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A note on bi-derivations of $\star$-prime rings

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Abstract

A classical result of Matej Brešar asserts that any bi-derivation on a noncommutative prime ring R is the form $B(x, y) = \lambda[x, y]$ where $\lambda \in C$ (the extended centroid of R). The main purpose of this work is to study this result in the case of $\star$- prime rings. Namely, we determine the bi-derivations in $\star$ prime rings. First, we present some results on the left Martindale quotiont ring and the extended centroid of a $\star$-prime ring with involution $\star$. Afterwards, we describe the bi-derivations in noncommutative $\star$-prime rings. This enable us to study the commuting maps in $\star$- prime rings with involution.

Keywords: *$\star$-prime ring, involution, bi-derivation, symmetric- Martindale ring, extended centroid.*

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1 Introduction

This research has been motivated by the work of Matej Brešar, V.S. Martindale and C. Robert Miers [3]. Throughout this article, R will represent an associative ring with center $Z(R)$. Recall that R is prime if $aRb = (0)$ implies that either $a = 0$ or $b = 0$. An additive mapping $x \mapsto x^\star$ of R into itself is called involution if $(x^\star)^\star = x$ and $(xy)^\star = y^\star x^\star$ holds for all $x, y \in R$. A ring equipped with an involution is known as ring with involution or $\star$-ring. A ring R with involution $\star$ is called a $\star$-prime ring if $aRb = (0)$ and $aRb^\star = (0)$ implies that either $a = 0$ or $b = 0$. It is clear that any prime ring with involution $\star$ is $\star$-prime but the converse is not true (see [1] for details). This is unifying notion in that it includes ordinary prime rings with involution and also prime rings R without involution. It suffices to map R into $R \times R^\circ$ via the embediding map $r \mapsto (r, r)$, where R° is the opposite ring. We note that if R is a prime ring then $R \times R^\circ$ is $\star$-prime ring where $\star$ is the exchange involution defined by $(x, y)^\star = (y, x)$. A ring R is called a semiprime ring if $aRa = (0)$ implies $a = 0$. We can see that every $\star$-prime ring is a semiprime ring. We denote by Q_r, Q_l, Q_s, and C, right, left, symmetric Martindale ring of quotients and extended centroid of $\star$-prime ring R , respectively (see [2]). We write $[x, y]$ for $xy - yx$. It follows that for all $x, y, z \in R$

$$[xy, z] = x[y, z] + [x, z]y,$$

and

$$[x, yz] = [x, y]z + y[x, z].$$

An additive mapping $D : R \mapsto R$ is called a derivation if

$$D(xy) = D(x)y + xD(y), \text{for all } x, y \in R.$$

An additive mapping $T : R \to R$ is called a left (right) centralizer in case $T(xy) = xT(y)$ ($T(xy) = xT(y)$) holds for all $x, y \in R$. An additive mapping $T : R \to R$ is called a two-sided centralizer if T is both a left and right centralizer. A bi-additive mapping $B : R \times R \to R$ is called a bi-derivation if for every $x \in R$ the map $y \to B(x, y)$ is a derivation of R and for every $y \in R$ the map $x \to B(x, y)$ is a derivation of R. For example, the mappings of the form $(x, y) \to \lambda[x, y]$ where λ is an element of the center of R are bi-derivations. In [3], it is shown that in the case of noncommutative prime ring R, the only bi-derivation in R are $B(x, y) = c[x, y]$ where $c \in C$ where C is the extended centroid of R . Brešar [4] has extended this result to 2- torsion free semiprime ring by proving that if R is a semiprime ring and $B : R \times R \to R$ a bi-derivation, then there exists an idempotent $\epsilon \in C$ and an element $\nu \in C$ such that the algebra $(1 - \epsilon)R$ is commutative and $\epsilon B(x, y) = \nu\epsilon[x, y]$, for all $x, y \in R$. Particularly, if R is prime, then the only bi-derivations of R are the forms $B(x, y) = c[x, y]$. Our goal is to generalize this result in the case of $\star$-prime rings. The notion of bi-derivation arise naturally in the study of additive commuting maps. Namely, if f is an additive commuting map, then the map $B : R \times R \to R$ given by $B(x, y) = [f(x), y]$ is a bi-derivation. It is shown in [3] that if f is an additive commuting map on a prime ring R, then $f(x) = \lambda x + \nu(x)$, where $\lambda \in C$ (the extended centroid of R) and $\nu : R \mapsto C$ an additive map. In this work, we describe the additive commuting maps on $\star$-prime rings.

2 The $\star$-extended centroid of a $\star$-prime ring

The Martindale ring of qotients of a prime ring was introduecd in [5] as a tool for studying rings satisfying a polynomial identity. The concept was extended to $\star$-prime rings in [2]. In this section we recall the construction of the Martindale ring of quotient and the $\star$-extended centroid of a $\star$-prime ring and we provide some essential results which will be nedded in the next.
Troughtout this section R will denote a $\star$-prime ring with involution $\star$. Let $\mathfrak{F}_*$ be the set of all nonzero $\star$-ideals of R. We consider the set

$$\mathfrak{L}_* = \{(I,f)\ /I \in \mathfrak{F}_*, f : I \to R \text{ a right centralizer}\}$$

Two elements (I,f) and (J,g) are equivalent if there exists $K \in \mathfrak{F}_*$ such that $K \subseteq I \cap J$ and $f = g$ on K. This is easily seen to yield an equivalence relation on $\mathfrak{L}_*$. Let $[I,f]$ denotes the equivalence class of (I,f) and $\mathfrak{Q}_{\mathfrak{r}}^\star$ the set of all equivalence classes. Let us define the addition and the multiplication on $\mathfrak{Q}_{\mathfrak{r}}^\star$ as follows

$$[I,f] + [J,g] = [I \cap J, f+g],$$

$$[I,f] \times [J,g] = [(I \cap J)^2, f \circ g].$$

Under this operations, $\mathfrak{Q}_{\mathfrak{r}}^\star$ is a ring with unit $[R, Id_R]$ wich contains R as a subring via the injection $r \mapsto [R, r_l]$ where $r_l(x) = rx$

Definition 2.1. The ring $\mathfrak{Q}_{\mathfrak{r}}^\star$ is called right Martindale ring of quotients of R and the centre C of $\mathfrak{Q}_{\mathfrak{r}}^\star$ is called the extended centroid of R.

In the following proposition, we provide some results which will be needed in the next.

Proposition 2.2 ([2], Lemma 3.1). *Let R be a $\star$-prime ring, Then*

1. *For $q \in \mathfrak{Q}_{\mathfrak{r}}^\star$, there exists $u \in \mathfrak{F}_\star$ such that $qu \in R$.*

2. *Let $f : I \to R$ be a right centralizer on $I \in \mathfrak{F}_\star$, then there exists $q \in \mathfrak{Q}_{\mathfrak{r}}^\star$ such that $f(x) = qx$.*

3. *An element $[I,f]$ of $\mathfrak{Q}_{\mathfrak{r}}^\star$ belongs to C if and only if f is a two sided centralizer.*

4. *Let $f : I \to R$ be a two sided centralizer on $I \in \mathfrak{F}_\star$, then there exists $q \in C$ such that $f(x) = qx$.*

For $\lambda = [I,f] \in C$ we put $\lambda^\star = [I,g]$ where $g(x) = (f(x^\star))^\star$ for all $x \in I$. It is clair that $\star$ is an involution on C.

Definition 2.3. The set of symmetric elements $C_\star$ of C under the above involution is called the $\star$-centroid of R.
The structure of $C_\star$ is given in [2] as follows:

Proposition 2.4. *Let R be a $\star$-prime ring. Then*

1. *$C_\star$ is the set of $[I, f] \in C$ such that f commutes with $\star$.*

2. *$C_\star$ is a field.*

Remark 2.5. Let $f : I \to R$ be a right and left centralizer on $I \in \mathfrak{F}_\star$ such that f commutes with $\star$. Then there exists $\lambda \in C_\star$ such that $f(x) = \lambda x$.

It is known that if R is a prime ring and $a, b \in R$ verify $axb = bxa$ for all $x \in R$, then either $a = 0$ or there exists $\lambda \in C$ such that $b = \lambda a$ (see [6], Theorem 1). Now we provide the corresponding result for $\star$-prime rings.

Theorem 2.6. *Let R be a $\star$-prime ring.*
Suppose that $axb = bxa$, $axb^\star = bxa^\star$, and $a^\star xb = b^\star xa$, for all $x \in R$, then either $a = 0$ or $b = \lambda a$ where $\lambda \in C_\star$

Proof. Suppose that $a \neq 0$. we define the map

$$\begin{array}{lcl} f: & RaR + Ra^\star R & \longrightarrow R \\ & \sum_i x_i a y_i + \sum_j x_j a^\star y_j & \longmapsto \displaystyle\sum_i x_i b y_i + \sum_j x_j b^\star y_j \end{array}$$

The mapping f is well defined. Indeed, suppose that

$$\sum_i x_i a y_i + \sum_j x_j a^\star y_j = 0.$$

Then, for all $r \in R$,

$$\begin{aligned} 0 &= br(\textstyle\sum_i x_i a y_i + \sum_j x_j a^\star y_j) \\ &= \textstyle\sum_i b r x_i a y_i + \sum_j b r x_j a^\star y_j \\ &= \textstyle\sum_i a r x_i b y_i + \sum_j a r x_j b^\star y_j \\ &= ar(\textstyle\sum_i x_i b y_i + \sum_j x_j b^\star y_j) \end{aligned}$$

It follows that

$$aR(\sum_i x_i b y_i + \sum_j x_j b^\star y_j) = (0) \tag{1}$$

Likewise, for all $r \in R$,

$$\begin{aligned} 0 &= b^\star r(\sum_i x_i a y_i + \sum_j x_j a^\star y_j) \\ &= \sum_i b^\star r x_i a y_i + \sum_j b^\star r x_j a^\star y_j \\ &= \sum_i a^\star r x_i b y_i + \sum_j a^\star r x_j b^\star y_j \\ &= a^\star r(\sum_i x_i b y_i + \sum_j x_j b^\star y_j) \end{aligned}$$

Hence

$$a^\star R(\sum_i x_i b y_i + \sum_j x_j b^\star y_j) = (0) \tag{2}$$

From (1) and (2) and the fact that R is $\star$-prime ring, we deduce that

$$\sum_i x_i b y_i + \sum_j x_j b^\star y_j = 0,$$

so that f is well defined.
In other hand , we can see that f commutes with $\star$. Indeed,

$$\begin{aligned} f((\sum_i x_i a y_i + \sum_j x_j a^\star y_j)^\star) &= f(\sum_i y_i^\star a^\star x_i^\star + \sum_j y_j^\star a x_j^\star) \\ &= \sum_i y_i^\star b^\star x_i^\star + \sum_j y_j^\star b x_j^\star \\ &= (\sum_i x_i b y_i + \sum_j x_j b^\star y_j)^\star \end{aligned}$$

Furthermore, f is a two-sided centralizer of R, then there exists $\lambda \in C_\star$ such that $f(x) = \lambda x$. As $f(a) = b$, we are done. $\square$

3 Bi-derivations in $\star$-prime rings

We start this section with the following result which provide the description of bi-derivations of $\star$-prime rings.

Theorem 3.1. *Let R be a $\star$-prime ring with involution $\star$ and $B : R\times R \longrightarrow R$ a bi-derivation such that $B^\star(x, y) = B(y^\star, x^\star)$ for all $x, y \in R$. Then there exists $\lambda \in C_\star$ such that $B(x, y) = \lambda[x, y]$, for all $x, y \in R$.*

For the proof of the theorem 3.1 we need the following lemmas.

Lemma 3.2. *Let R be a ring and $B : R \longrightarrow R$ a bi-derivation. Then, for all $x, y, z, u, v \in R$,*

$$B(x, y)z[u, v] = [x, y]zB(u, v)$$

Proof. As B is a bi-derivation then, for all x, y, u, v in R

$$B(xu, yv) = B(x, yv)u + xB(u, yv).$$

Then

$$B(xu, yv) = B(x, y)vu + yB(x, v)uxB(u, y)v + xyB(u, v). \tag{3}$$

As $B(xu, yv) = B(xu, y)v + yB(xu, v)$, we deduce that

$$B(xu, yv) = B(x, y)uv + xB(u, y)v + yB(x, v)u + yxB(u, v). \tag{4}$$

From (3) and (4), we obtain

$$B(x, y)[u, v] = [x, y]B(u, v), \forall\, x, y, u, v \in R. \tag{5}$$

Replacing u by zu in (5), we get

$$B(x, y)([z, v]u + z[u, v]) = [x, y](B(z, v)u + zB(u, v)), \tag{6}$$

so that

$$B(x, y)[z, v]u + B(x, y)z[u, v] = [x, y]B(z, v)u + [x, y]zB(u, v). \tag{7}$$

By virtue of relation (5) we have

$$B(x, y)[z, v]u = [x, y]B(z, v)u,\ \forall x, y, z, u, v \in R,$$

and we obtain the assertion of the lemma. □

Lemma 3.3. *Let S be any set and R be a $\star$-prime ring with involution $\star$. Suppose that functions $f; S \longrightarrow R$ and $g : S \longrightarrow R$ satisfy*

$$F(s)xG(t) = G(s)xF(t),\ \forall\, s, t \in S, \forall x \in R,$$

$$F^\star(s)xG(t) = G^\star(s)xF(t),\ \forall\, s, t \in S, \forall x \in R,$$

and

$$F(s)xG^\star(t) = G(s)xF^\star(t),\ \forall\, s, t \in S, \forall x \in R.$$

Then there exists $\lambda \in C_\star$, the $\star$-extended centroid of R such that

$$G(s) = \lambda F(s), \forall s \in S.$$

Proof. Let us consider $s \in S$. Then, for all $x \in R$,

$$F(s)xG(s) = G(s)xF(s),$$

$$F^\star(s)xG(s) = G^\star(s)xF(s),$$

and

$$F(s)xG^\star(s) = G(s)xF^\star(s)$$

If $F(s) \neq 0$, then from Theorem 2.6 there exists $\lambda(s) \in C_\star$ such that

$$G(s) = \lambda(s)F(s).$$

Let $t \in S$ such that $F(t) \neq 0$. For all $x \in R$,

$$\begin{array}{rcl} \lambda(t)F(s)xF(t) & = & F(s)x\lambda(t)F(t) \\ & = & F(s)xG(t) \\ & = & G(s)xG(t) \\ & = & \lambda(t)F(s)xF(t). \end{array}$$

Hence

$$(\lambda(t) - \lambda(s))F(s)xF(t) = F(s)xF(t), \ \forall x \in R. \tag{8}$$

In the other hand , the relation $F(s)xG^\star(t) = G(s)xF^\star(t)$ implies

$$\begin{array}{rcl} \lambda(t)F(s)xF^\star(t) & = & F(s)x\lambda(t)F^\star(t) \\ & = & F(s)(\lambda(t)F(t))^\star \\ & = & F(s)xG^\star(t) \\ & = & G(s)xF^\star(t) \\ & = & \lambda(s)F(s)xF^\star(t). \end{array}$$

It follows that

$$(\lambda(t) - \lambda(s))F(s)xF^\star(t) = 0, \forall x \in R. \tag{9}$$

From (8) and (9), the ⋆-primeness of R yields $\lambda(t) = \lambda(s)$, for all $t, s \in S$. Hence, there exists $\lambda \in C_\star$ such that $G(s) = \lambda F(s)$, for all $s \in S$ where $F(s) \neq 0$. However, if $F(s) = 0$, then $G(s)xF(t) = 0$ and $G(s)xF^\star(t) = 0$ so that $G(s) = 0$. Finally, there exists $\lambda \in C_\star$ such that $G(s) = \lambda F(s)$, for all $s \in S$.

□

Now, we are ready to prove the theorem 3.1.

Proof. Let $S = R \times R$, and define $F; S \longrightarrow R$ such that

$$F(x,y) = B(x,y),$$

and define $G; S \longrightarrow R$ such that

$$G(x,y) = [x,y].$$

From Lemma 3.2, for all $x, y, z, u, v \in R$

$$\begin{aligned} F(x,y)zG(u,v) &= B(x,y)z[u,v] \\ &= [x,y]zB(u,v) \\ &= G(x,y)zF(u,v), \end{aligned}$$

$$\begin{aligned} F(x,y)zG^\star(u,v) &= B(x,y)z[u,v]^\star \\ &= [x,y]z[v^\star,u^\star] \\ &= [x,y]zB(v^\star,u^\star) \\ &= [x,y]zB^\star(u,v) \\ &= G(x,y)zF^\star(u,v) \end{aligned}$$

and

$$\begin{aligned} F^\star(x,y)zG(u,v) &= B^\star(x,y)z[u,v] \\ &= B(y^\star,x^\star)z[u,v] \\ &= [y^\star,x^\star]zB(u,v) \\ &= G^\star(x,y)zF(u,v) \end{aligned}$$

By virtue of Lemma 3.3 there exists $\lambda \in C_\star$ such that

$$F(x,y) = \lambda G(x,y), \ \forall x, y \in R,$$

so that

$$B(x,y) = \lambda[x,y], \ \forall x, y \in R.$$

□

Corollary 3.4. *Let R a $\star$-prime ring with involution $\star$ and $B : R \times R \longrightarrow R$ a bi-derivation. Then there exists $\lambda \in C_\star$ such that*

$$B(x,y) + B^\star(y^\star,x^\star) = \lambda[x,y], \forall x, y \in R$$

Proof. Let B be a bi-derivation on R. We consider the map $B_1 : R \times R \longrightarrow R$ defined by

$$B_1(x,y) = B(x,y) + B^\star(y^\star,x^\star), \forall x, y \in R.$$

For all elements x, y, z in R

$$\begin{aligned} B_1(xy, z) &= B(xy, z) + B^{\star}(z^{\star}, y^{\star}x^{\star}) \\ &= B(x, z)y + xB(y, z) + [B(z^{\star}, y^{\star})x^{\star} + y^{\star}B(z^{\star}, x^{\star})]^{\star} \\ &= B(x, z)y + xB(y, z) + xB^{\star}(z^{\star}, y^{\star}) + B^{\star}(z^{\star}, x^{\star})y \\ &= B_1(x, z)y + xB_1(y, z). \end{aligned}$$

In the other hand

$$\begin{aligned} B_1(x, yz) &= B(x, yz) + B^{\star}(z^{\star}y^{\star}, x^{\star}) \\ &= B(x, y)z + yB(x, z) + [B(z^{\star}, x^{\star})y^{\star} + z^{\star}B(y^{\star}, x^{\star})]^{\star} \\ &= B_1(x, y)z + yB_1(x, z). \end{aligned}$$

It follows that the mapping B_1 is a bidrivation on R. Furthermore,

$$B_1^{*}(x, y) = B_1(y^{*}, x^{*}), \forall x, y \in R.$$

By virtue of Theorem 3.1, there exists $\lambda \in C_{*}$ such that $B_1(x, y) = \lambda[x, y]$ for all $x, y \in R$.

□

Corollary 3.5. *Let R a $\star$-prime ring with involution $\star$ and f be an additive commuting mapping on R which commutes with $\star$. Then there exists an element $\lambda \in C_{\star}$ and an additive mapping $\nu : R \longrightarrow C$ such $f(x) = \lambda x + \nu(x)$.*

Proof. Let f be an additive commuting mapping in R which commutes with $\star$. We define the mapping B on $R \times R$ as follows

$$B(x, y) = [f(x), y], \forall x, y \in R.$$

As f is commutting, then $[f(x) + f(y), x + y] = 0$ for all $x, y \in R$.
Hence

$$[f(x), y] = [x, f(y)], \forall x, y \in R.$$

We can easily verify that B is a bi-derivation and

$$\begin{aligned} B^{\star}(x, y) &= [f(x), y]^{\star} \\ &= [y^{\star}, f(x^{\star})] \\ &= -[f(x^{\star}), y^{\star}] \\ &= -[x^{\star}, f(y^{\star})] \\ &= [f(y^{\star}), x^{\star}] \\ &= B(y^{\star}, x^{\star}). \end{aligned}$$

From Theorem 3.1, there exists an element λ in $C_\star$ such that

$$[f(x), y] = \lambda[x, y] \text{ for all } x, y \in R.$$

Then

$$[f(x) - \lambda x, y] = 0 \text{ for all } x, y \in R.$$

It follows that for any $x \in R$ $\nu(x) = f(x) - \lambda x \in C$, and the proof is complete. □

Corollary 3.6. *Let R a $\star$-prime ring with involution $\star$ and f be an additive commuting mapping in R. Then there exists an element $\lambda \in C_\star$ and an additive mapping $\nu : R \longrightarrow C$ such $f(x) + f^\star(x^\star) = \lambda x + \nu(x)$*

Proof. It suffies to remark that the mapping $g(x) = f(x)+f^\star(x^\star)$ is an additive commuting map which commutes with $\star$. □

Corollary 3.7. *Let R be a noncommutative $\star$-prime ring with involution $\star$ such that char$(R) \neq 2$. Then R can not admits a nonzero symmetric bi-derivation .*

Proof. Suppose that R is a noncommutative $\star$-prime ring and B a nonzero symmetric bi-derivation on R. By virtue of theorem 3.4, there exists $\lambda \in C_\star$ such that $B(x, y) + B^\star(y^\star, x^\star) = \lambda[x, y]$ for all $x, y \in R$. As B is symmetric then $\lambda[y, x] = B(y, x) + B^\star(x^\star, y^\star) = \lambda[x, y]$ for all $x, y \in R$. It follows that $2\lambda[x, y] = 0$ so that $[x, y] = 0$ for all $x, y \in R$.This result is impossible because R is not commutative. □

4 Open Problem

In this work we have described the bi-derivations $B : R \times R \longrightarrow R$ where R is a $\star$-prime ring under the condition (c): $B^\star(x, y) = B(y^\star, x^\star),\ \forall x, y \in R$. This result enabled us to deduce the additive commuting maps on R which commute with $\star$.
The open problems here are:

P1: Does the theorem 3.1 remain true without the condition (c)?

P2: What is the forms of the additive commuting mapping on $\star$-prime rings?

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