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Two higher-dimensional Lie algebras and their applications

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Abstract

Two higher-dimensional Lie algebras H ***and*** E ***are introduced firstly. Then the corresponding loop algebras*** $\widetilde{H}$ ***and*** $\widetilde{E}$ ***are introduced, whose commutation operations defined as simple as that in the loop algebra*** $\widetilde{A}_1$***. By taking advantage of*** $\widetilde{H}$ ***and*** $\widetilde{E}$***, two types of coupling integrable couplings of a hierarchy obtained. The coupling integrable couplings of the new hierarchy got in the paper can reduce to two types of the coupling integrable couplings of an interesting equation, which is different from GBK equation.***

Keywords: *loop algebra, integrable couplings, GBK equation.*

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1 Introduction

Search for new integrable (Lax integrable and Liouville integrable) systems has been of quite interest and importance in soliton theory[1-3]. The notion of integrable couplings was first introduced when study of Virasoro symmetric algebras[4,5]. Integrable couplings are coupled systems of integrable equations which contain given integrable equations as their sub-systems[6]. And

integrable couplings have much richer mathematical structures than scalar integrable equations[7-14]. So it is important to study integrable couplings in solition theory.
Let us consider an integrable evolution equation

$$u_t = K(u) = K(x, t, u, u_x, u_{xx}, \cdots), \tag{1}$$

where u is a column vector of dependent variables. Suppose that it has a zero curvature representation[15,16]

$$U_t - V_x + [U, V] = 0, \tag{2}$$

where the Lax pair, U and V, belongs to a matrix loop algebra.
Given two integrable couplings of the integrable equation (1)[4,6] as follows

$$\overline{u}_{1,t} = \overline{K}_1(\overline{u}_1) = \begin{pmatrix} K(u) \\ S(u,v) \end{pmatrix}, \overline{u}_1 = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{3}$$

and

$$\overline{u}_{2,t} = \overline{K}_2(\overline{u}_2) = \begin{pmatrix} K(u) \\ T(u,w) \end{pmatrix}, \overline{u}_2 = \begin{pmatrix} u \\ w \end{pmatrix}, \tag{4}$$

we can form a new bigger system

$$\widehat{u}_t = \widehat{K}_2(\widehat{u}_2) = \begin{pmatrix} K(u) \\ S(u,w) \\ T(u,w) \end{pmatrix}, \widehat{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \tag{5}$$

(5) is called coupling integrable couplings of (3) and (4).
In [17] 4 higher-dimensional Lie algebras were constructed. And Zhang gave two types of coupling integrable couplings of the AKNS hierarchy and the KN hierarchy with the help of the corresponding loop algebras. In this paper, we introduce 2 higher-dimensional Lie algebras and their corresponding loop algebras. With the help of the loop algebras, we obtain two types of a hierarchy, which can reduce to two types of the coupling integrable couplings of the interesting equations.

2 higher-dimensional Lie algebras

In [1] Tu introduced Lie algebra A_1, whose basis was

$$\overline{h} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

And their commutation operation are defined as follows

$$[\overline{h}, e] = e, [\overline{h}, f] = -f, [e, f] = 2\overline{h}.$$

Let$x(n) = x\otimes\lambda^n$ for $x = \overline{h}, e, f$. The basis for $\widetilde{A}_1$ is taken to be $\{\overline{h}(n), e(n), f(n)\}$, for which it holds that

$$[\overline{h}(m), e(n)] = e(m+n), [\overline{h}(m), f(n)] = -f(m+n), [e(m), f(n)] = 2\overline{h}(m+n).$$

With the help of $\widetilde{A}_1$ and its corresponding loop algebras, many integrable hierarchies an be obtained[18-21]. Zhang constructed 4 higher-dimension Lie algebras in [17]. In this section we introduce 2 of them, whose commutation operation defined as simple as that in $\widetilde{A}_1$.
We set $H = span\{h_1, h_2, h_3, h_4, h_5, h_6, h_7\}$, where

$$h_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, h_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, h_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$
$$h_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, h_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, h_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$
$$h_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The computation in H as $[a, b] = ab - ba, \forall a, b \in H$. Then we have
$[h_1, h_2] = 2h_2, [h_1, h_3] = -2h_3, [h_1, h_4] = h_4, [h_1, h_5] = -h_5, [h_1, h_6] = h_6, [h_1, h_7] = -h_7,$
$[h_2, h_3] = h_1, [h_2, h_4] = 0, [h_2, h_5] = h_4, [h_2, h_6] = 0, [h_2, h_7] = h_6, [h_3, h_4] = h_5,$
$[h_3, h_5] = 0, [h_3, h_6] = h_7, [h_3, h_7] = [h_4, h_5] = [h_4, h_6] = [h_4, h_7] = [h_5, h_6] = [h_5, h_7] = [h_6, h_7] = 0.$
H is a Lie algebra which is a subalgebra of the Lie algebra $sl(5, R)$.
Set $E = span\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$ in which

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$e_4=\begin{pmatrix}0&0&0&0&1&0\\0&0&0&0&0&-1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},e_5=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&-1\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},e_6=\begin{pmatrix}0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},$$

$$e_7=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},e_8=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},e_9=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}.$$

The commutation read that
$[e_1,e_2]=2e_2,[e_1,e_3]=-2e_3,[e_1,e_4]=[e_1,e_5]=0,[e_1,e_6]=2e_6,[e_1,e_7]=-2e_7,$
$[e_1,e_8]=2e_8,[e_1,e_9]=-2e_9,[e_2,e_3]=e_1,[e_2,e_4]=-2e_6,[e_2,e_5]=-2e_8,[e_2,e_6]=0,$
$[e_2,e_7]=e_4,[e_2,e_8]=0,[e_2,e_9]=e_5,[e_3,e_4]=2e_7,[e_3,e_5]=2e_9,[e_3,e_6]=-e_4,$
$[e_3,e_7]=0,[e_3,e_8]=-e_5,[e_3,e_9]=[e_4,e_5]=[e_4,e_6]=\cdots=[e_8,e_9]=0.$
The Lie algebra E is a subalgebra of the Lie algebra $sl(6,R)$.
Let $H_1=span\{h_1,h_2,h_3\},H_2=span\{h_4,h_5,h_6,h_7\}$, then we have that $H=H_1\uplus H_2,[H_1,H_2]\subset H_2$, where H_1,H_2 are two subalgebras of H and the symbol $\uplus$ stands for semi-direct sum [5]. Similarly, $E=E_1\uplus E_2,[E_1,E_2]\subset E_2$, where $E_1=span\{e_1,e_2,e_3\},E_2=span\{e_4,e_5,e_6,e_7,e_8,e_9\}$.
With the help of Lie algebra H,E, we can get integrable couplings.

3 Two types of coupling integrable couplings of the new hierarchy

In this section, we can obtain two types of coupling integrable couplings of the new hierarchy respectively with the help of the loop algebras $\widetilde{H}$ and $\widetilde{E}$ by using Tu scheme, which can reduce to two types of the coupling integrable couplings of the interesting equation.

(I)The first type of coupling integrable couplings of the new hierarchy

We introduce one loop algebra of Lie algebra H:

$$\widetilde{H}=span\{h_i(n)=h_i\lambda^n,i=1,2,3,4,5,6,7\},$$

the resulting commutators are defined as :$[h_i(m),h_j(n)]=[h_i,h_j]\lambda^{m+n},m,n\in Z,i,j=1,2,3,4,5,6,7$. With the help of the loop algebra $\widetilde{H}$, we introduce a Lax pair for zero curvature equation as follows

$$U = h_1(1) + qh_1(0) + h_2(0) + rh_3(0) + u_1h_4(0) + u_2h_5(0) + s_1h_6(0) + s_2h_7(0), \quad (6)$$

$$V = \sum_{m\geq 0}(V_{1m}h_1(2-m) + qV_{1m}h_1(1-m) + V_{2mx}h_1(1-m) + V_{1m}h_2(1-m) + V_{3mx}h_3(-m)$$
$$+rV_{1m}h_3(1-m) + V_{4m}h_4(-m) + V_{5m}h_5(-m) + V_{6m}h_6(-m) + V_{7m}h_7(-m)). \quad (7)$$

A solution to the stationary zero curvature equation

$$V_x = [U, V]$$

for V presents

$$V_{1,m+2x} + (qV_{1,m+1})_x + V_{2,m+1xx} = V_{3mx}, \quad (8)$$
$$V_{1,m+1x} = -2V_{2,m+1x},$$
$$V_{3,mxx} + (rV_{1,m+1})_x = -2V_{3,m+1x} - 2qV_{3mx} + 2rV_{2m+1x}, \quad (9)$$
$$V_{4,mx} = V_{4,m+1} + qV_{4m} + V_{5m} - u_1V_{1,m+2} - u_1qV_{1,m+1} - u_1V_{2,m+1x} - u_2V_{1,m+1}, (10)$$
$$V_{5,mx} = -V_{5,m+1} - qV_{5m} + rV_{4m} - u_1V_{3mx} - u_1rV_{1,m+1} + u_2V_{1,m+2}$$
$$+u_2qV_{1,m+1} + u_2V_{2,m+1x}, (11)$$
$$V_{6,mx} = V_{6,m+1} + qV_{6m} + V_{7m} - s_1V_{1,m+2} - s_1qV_{1,m+1} - s_1V_{2,m+1x} - s_2V_{1,m+1}, (12)$$
$$V_{7,mx} = -V_{7,m+1} - qV_{7m} + rV_{6m} - s_1V_{3mx} - s_1rV_{1,m+1} + s_2V_{1,m+2}$$
$$+s_2qV_{1,m+1} + s_2V_{2,m+1x}. (13)$$

Denote $V_+^{(n)} = \sum_{m=0}^{n}(V_{1m}h_1(2+n-m) + qV_{1m}h_1(1+n-m) + V_{2mx}h_1(1+n-m)$
$+ V_{1m}h_2(1+n-m) + V_{3mx}h_3(n-m) + rV_{1m}h_3(1+n-m) + V_{4m}h_4(n-m) +$
$V_{5m}h_5(n-m)$
$+ V_{6m}h_6(n-m) + V_{7m}h_7(n-m))$,
the stationary zero curvature equation

$$V_x = [U, V]$$

can be decomposed into

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = V_{-x}^{(n)} - [U, V_-^{(n)}].$$

It is easy to compute that
$-V_{+x}^{(n)} + [U, V_+^{(n)}] = [V_{1,n+2x} + (qV_{1,n+1})_x + V_{2,n+1xx}]h_1(0) + [2V_{3,n+1x} + (rV_{1,n+1})_x -$
$2rV_{2,n+1x}]h_3(0)$
$+ (-V_{4,n+1} + u_1V_{1,n+2} + u_1qV_{1,n+1} + u_1V_{2,n+1x} + u_2V_{1,n+1})h_4(0)$
$+ (V_{5,n+1} - u_2V_{1,n+2} + u_1rV_{1,n+1} - u_2qV_{1,n+1} - u_2V_{2,n+1x})h_5(0)$
$+ (-V_{6,n+1} + s_1V_{1,n+2} + s_1qV_{1,n+1} + s_1V_{2,n+1x} + s_2V_{1,n+1})h_6(0)$
$+ (V_{7,n+1} - s_2V_{1,n+2} + s_1rV_{1,n+1} - s_2qV_{1,n+1} - s_2V_{2,n+1x})h_7(0).$
Taking $V^{(n)} = V_+^{(n)}$, we have

$-V_x^{(n)}+[U,V^{(n)}]=[V_{1,n+2x}+(qV_{1,n+1})_x+V_{2,n+1xx}]h_1(0)+[2V_{3,n+1x}+(rV_{1,n+1})_x-2rV_{2,n+1x}]h_3(0)$
$+(-V_{4,n+1}+u_1V_{1,n+2}+u_1qV_{1,n+1}+u_1V_{2,n+1x}+u_2V_{1,n+1})h_4(0)$
$+(V_{5,n+1}-u_2V_{1,n+2}+u_1rV_{1,n+1}-u_2qV_{1,n+1}-u_2V_{2,n+1x})h_5(0)$
$+(-V_{6,n+1}+s_1V_{1,n+2}+s_1qV_{1,n+1}+s_1V_{2,n+1x}+s_2V_{1,n+1})h_6(0)$
$+(V_{7,n+1}-s_2V_{1,n+2}+s_1rV_{1,n+1}-s_2qV_{1,n+1}-s_2V_{2,n+1x})h_7(0).$
Hence the zero curvature equation

$$U_t-V_x^{(n)}+[U,V^{(n)}]=0 \tag{14}$$

gives rise to the Lax integrable hierarchy

$$u_t=\begin{pmatrix} q\\ r\\ u_1\\ u_2\\ s_1\\ s_2\end{pmatrix}_t=\begin{pmatrix} -V_{1,n+2x}-(qV_{1,n+1})_x-V_{2,n+1xx}\\ -2V_{3,n+1x}-(rV_{1,n+1})_x+2rV_{2,n+1x}\\ V_{4,n+1}-u_1V_{1,n+2}-u_1qV_{1,n+1}-u_1V_{2,n+1x}-u_2V_{1,n+1}\\ -V_{5,n+1}+u_2V_{1,n+2}-u_1rV_{1,n+1}+u_2qV_{1,n+1}+u_2V_{2,n+1x}\\ V_{6,n+1}-s_1V_{1,n+2}-s_1qV_{1,n+1}-s_1V_{2,n+1x}-s_2V_{1,n+1}\\ -V_{7,n+1}+s_2V_{1,n+2}-s_1rV_{1,n+1}+s_2qV_{1,n+1}+s_2V_{2,n+1x}\end{pmatrix}$$

$$=\begin{pmatrix} -V_{3nx}\\ V_{3nxx}+2qV_{3nx}\\ V_{4nx}-qV_{4n}-V_{5n}\\ V_{5nx}+qV_{5n}-rV_{4n}+u_1V_{3nx}\\ V_{6nx}-qV_{6n}-V_{7n}\\ V_{7nx}+qV_{7n}-rV_{6n}+s_1V_{3nx}\end{pmatrix}. \tag{15}$$

When take $s_1=s_2=0$,(9) can reduces to an integrable coupling of the hierarchy

$$u_t=\begin{pmatrix} q\\ r\\ u_1\\ u_2\end{pmatrix}_t=\begin{pmatrix} -V_{3nx}\\ V_{3nxx}+2qV_{3nx}\\ V_{4nx}-qV_{4n}-V_{5n}\\ V_{5nx}+qV_{5n}-rV_{4n}+u_1V_{3nx}\end{pmatrix}. \tag{16}$$

And when take $u_1=u_2=0$,(9) can reduces to another integrable coupling of the hierarchy

$$u_t=\begin{pmatrix} q\\ r\\ s_1\\ s_2\end{pmatrix}_t=\begin{pmatrix} -V_{3nx}\\ V_{3nxx}+2qV_{3nx}\\ V_{6nx}-qV_{6n}-V_{7n}\\ V_{7nx}+qV_{7n}-rV_{6n}+s_1V_{3nx}\end{pmatrix}. \tag{17}$$

So we call (9) the first type of coupling integrable couplings of the hierarchy.
When set $V_{1,0}=V_{2,0}=V_{3,0}=V_{4,0}=V_{5,0}=V_{6,0}=V_{7,0}=0, V_{1,1}=\alpha$,
we can obtain
$V_{2,1}=-\frac{1}{2}\alpha, V_{3,1}=-\frac{1}{2}\alpha r, V_{4,1}=\alpha u_2, V_{5,1}=-\alpha u_1 r, V_{6,1}=\alpha s_2, V_{7,1}=-\alpha s_1 r,$

$V_{1,2}=-\alpha q, V_{2,2}=\frac{1}{2}\alpha q, V_{3,2}=\alpha qr+\frac{1}{4}\alpha r_x, V_{4,2}=\alpha u_{2x}-2\alpha u_2 q+\frac{1}{2}\alpha u_1 r,$
$V_{5,2}=\alpha u_{1x}r+\frac{3}{2}\alpha u_1 r_x+2\alpha u_1 qr+\frac{1}{2}\alpha u_2 r, V_{6,2}=\alpha s_{2x}-2\alpha s_2 q+\frac{1}{2}\alpha s_1 r,$
$V_{7,2}=\alpha s_{1x}r+\frac{3}{2}\alpha s_1 r_x+2\alpha s_1 qr+\frac{1}{2}\alpha s_2 r.$
When take $n=2$ in (9), we can get the first type of coupling of integrable coupling of the interesting equation

$$q_t=-\alpha q_x r-\alpha q r_x-\frac{1}{4}\alpha r_{xx}, \quad (18)$$

$$r_t=\alpha q_{xx}r+2\alpha q_x r_x+\frac{3}{2}\alpha q r_{xx}+\frac{1}{4}\alpha r_{xxx}+2\alpha q q_x r+2\alpha q^2 r_x, \quad (19)$$

$$u_{1t}=\alpha u_{2xx}-3\alpha u_{2x}q-2\alpha u_2 q_x+2\alpha u_2 q^2-\frac{5}{2}\alpha u_1 qr-\frac{1}{2}u_{1x}r-\alpha u_1 r_x-\frac{1}{2}\alpha u_2 r, \quad (20)$$

$$u_{2t}=\alpha u_{1xx}r+\frac{5}{2}\alpha u_{1x}r_x+\frac{7}{4}\alpha u_1 r_{xx}+3\alpha u_{1x}qr+3\alpha u_1 q_x r+\frac{9}{2}\alpha u_1 q r_x-\frac{1}{2}\alpha u_{2x}r$$
$$+\frac{5}{2}\alpha u_2 qr+\frac{1}{2}\alpha u_2 r_x+2\alpha u_1 q^2 r-\frac{1}{2}\alpha u_1 r^2, \quad (21)$$

$$s_{1t}=\alpha s_{2xx}-3\alpha s_{2x}q-2\alpha s_2 q_x+2\alpha s_2 q^2-\frac{5}{2}\alpha s_1 qr-\frac{1}{2}s_{1x}r-\alpha s_1 r_x-\frac{1}{2}\alpha s_2 r, \quad (22)$$

$$s_{2t}=\alpha s_{1xx}r+\frac{5}{2}\alpha s_{1x}r_x+\frac{7}{4}\alpha s_1 r_{xx}+3\alpha s_{1x}qr+3\alpha s_1 q_x r+\frac{9}{2}\alpha s_1 q r_x-\frac{1}{2}\alpha s_{2x}r \quad (23)$$
$$+\frac{5}{2}\alpha s_2 qr+\frac{1}{2}\alpha s_2 r_x+2\alpha s_1 q^2 r-\frac{1}{2}\alpha s_1 r^2. \quad (24)$$

(12) is an interesting equation

$$q_t=q_x r+q r_x+\frac{1}{4}r_{xx}, \quad (25)$$

$$r_t=-q_{xx}r-2q_x r_x-\frac{3}{2}q r_{xx}-\frac{1}{4}r_{xxx}-2q q_x r-2q^2 r_x, \quad (26)$$

in the case when $u_1=u_2=s_1=s_2=0$ and $\alpha=-1$,
(13) can be written as

$$q_t=(qr+\frac{1}{4}r_x)_x, \quad (27)$$

$$r_t=-(qr+\frac{1}{4}r_x)_{xx}-2q(qr+\frac{1}{4}r_x)_x, \quad (28)$$

which is different from the famous GBK equation

$$q_t=\frac{1}{2}(q^2+2r-q_x)_x, \quad (29)$$

$$r_t=(qr+\frac{1}{2}r_x)_x. \quad (30)$$

(II)The second type of coupling integrable couplings of the new hierarchy

We have one loop algebra of Lie algebra E:

$$\widetilde{E}=span\{e_i(n)=e_i\lambda^n, i=1,2,3,4,5,6,7,8,9\},$$

the resulting commutators are defined as :$[e_i(m), e_j(n)] = [e_i, e_j]\lambda^{m+n}, m, n \in Z, i, j = 1, 2, 3, 4, 5, 6, 7, 8, 9$.
In terms of the loop algebra $\widetilde{E}$, we introduce a Lax pair for zero curvature equation as follows

$$U = e_1(1) + qe_1(0) + e_2(0) + re_3(0) + u_1e_6(0) + u_2e_7(0) + s_1e_8(0) + s_2e_9(0), \quad (31)$$

$$V = \sum_{m\geq 0}(V_{1m}e_1(2-m) + qV_{1m}e_1(1-m) + V_{2mx}e_1(1-m) + V_{1m}e_2(1-m)$$
$$+V_{3mx}e_3(-m) + rV_{1m}e_3(1-m) + V_{4m}e_4(-m) + V_{5m}e_5(-m) \quad (32)$$
$$+V_{6m}e_6(-m) + V_{7m}e_7(-m) + V_{8m}e_8(-m) + V_{9m}e_9(-m)). \quad (33)$$

The stationary zero curvature equation

$$V_x = [U, V]$$

leads to the following recursive relation

$$V_{1,m+2x} + (qV_{1,m+1})_x + V_{2,m+1xx} = V_{3mx}, \quad (34)$$
$$V_{1,m+1x} = -2V_{2,m+1x},$$
$$V_{3,mxx} + (rV_{1,m+1})_x = -2V_{3,m+1x} - 2qV_{3mx} + 2rV_{2m+1x}, \quad (35)$$
$$V_{4,mx} = V_{7m} - rV_{6m} + u_1V_{3mx} + u_1rV_{1,m+1} - u_2V_{1,m+1}, \quad (36)$$
$$V_{5,mx} = V_{9m} - rV_{8m} + s_1V_{3mx} + s_1rV_{1,m+1} - s_2V_{1,m+1}, \quad (37)$$
$$V_{6,mx} = 2V_{6,m+1} + 2qV_{6m} - 2V_{4m} - 2u_1V_{1,m+2} - 2u_1qV_{1,m+1} - 2u_1V_{2,m+1x}, \quad (38)$$
$$V_{7,mx} = -2V_{7,m+1} - 2qV_{7m} + 2rV_{4m} + 2u_2V_{1,m+2} + 2u_2qV_{1,m+1} + 2u_2V_{2,m+1x}, \quad (39)$$
$$V_{8,mx} = 2V_{8,m+1} + 2qV_{8m} - 2V_{5m} - 2s_1V_{1,m+2} - 2s_1qV_{1,m+1} - 2s_1V_{2,m+1x}, \quad (40)$$
$$V_{9,mx} = -2V_{9,m+1} - 2qV_{9m} + 2rV_{5m} + 2s_2V_{1,m+2} + 2s_2qV_{1,m+1} + 2s_2V_{2,m+1x}. \quad (41)$$

Note $V_+^{(n)} = \sum_{m=0}^{n}(V_{1m}e_1(2+n-m) + qV_{1m}e_1(1+n-m) + V_{2mx}e_1(1+n-m)$
$+ V_{1m}e_2(1+n-m) + V_{3mx}e_3(n-m) + rV_{1m}e_3(1+n-m) + V_{4m}e_4(n-m) +$
$V_{5m}e_5(n-m)$
$+ V_{6m}e_6(n-m) + V_{7m}e_7(n-m) + V_{8m}e_8(n-m) + V_{9m}e_9(n-m))$,
the stationary zero curvature equation

$$V_x = [U, V]$$

can be decomposed into

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = V_{-x}^{(n)} - [U, V_-^{(n)}].$$

It is easy to infer that
$-V_{+x}^{(n)} + [U, V_+^{(n)}] = [V_{1,n+2x} + V_{2,n+1xx} + (qV_{1,n+1x})]e_1(0) + [(rV_{1,n+1})_x + 2V_{3,n+1x} -$

$2rV_{2,n+1x}]e_3(0)$
$+(-u_1rV_{1,n+1}+u_2V_{1,n+1})e_4(0)+(-s_1rV_{1,n+1}+s_2V_{1,n+1})e_5(0)$
$+(-2V_{6,n+1}+2u_1V_{1,n+2}+2u_1qV_{1,n+1}+2u_1V_{2,n+1x})h_6(0)$
$+(2V_{7,n+1}-2u_2V_{1,n+2}-2u_2qV_{1,n+1}-2u_2V_{2,n+1x})e_7(0)$
$+(-2V_{8,n+1}+2s_1V_{1,n+2}+2s_1qV_{1,n+1}+2s_1V_{2,n+1x})e_8(0)$
$+(2V_{9,n+1}-2s_2V_{1,n+2}-2s_2qV_{1,n+1}-2s_2V_{2,n+1x})e_9(0))$.
Take a modified term $\Delta n=-u_1V_{1,n+1}e_6(0)-u_2V_{1,n+1}e_7(0)-s_1V_{1,n+1}e_8(0)-s_2V_{1,n+1}e_9(0)$ for $V_+^{(n)}$, namely take $V^{(n)}=V_+^{(n)}+\Delta n$,
after a calculation, we get
$-V_x^{(n)}+[U,V^{(n)}]=[V_{1,n+2x}+V_{2,n+1xx}+(qV_{1,n+1x})]e_1(0)+[(rV_{1,n+1})_x+2V_{3,n+1x}-2rV_{2,n+1x}]e_3(0)$
$+(-2V_{6,n+1}+2u_1V_{2,n+1x}+(u_1V_{1,n+1})_x)e_6(0)+(2V_{7,n+1}-2u_2V_{2,n+1x}+(u_2V_{1,n+1})_x)e_7(0)$
$+(-2V_{8,n+1}+2s_1V_{2,n+1x}+(s_1V_{1,n+1})_x)e_8(0)+(2V_{9,n+1}-2s_2V_{2,n+1x}+(s_2V_{1,n+1})_x)e_9(0)$.
The zero curvature equation

$$U_t-V_x^{(n)}+[U,V^{(n)}]=0 \tag{42}$$

admits that

$$u_t=\begin{pmatrix} q\\ r\\ u_1\\ u_2\\ s_1\\ s_2\end{pmatrix}_t=\begin{pmatrix} -V_{1,n+2x}-(qV_{1,n+1})_x-V_{2,n+1xx}\\ -2V_{3,n+1x}-(rV_{1,n+1})_x+2rV_{2,n+1x}\\ 2V_{6,n+1}-2u_1V_{2,n+1x}-(u_1V_{1,n+1})_x\\ -2V_{7,n+1}+2u_2V_{2,n+1x}-(u_2V_{1,n+1})_x\\ 2V_{8,n+1}-2s_1V_{2,n+1x}-(s_1V_{1,n+1})_x\\ -2V_{9,n+1}+2s_2V_{2,n+1x}-(s_2V_{1,n+1})_x\end{pmatrix}$$

$$=\begin{pmatrix} -V_{3nx}\\ V_{3nxx}+2qV_{3nx}\\ 2V_{6,n+1}-2u_1V_{2,n+1x}-(u_1V_{1,n+1})_x\\ -2V_{7,n+1}+2u_2V_{2,n+1x}-(u_2V_{1,n+1})_x\\ 2V_{8,n+1}-2s_1V_{2,n+1x}-(s_1V_{1,n+1})_x\\ -2V_{9,n+1}+2s_2V_{2,n+1x}-(s_2V_{1,n+1})_x\end{pmatrix}. \tag{43}$$

When take $s_1=s_2=0$,(17) can reduces to an integrable coupling of the new hierarchy

$$u_t=\begin{pmatrix} q\\ r\\ u_1\\ u_2\end{pmatrix}_t=\begin{pmatrix} -V_{3,nx}\\ V_{3,nxx}+2qV_{3,nx}\\ 2V_{6,n+1}-2u_1V_{2,n+1x}-(u_1V_{1,n+1})_x\\ -2V_{7,n+1}+2u_2V_{2,n+1x}-(u_2V_{1,n+1})_x\end{pmatrix}. \tag{44}$$

And when take $u_1=u_2=0$,(17) can reduces to another integrable coupling of the new hierarchy

$$u_t=\begin{pmatrix} q\\ r\\ s_1\\ s_2\end{pmatrix}_t=\begin{pmatrix} -V_{3,nx}\\ V_{3,nxx}+2qV_{3,nx}\\ 2V_{8,n+1}-2s_1V_{2,n+1x}-(s_1V_{1,n+1})_x\\ -2V_{9,n+1}+2s_2V_{2,n+1x}-(s_2V_{1,n+1})_x\end{pmatrix}. \tag{45}$$

So we call (19) the second type of coupling integrable couplings of the new hierarchy.
When set $V_{1,0} = \beta, V_{2,0} = V_{3,0} = V_{4,0} = V_{5,0} = V_{6,0} = V_{7,0} = 0$, we can obtain
$V_{1,1} = 0, V_{2,1} = 2\beta r, V_{3,1} = 2\beta, V_{4,1} = V_{5,1} = 0, V_{6,1} = -2\beta u_1, V_{7,1} = -2\beta u_2,$
$V_{8,1} = -2\beta s_1, V_{9,1} = -2\beta s_2, V_{1,2} = 2\beta r, V_{2,2} = 2\beta qr - 2\beta r_x, V_{3,2} = 2\beta q, V_{4,2} = 2\beta u_2 r - 2\beta u_1,$
$V_{5,2} = 2\beta s_2 r - 2\alpha s_1, V_{6,2} = 2\beta u_{1x} - 2\beta u_1 q, V_{7,2} = -2\beta u_{2x} - 2\beta u_2 q, V_{8,2} = 2\beta s_{1x} - 2\beta s_1 q,$
$V_{9,2} = -2\beta s_{2x} - 2\beta s_2 q, V_{1,3} = -2\beta r_x + 4\beta qr,$
$V_{2,3} = 2\beta r_{xx} - 2\beta q_x r - 4\beta q r_x + 2\beta q^2 r + 4\alpha r^2, V_{3,3} = 2\beta q_x + 4\beta r + 2\beta q^2,$
$V_{4,3} = 2\beta u_{1x} - 4\beta q u_1 + 4\beta q r u_2 + 2\beta u_{2x} r - 2\beta u_2 r_x,$
$V_{5,3} = 2\beta s_{1x} - 4\beta q s_1 + 4\beta q r s_2 + 2\beta s_{2x} r - 2\beta s_2 r_x,$
$V_{6,3} = -8\alpha u_{1xx} + 8\alpha u_{1x} q + 4\alpha u_1 q_x - 2\alpha u_1 q^2 + 8\alpha u_{2x} r - 4\alpha u_1 r + 4\alpha u_2 r_x,$
$V_{7,3} = -8\alpha u_{2xx} - 8\alpha u_{2x} q - 4\alpha u_2 q_x - 2\alpha u_2 q^2 - 8\alpha u_{1x} - 4\alpha u_2 r,$
$V_{8,3} = -8\alpha s_{1xx} + 8\alpha s_{1x} q + 4\alpha s_1 q_x - 2\alpha s_1 q^2 + 8\alpha s_{2x} r - 4\alpha s_1 r + 4\alpha s_2 r_x,$
$V_{9,3} = -8\alpha s_{2xx} - 8\alpha s_{2x} q - 4\alpha s_2 q_x - 2\alpha s_2 q^2 - 8\alpha s_{1x} - 4\alpha s_2 r.$
When take $n = 2$ in (17), we can get the first type of coupling of integrable coupling of the DLW equation

$$
\begin{aligned}
q_t &= -2\beta q_{xx} - 4\beta r_x - 4\beta q q_x, && (46)\\
r_t &= 2\beta r_{xx} - 4\beta q_x r - 4\beta q r_x, \\
u_{1t} &= 2\beta u_{1xx} - 4\beta u_{1x} q - 4\beta u_1 q_x + 4\beta u_1 r - 4\beta u_2 r^2, && (47)\\
u_{2t} &= -2\beta u_{2xx} - 4\beta u_{2x} q - 4\beta u_2 r + 4\beta u_1, && (48)\\
s_{1t} &= 2\beta s_{1xx} - 4\beta s_{1x} q - 4\beta s_1 q_x + 4\beta s_1 r - 4\beta s_2 r^2, && (49)\\
u_{2t} &= -4\beta s_{2xx} - 4\beta s_{2x} q - 4\beta s_2 r + 4\beta s_1. && (50)
\end{aligned}
$$

(20) is exactly the DLW equation

$$
q_t = -\alpha(qr + \frac{1}{4} r_x)_x, \tag{51}
$$

$$
r_t = \alpha(qr + \frac{1}{4} r_x)_{xx} + 2q(qr + \frac{1}{4} r_x)_x, \tag{52}
$$

in the case when $u_1 = u_2 = s_1 = s_2 = 0$ and $\beta = -\frac{1}{2}$.

Open problem. The loop algebras presented in this paper can be used to obtain coupling integrable couplings of other known integrable hierarchies. But how to deduce the Hamiltonian structures of the coupling integrable couplings of the DLW equation is one open problem which is worthwhile studying in the future.

4 Open Problem

Open problem. The loop algebras presented in this paper can be used to obtain coupling integrable couplings of other known integrable hierarchies. But how to deduce the Hamiltonian structures of the coupling integrable couplings of the DLW equation is one open problem which is worthwhile studying in the future.

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