = \left(\frac{{}^a p_n}{{}^a q_n} + \frac{{}^b p_n}{{}^b q_n} \right).$$

The set $({}^c p_n / {}^c q_n)$ is not stationary. In fact, for all even positive integer n , we have

$$A + B - \frac{{}^c p_n}{{}^c q_n} = A - \frac{{}^a p_n}{{}^a q_n} + B - \frac{{}^b p_n}{{}^b q_n} > 0.$$

It is not hard to show that one has

$$\begin{aligned} \left| A + B - \frac{{}^c p_n}{{}^c q_n} \right| &\leq \left| A - \frac{{}^a p_n}{{}^a q_n} \right| + \left| B - \frac{{}^b p_n}{{}^b q_n} \right| \\ &< \frac{1}{{}^a q_n} \frac{1}{{}^a q_{n+1}} + \frac{1}{{}^b q_n} \frac{1}{{}^b q_{n+1}}. \end{aligned}$$

Then one can observe that, by Lemma 2, ${}^c q_n = {}^a q_n {}^b q_n < {}^a q_n^{2+\varepsilon}$ and for any $\varepsilon > 0$

$$\frac{1}{{}^a q_n} \frac{1}{{}^a q_{n+1}} < \frac{1}{{}^a q_n} \frac{1}{{}^a q_n^{\frac{\alpha-\varepsilon}{+\varepsilon}}} = \frac{1}{{}^a q_n^{\frac{\alpha+1}{+\varepsilon}}} < \frac{1}{{}^c q_n^{\frac{\alpha+1}{(+\varepsilon)(2+\varepsilon)}}}.$$

Similarly, ${}^c q_n^{\alpha-\varepsilon} = {}^a q_n^{\alpha-\varepsilon} {}^b q_n^{\alpha-\varepsilon} < {}^b q_{n+1} {}^b q_n^{\alpha-\varepsilon}$, whence

$$\frac{1}{{}^b q_n} \frac{1}{{}^b q_{n+1}} < \frac{{}^b q_n^{\alpha-1-\varepsilon}}{{}^c q_n^{\alpha-\varepsilon}} = \frac{1}{{}^c q_n} \frac{1}{{}^a q_n^{\alpha-1-\varepsilon}} < \frac{1}{{}^c q_n^{1+\frac{\alpha-1-\varepsilon}{2+\varepsilon}}}.$$

for any sufficiently large n . By choosing $\varepsilon \rightarrow 0$, we obtain

$$\left| A + B - \frac{{}^c p_n}{{}^c q_n} \right| < \frac{1}{{}^c q_n^{\frac{1+\alpha}{2}}},$$

for any sufficiently large n . Theorem of Roth implies that $A + B$ is transcendental.

iii) We will prove that AB is a transcendental number.

We begin by justifying that the sequence

$$\left(\frac{{}^c p_n}{{}^c q_n} \right) = \left(\frac{{}^a p_n}{{}^a q_n} \frac{{}^b p_n}{{}^b q_n} \right) = (A_n B_n)$$

is not stationary. Suppose that

$$A_n B_n = A_{n-1} B_{n-1} = A_{n-2} B_{n-2} = A_{n-3} B_{n-3}.$$

Then

$$\begin{aligned} 0 = A_n B_n - A_{n-1} B_{n-1} &= (A_n - A_{n-1}) B_n + A_{n-1} (B_n - B_{n-1}) \\ &= \frac{(-1)^{n+1} {}^b p_n}{{}^a q_n {}^a q_{n-1} {}^b q_n} + \frac{{}^a p_{n-1} (-1)^{n+1}}{{}^a q_{n-1} {}^b q_n {}^b q_{n-1}}, \end{aligned}$$

whence, by appropriate iteration,

$$\frac{{}^b p_n}{{}^a q_n} = -\frac{{}^a p_{n-1}}{{}^b q_{n-1}} = \frac{{}^b p_{n-2}}{{}^a q_{n-2}} = -\frac{{}^a p_{n-3}}{{}^b q_{n-3}}.$$

and similarly

$$\frac{{}^a p_n}{{}^b q_n} = -\frac{{}^b p_{n-1}}{{}^a q_{n-1}} = \frac{{}^a p_{n-2}}{{}^b q_{n-2}} = -\frac{{}^b p_{n-3}}{{}^a q_{n-3}}.$$

On the other hand,

$$\begin{aligned} 0 = A_n B_n - A_{n-2} B_{n-2} &= (A_n - A_{n-2}) B_n + A_{n-2} (B_n - B_{n-2}) \\ &= \frac{(-1)^n a_n {}^b p_n}{{}^a q_n {}^a q_{n-2} {}^b q_n} + \frac{{}^a p_{n-2} (-1)^n b_n}{{}^a q_{n-2} {}^b q_n {}^b q_{n-2}}, \end{aligned}$$

hence

$$\frac{{}^b p_n}{{}^a q_n} = -\frac{b_n}{{}^a q_n} \frac{{}^a p_{n-2}}{{}^b q_{n-2}} \quad \text{and similarly} \quad \frac{{}^b p_{n-1}}{{}^a q_{n-1}} = -\frac{b_{n-1}}{{}^a q_{n-1}} \frac{{}^a p_{n-3}}{{}^b q_{n-3}}.$$

It follows that $a_n a_{n-1} = b_n b_{n-1}$, against $a_n > b_n > 0$ for any sufficiently large n . Therefore $(A_n B_n)$ is not stationary. Since $|A_n| \leq 2|A|$ and $\frac{1}{2b_1} \leq |B_n|$, we have

$$\left| AB - \frac{{}^c p_n}{{}^c q_n} \right| \leq |A - A_n| |B| + |B - B_n| |A_n| < \frac{|B|}{{}^a q_n {}^a q_{n+1}} + \frac{2|A|}{{}^b q_n {}^b q_{n+1}}$$

for any sufficiently large n . It follows again that AB is a transcendental number.

iv) The sequence

$$\left(\frac{{}^c p_n}{{}^c q_n} \right) = \left(\frac{{}^a p_n / {}^b p_n}{{}^a q_n / {}^b q_n} \right) = (A_n / B_n)$$

is not stationary. This can be proved along the same lines as in the proof in *iii*). Suppose that

$$\frac{A_n}{B_n} = \frac{A_{n-1}}{B_{n-1}} = \frac{A_{n-2}}{B_{n-2}} = \frac{A_{n-3}}{B_{n-3}}.$$

Then, from $(A_m - A_{m-1})/B_m = A_{m-1}(1/B_{m-1} - 1/B_m)$ for $m = n, n-1, n-2$ one gets

$$\frac{{}^b q_n}{{}^a q_n} = -\frac{{}^a p_{n-1}}{{}^b p_{n-1}} = \frac{{}^b q_{n-2}}{{}^a q_{n-2}} = -\frac{{}^a p_{n-3}}{{}^b p_{n-3}}.$$

and similarly

$$\frac{{}^a p_n}{{}^b p_n} = \frac{{}^b q_{n-1}}{{}^a q_{n-1}} = \frac{{}^a p_{n-2}}{{}^b p_{n-2}} = -\frac{{}^b q_{n-3}}{{}^a q_{n-3}}.$$

On the other hand, from $(A_m - A_{m-1})/B_m = A_{m-2}(1/B_{m-2} - 1/B_m)$ for $m = n, n-1$ one obtains

$$\frac{{}^b q_n}{{}^a q_n} = \frac{b_n {}^a p_{n-2}}{a_n {}^b p_{n-2}}, \quad \frac{{}^b p_{n-1}}{{}^a q_{n-1}} = -\frac{b_{n-1} {}^a p_{n-3}}{a_{n-1} {}^b q_{n-3}}.$$

That gives $a_n a_{n-1} = b_n b_{n-1}$, however $a_n > b_n > 0$ for any sufficiently large n . Therefore (A_n/B_n) is not stationary.

Then, we have

$$\left| \frac{A}{B} - \frac{{}^c p_n}{{}^c q_n} \right| = \frac{1}{|B| |{}^b p_n / {}^b q_n|} \left| \frac{{}^b p_n}{{}^b q_n} \left(A - \frac{{}^a p_n}{{}^a q_n} \right) + \frac{{}^a p_n}{{}^a q_n} \left(\frac{{}^b p_n}{{}^b q_n} - B \right) \right|.$$

As $1/2b_1 \leq |{}^b p_n / {}^b q_n|$ for $n \geq 2$, then we have

$$\begin{aligned} \left| \frac{A}{B} - \frac{c p_n}{c q_n} \right| &= \frac{1}{|B|} \left| A - \frac{a p_n}{a q_n} \right| + \frac{4|A|b_1}{|B|} \left| B - \frac{b p_n}{b q_n} \right| \\ &< \frac{1}{a q_n a q_{n+1}} + \frac{1}{b q_n b q_{n+1}} \end{aligned}$$

for any sufficiently large n . Therefore A/B is transcendental.

v) The sequence $(A_n - B_n)$ is not stationary. Indeed, suppose that

$$A_n - B_n = A_{n-1} - B_{n-1} = A_{n-2} - B_{n-2} = A_{n-3} - B_{n-3}.$$

From $A_m - A_{m-1} = B_m - B_{m-1}$ for $m = n, n-1, n-2$ one obtains

$$\frac{b q_n}{a q_n} = \frac{a q_{n-1}}{b q_{n-1}} = \frac{b q_{n-2}}{a q_{n-2}} = \frac{a q_{n-3}}{b q_{n-3}}$$

On the other hand, from $A_m - A_{m-2} = B_m - B_{m-2}$ for $m = n, n-1$ one gets

$$\frac{b q_n}{a q_n} = \frac{b_n a q_{n-2}}{a_n b q_{n-2}}, \quad \frac{b q_{n-1}}{a q_{n-1}} = \frac{b_{n-1} a q_{n-3}}{a_{n-1} b q_{n-3}}.$$

Once again, it follows that $a_n a_{n-1} = b_n b_{n-1}$, against $a_n > b_n > 0$ for any sufficiently large n . Therefore $(A_n - B_n)$ is not stationary.

So, one has

$$\begin{aligned} \left| A - B - \frac{c p_n}{c q_n} \right| &= \left| A - \frac{a p_n}{a q_n} + B - \frac{a p_n}{a q_n} \right| \leq \left| A - \frac{a p_n}{a q_n} \right| + \left| B - \frac{b p_n}{b q_n} \right| \\ &< \frac{1}{a q_n a q_{n+1}} + \frac{1}{b q_n b q_{n+1}}. \end{aligned}$$

Hence $A - B$ is transcendental.

Example. Let

$$\begin{cases} a_n = 2^{n!+1}, & n \geq 1 \\ b_n = 2^{4(n-1)!}, & n \geq 1. \end{cases}$$

We can see that

$$a_n > b_n > a_{n-1}^3 \text{ for all } n \geq 4.$$

By Theorem 1, the continued fractions A , B , $A \pm B$, and $AB^{\pm 1}$ are all transcendental numbers.

Remark. It is worth mentioning that, by the same arguments, the numbers A , B , $A \pm B$, and $AB^{\pm 1}$ are irrational under the weaker assumption $\alpha > 1$.

3.2. Approximation measure of a continued fraction A

In this subsection, we give the second main result of this article.

Theorem 2. *Let k be a real number > 1 , $(\alpha_n)_{n \geq 1}$ be an increasing sequence of real numbers > 1 and extending to $+\infty$ such that*

$$a_{n-1}^{\alpha(n-1)} < a_n < a_{n-1}^{k\alpha(n-1)}, \quad \text{for all } n \geq 2.$$

Then, for every algebraic number ξ of degree $d \geq 2$ and height $H \geq q_2^{\alpha_2}$ there exists a positive real constant $C = C(A, d, k, \alpha_1) > 0$ such that

$$|A - \xi| > C \cdot H^{-\gamma} \quad \text{where } \gamma = 2kd \left(\frac{\alpha_1}{\alpha_1 - 1} \right)^2.$$

Throughout this section, we adopt the following notations $p_n = {}^a p_n$ and $q_n = {}^a q_n$. To prove Theorem 2 we need the following lemma:

Lemma 3.

i) With the above hypothesis of Theorem 2, if $a_n > a_{n-1}^{\alpha(n-1)}$ for all $n \geq 2$, then

$$q_n < 2a_n^{\frac{\alpha_1}{\alpha_1 - 1}}.$$

ii) The hypothesis $a_n < a_{n-1}^{k\alpha(n-1)}$ for all $n \geq 2$ implies that

$$q_n < 2q_{n-1}^{\frac{k\alpha_1}{\alpha_1 - 1} \alpha(n-1)}.$$

Proof. By the same method as in Lemma 1, we prove Lemma 3. Details are left to the reader.

Proof of Theorem 2. Let ξ be an algebraic number of degree $d \geq 2$ and height $H \geq q_2^{\alpha_2}$. By Theorem 1, part *i)* A is transcendental, whence for any large n

$$|A - \xi| > |\xi - A_n| - |A - A_n|.$$

To obtain an approximation measure of A , it is sufficient to minimize $|A - \xi|$ by function of H and d .

According to Liouville's Theorem, it is not hard to see that

$$\left| \xi - \frac{p_n}{q_n} \right| \geq \frac{1}{Hd^2 \max(1, |\xi| + \theta^{d-1})q_n^d}. \quad (1)$$

It follows from lemma 3, that

$$|A - A_n| < \frac{1}{a_{n+1}q_n^2} < \frac{2^{\frac{\alpha_1-1}{\alpha_1}\alpha(n)}}{2 + \frac{\alpha_1-1}{\alpha_1}\alpha(n)} \frac{1}{q_n}. \quad (2)$$

Now, we will search the smallest integer n such that

$$\frac{1}{2Hd^2q_n^d} \quad \text{and} \quad \frac{2^{\frac{\alpha_1-1}{\alpha_1}\alpha(n)}}{2 + \frac{\alpha_1-1}{\alpha_1}\alpha(n)} \frac{1}{q_n}$$

are of the same order.

Due to inequalities (1) and (2) we obtain

$$|A - \xi| > \frac{1}{Hd^2q_n^d} - \frac{2^{\frac{\alpha_1-1}{\alpha_1}\alpha(n)}}{2 + \frac{\alpha_1-1}{\alpha_1}\alpha(n)} \frac{1}{q_n}.$$

In order to prove

$$|A - \xi| > \frac{1}{2Hd^2q_n^d}, \quad (3)$$

it is sufficient to have

$$\frac{1}{Hd^2q_n^d} - \frac{2^{\frac{\alpha_1-1}{\alpha_1}\alpha(n)}}{2 + \frac{\alpha_1-1}{\alpha_1}\alpha(n)} \frac{1}{q_n} > \frac{1}{2Hd^2q_n^d}.$$

Which is equivalent to the following inequality

$$\frac{1}{2Hd^2q_n^d} > \frac{2^{\frac{\alpha_1-1}{\alpha_1}\alpha(n)}}{2 + \frac{\alpha_1-1}{\alpha_1}\alpha(n)} \frac{1}{q_n},$$

that is

$$q_n^{\frac{\alpha_1-1}{\alpha_1}\alpha(n)} > 2Hd^2q_n^{d-2} 2^{\frac{\alpha_1-1}{\alpha_1}\alpha(n)}. \quad (4)$$

To realize the inequality (4), it suffices that the integer n satisfies

$$q_n^{\frac{\alpha_1-1}{2\alpha_1}\alpha(n)} \geq 2^{\frac{\alpha_1-1}{\alpha_1}\alpha(n)} \quad \text{and} \quad q_n^{\frac{\alpha_1-1}{2\alpha_1}\alpha(n)} > 2Hd^2q_n^{d-2}. \quad (5)$$

The first part of (5) is at once clear, because $q_n > 2$ for $n \geq 2$.

Therefore, the second part of (5) is equivalent to

$$q_n^{\frac{\alpha_1-1}{2\alpha_1}\alpha(n)+2-d} > 2Hd^2.$$

Now, let n_1 be the smallest integer ≥ 2 which verifies

$$q_{n_1-1}^{\frac{\alpha_1-1}{2\alpha_1}\alpha(n_1-1)+2-d} \leq 2Hd^2 < q_{n_1}^{\frac{\alpha_1-1}{2\alpha_1}\alpha(n)+2-d}. \quad (6)$$

The integer n_1 exists because we have assumed that

$$H \geq q_2^{\alpha(2)} > q_2^{\frac{\alpha_1-1}{2\alpha_1}\alpha(2)+2-d}$$

and that the sequence

$$\left(q_n^{\frac{\alpha_1-1}{2\alpha_1}\alpha(n)+2-d} \right) \text{ is increasing and extending to } +\infty.$$

Using again ii) of Lemma 3 and the left part of (6), we obtain

$$\frac{1}{q_{n_1}^d} > \frac{1}{2^d q_{n_1}^{\frac{kd\alpha_1}{\alpha_1-1}\alpha(n_1-1)}} \quad \text{and} \quad \frac{1}{q_{n_1-1}} > \frac{1}{(2Hd^2)^{\frac{\alpha_1-1}{2\alpha_1}\alpha(n_1-1)+2-d}}.$$

Therefore, we shall have

$$\begin{aligned} |A - \xi| &> \frac{1}{2Hd^2 q_{n_1}^d} > \frac{1}{2^d} \frac{1}{(2Hd^2)^{1 + \frac{\frac{kd\alpha_1}{\alpha_1-1}\alpha(n_1-1)}{\frac{\alpha_1-1}{2\alpha_1}\alpha(n_1-1)+2-d}}} \\ &> 2^{-d} (2Hd^2)^{-\left[1+2kd\left(\frac{\alpha_1}{\alpha_1-1}\right)^2 \left(1 + \frac{d-2}{\frac{\alpha_1-1}{\alpha_1}\alpha(n_1)+2-d}\right)\right]}. \end{aligned}$$

Then, we have

$$\frac{1}{\frac{\alpha_1-1}{2\alpha_1}\alpha(n_1-1)+2-d} < \frac{\log(2Hd^2)}{\log 2}. \quad (7)$$

Introducing the inequality (7) in the above relationship, we found

$$|A - \xi| > \frac{1}{2^d} \frac{1}{(2Hd^2 \text{ II})^{1+2kd\left(\frac{\alpha_1}{\alpha_1-1}\right)^2 \left(1 + \frac{(d-2)\log 2}{\log(2Hd^2)}\right)}}.$$

This yields that

$$|A - \xi| > \frac{1}{4} \frac{1}{(2Hd^2)^{1+2kd\left(\frac{\alpha_1}{\alpha_1-1}\right)^2}}.$$

Which proves the result.

Example. Let us consider the following example

$$\begin{cases} a_n = 2^{\frac{(2n)!}{2^n}}, n \geq 2 \\ \alpha_n = (2n+1)(n+1), n \geq 2. \end{cases}$$

We can see that

$$a_n = \left(2^{\frac{(2n-2)!}{2^{n-1}}}\right)^{\frac{(2n)(2n-1)}{2}} = a_{n-1}^{n(2n-1)}.$$

So, for all $n \geq 2$, we have

$$a_{n-1}^{\alpha_{n-1}} < a_n < a_{n-1}^{2\alpha_{n-1}}.$$

Then, there exists a positive real constant C such that for every $(p, q) \in \mathbb{N} \times \mathbb{N}^*$ with $q \geq (q_2)^{12}$, we have

$$\left|A - \sqrt{2} \frac{p}{q}\right| > Cq^{-14,22}.$$

4 Open Problem

1. In 2 of Theorem 1, in order to have the transcendence of the six continued fractions A , B , $A + B$, $A - B$, AB and A/B , the real constant $\alpha > 3$ that intervenes in the hypothesis $a_n > b_n > a_{n-1}^\alpha$ is not optimal.

Can we improve this exponent α (to lower the exponent) by taking for example $\alpha > 1$? Knowing that, according to the result 1 of Theorem 1, we note that we must have $\alpha > 1$.

2. In order to prove the algebraic independence of the continued fractions A and B , can we find a positive real constant α such that $a_n > b_n > a_{n-1}^\alpha$ for all $n \geq n_0$?

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