Int. J. Open Problems Compt. Math., Vol. 12, No. 1, March 2019
ISSN 1998-6262; Copyright ©ICSRS Publication, 2019
www.i-csrs.org

Arithmetic Entire Functions

Chahrazed Harrat

University of Sciences and Technology "M B"of Oran,Department of Mathematics,
Faculty of Mathematics and computer sciences, Bp 1505 El M'naouar Oran 31000, Algeria.
E-mail: chahrazed.harrat@univ-usto.dz

Received 05 April 2018; Accepted 15 December 2018

(Communicated by Iqbal Jebril)

Abstract

The aim of this paper is solved the following problem: Let P_n be a sequence of distinct polynomials (of bounded degree) with some arithmetic properties, what can be said of an entire function f, the Lagrange interpolation polynomial P_n of f with respect to roots of P_n is in $\mathbb{Z}[X]$ for all n?

Keywords: *Entire function, Holomorphic function, Arithmetic function, Integral of Cauchy, Divided Differences, Lagrange interpolation polynomial.*
2010 Mathematics Subject Classification: 32A10 32A15.

1 Introduction

A well-known theorem of Hardy and Polya about the functions arithmetic: if f is arithmetic and of exponential type less than $\log 2$ ($f(z) = O(\exp cz)$ for some $c < \log 2$)then f is necessarily a polynomial. Futher the condition cannot be relaxed as seen by considering $f(z) = 2^z$. Later Guelfond studied a relative problem. Let β be a positive integer. F. Gramain [8] presente the situation in 1988 an attempt of the same kind: how to show by the method of transcendence, the result obtained in 1933 by A. O. Guelfond on entire functions taking integer values in all points of geometric progression, like the previous multiplicative additive problem (see [7] theorem VIII). On the other hand, the same type of results for entire functions of several variables were obtained by P. Bundschuh in 1980 and J. P. Bezivin in 1983 the first uses newton interpolation series in several variables and the second linear récurentes suites. P. Bundschuh and J. P. Bezivin have studied the case or first s derivatives of the function f are also integers. J. P. Bezivin [1] in 1984 studied multivariate generalization of result the Guelfond. Tanguy Rivoal and Michael Welter [9] have found the following result: let F be a holomorphic function on $\left\{z \in \mathbb{C}^d : R(z_j) > 0, j = 1, ..., d\right\}$, suppose that the following conditions are satisfied,

$(i) F(\mathbb{N}^d) \subset \mathbb{Z}$.

(ii)There are real $c > 0, \alpha \geq 0$ and β such that we have for all z satisfying $R(z_j) > 0$:

$$|F(z)| \leq c \prod_{j=1}^{d} \frac{e^{|z_j|\Psi(\theta_j)}}{R(z_j)^{\alpha}(1 + |z_j|)^{\beta}}, or\ \theta_j \in [\frac{-\pi}{2}, \frac{\pi}{2}]$$

$$\Psi(\theta_j) = \cos(\theta)\log(2\cos(\theta)) + \theta\sin(\theta)$$

So if $\frac{1}{2} - \beta < \alpha < \frac{1}{2}$, the function F is a polynomial with rational coefficients.

B. Djebbar [3] says that if $h(z) = 0$, for $z = 0, 1, 2....$(h the entire harmonic function on $\mathbb{C}$ of exponential type $\tau < \pi$),and $h(z) = h(\overline{z})$,then $h \equiv 0$.

What can be said of an entire function which takes on integral value at each point of the sequence $\beta^n, n \in \mathbb{N}$.

Again Guelfond showed that if the function doesn't grow too fast near infinity, see below for details, then it is necessarily a polynomial.

Let us look a little bit differently these results. We can, in an a priori complicated way, describe the conditions $f(n) \in \mathbb{Z}$, (resp. $f(\beta^n) \in \mathbb{Z}$) in the following manner:the Lagrange interpolation polynomial of degree 0 of f with respect to the point $n (resp. \beta^n)$ lies in $\mathbb{Z}[X]$. However this suggests to consider the following question. Let P_n be a sequence of distinct polynomials (of bounded degree) with some arithmetic properties, what can be said of an entire function f the Lagrange interpolation polynomial P_n of f with respect to roots of P_n is in $\mathbb{Z}[X]$ for all n?

It is expected that under restriction on the growth of f, f is a polynomial. We study this question in the very particular cases where $P_n(X) = P(X) - r_n$ or $P_n(X) = P(X) - \beta^n$. $P(X)$ being any non constant polynomial. We precise once and for all that the game played by $\mathbb{Z}$ is secondary,it can be replaced by any discrete sub-ring of $\mathbb{C}$.

2 Preliminaries

2.1 Order and type of an entire function

Let $\chi : [0\ ,\ +\infty[\ \longrightarrow [0\ ,\ +\infty[\cup\{+\infty\}$ be an increasing function. We define the growth order $\rho(\chi)$ of χ by the formula :

$$\varrho(\chi) = \limsup_{r\to\infty} \frac{\log \chi(r)}{\log r}$$

If χ is of finite order, $(0 < \varrho(\chi) < \infty)$ the growth type $\sigma(\chi)$ is defined by :

$$\sigma(\chi) = \limsup_{r\to\infty} \frac{\chi(r)}{r^{\varrho}}$$

For any entire function f on $\mathbb{C}^N$, the N-growth of f is defined by the growth of the function $\chi(r) = \log^+ M_N(f,r)$ where N is a norm on $\mathbb{C}^N$ and $M_N(f,r) = \sup_{N(z)\leq r} |f(z)|$. For any complete bounded domain Δ of center 0 in $\mathbb{C}^N$, the $\Delta-$growth of f is defined by the growth of the function $\chi(r) = \log^+ M_\Delta(f,r)$ where $M_\Delta(f,r) = \sup_{z\in r.\Delta} |f(z)|$.(see [15]. for more detail)

2.2 Holomorphic function

In complex analysis, a holomorphic function is a complex-valued function defined and differentiable at every point of an open subset of the complex plane $\mathbb{C}$.

2.3 Arithmetic function

Let f be an entire function on $\mathbb{C}$, f is said to be arithmetic if for each non negative integer n, $f(n)$ is an integre.

2.4 Residue

In complex analysis, the residue is a complex number which describes the behavior of the line integral of a holomorphic function in the neighborhood of a singularity. The residuals are calculated quite easily and, once known, can calculate more complicated curvilinear integrals with the residue theorem.

3 Holomorphic functions with complex variables

Theorem 3.1 *(H.D) [6] Let g an holomorphic function on* $\mathbb{C}^2$, *with* $g(\beta^n, \beta^m)$ *are integers,* $\forall n, m \in \mathbb{N}$ *(*β*, integer greater than unity) and* $g(z_1, z_2)$ *satisfies the inequality:*

$$\ln \mid g(z_1, z_2) \mid < \frac{\ln^2 r_1 r_2}{4 \ln \beta} - \frac{1}{2} \ln r_1 r_2 - \omega(r_1, r_2),$$

or

$$r_1 = |z_1|, r_2 = |z_2|, \quad \lim_{\min(r_1, r_2) \to +\infty} \omega(r_1, r_2) = \infty.$$

Then g is a polynomial.

Theorem 3.2 *(H.D) [6] Let g an holomorphic function on* $\mathbb{C}^d$,
with: $g(\beta^{n1}, ..., \beta^{nd})$ *are integers,*$\forall n_1, .., n_d \in \mathbb{N}^d$*(*$\beta$*, integer greater than unity), and* $g(z_1, z_2, ..., z_d)$ *satisfies the inequality:*

$$\ln \mid g(z_1, z_2, ..., z_d) \mid < \frac{\ln^2 r_1 r_2 ... r_d}{4 \ln \beta} - \frac{1}{2} \ln r_1 r_2 ... r_d - \omega(r_1, r_2, ..., r_d)$$

or

$$r_j = |z_j|, j = 1, ..., d, \quad \lim_{\min(r_1, r_2, ..., r_d) \to +\infty} \omega(r_1, r_2, ..., r_d) = \infty.$$

Then g is a polynomial.

Corollary 3.3 *(H.D) [6]Let F the holomorphic function on* $\mathbb{C}^d$ *, such that* $F(\beta^{n1}, ..., \beta^{nd})$ *are integers,*$\forall n_1, ..., n_d \in N^d$*(*$\beta$*, integer greater than unity),*
and $F(z_1, z_2, ..., z_d)$ *satisfies the inequality:*

$$\ln \mid F(z_1, z_2, ..., z_d) \mid < \frac{\ln^2 r_1 r_2 ... r_d}{4 \ln \beta} - \frac{1}{2} \ln r_1 r_2 ... r_d - \omega(r_1, r_2, ..., r_d)$$

or

$$r_j = |z_j|, j = 1, ..., d, \quad \lim_{\min(r_1, r_2, ..., r_d) \to +\infty} \omega(r_1, r_2, ..., r_d) = \infty.$$

If $F(\mathbb{N}^d) = 0$*, then* $F \equiv 0$ *on* $\mathbb{C}^d$.

4 Main results

4.1 GLOBAL DIVIDED DIFFERENCE

Let $P(X)$ is a non constant polynomial and (x_n) a sequence of pairwise distinct complex numbers.We set $P_n(X) = P(X) - x_n$.

For each n , the unique polynomial of degree not greater than $\sum_{i=1}^{n} \deg p_i - 1 = n \deg p - 1$ which interpolates(in the general sense) f on the set $\cup_{i=1}^{n} p_i^{-1}(0)$ is denoted by $H[f, p_1 ... p_n]$.

Here we only assume f to be conveniently defined on $\cup_{i=1}^{n} p_i^{-1}(0)$. Note that the polynomials P_n are relatively prime.

Definition 4.1 *We define (and call it a global divided defference) $f[p_1, ..., p_n]$ to by the quotient of $H[f, p_1 ... p_n]$ in the Euclidean division by $p_1 ... p_{n-1}$. Note that it is a polynomial of degree not greater than $\deg p - 1$. In case $n = 1$ it is only $H[f, p_1]$.*

When $P(X) = X$ then $f[p_1, ..., p_n]$ is a usual divided difference of f ,often(differently but similarly) denoted by $f[x_1, ..., x_n]$.the reader is invited to examine the very analogy between the lemmas below and the corresponding classical ones.

Lemma 4.2 *[5](Newton type formula)*

$$H[f, p_1 ... p_n] = \sum_{i=1}^{n} f[p_1, ..., p_i] p_1 ... p_{i-1} \tag{4.1}$$

Proof. Let : $X = \cup_{i=1}^{n-1} p_i^{-1}(0)$

$$T = H[f, p_1 ... p_n] - f[p_1, ..., p_n] p_1 ... p_{n-1}$$

$$T(x) = H[f, p_1 ... p_n](x) - f[p_1, ..., p_n] p_1(x) ... p_{n-1}(x); \forall x \in X$$

$H[f, p_1 ... p_n]$ is the unique polynomial of degree not greater than $\deg p - 1$, $when$:

$$H[f, p_1 ... p_n](x) = f(x); \forall x \in \cup_{i=1}^{n-1} p_i^{-1}(0)$$

x is a root of P_i :

$$f[p_1, ..., p_n] p_1(x) ... p_{n-1}(x) = 0$$

$$T(x) = H[f, p_1 ... p_n](x) = f(x); \forall x \in \cup_{i=1}^{n-1} p_i^{-1}(0)$$

and $\deg T \leq (n-1) \deg p - 1$

when $cardX = (n-1) \deg p$

Then

$$T = H[f, p_1 ... p_{n-1}]$$

$$H[f, p_1 ... p_n] = f[p_1, ..., p_n] p_1 ... p_{n-1} + H[f, p_1 ... p_{n-1}]$$

∎

Lemma 4.3 *(Neville -Aitken type formula)*

$$p_nH[f,p_1...p_{n-1}] - p_1H[f,p_2...p_n] = (x_1 - x_n)H[f,p_1...p_n] \tag{4.2}$$

Proof. *We remark that*

$$H[f,p_1...p_n]{\equiv}H[f,p_i] \quad [p_i];(i = 1,...,n)$$

Show that

$$H[f,p_1...p_n] - H[f,p_i] = \alpha p_i + R_i$$

when $\deg R_i \leq d-1$,*let* $x \in p_i^{-1}(0)$

$$H[f,p_1...p_n](x) = f(x); x \in p_i^{-1}(0) \subset \cup_{i=1}^{n} p_i^{-1}(0)$$

$$H[f,p_i](x) = f(x)$$

$$\alpha p_i(x) = 0$$

Then

$$R_i(x) = 0; \forall x \in p_i^{-1}(0)$$

but $\deg R_i \leq \deg P - 1$, *and* $R_i(x)$ *admits* d *rootswhere* :$R_i = 0$ *and after the remark*

$$H[f,p_1...p_n]{\equiv}H[f,p_i] \quad [p_i];(i = 1,...,n)$$

$$p_nH[f,p_1...p_{n-1}]{\equiv}p_nH[f,p_i] \quad [p_i];(i = 1,...,n) \Leftrightarrow p_nH[f,p_1...p_{n-1}]{-}p_nH[f,p_i] = \alpha p_n p_i$$

$$p_1H[f,p_2...p_n]{\equiv}p_1H[f,p_i] \quad [p_i];(i = 1,...,n) \Leftrightarrow p_1H[f,p_2...p_n]{-}p_1H[f,p_i] = \beta p_1 p_i$$

Then

$$p_nH[f,p_1...p_{n-1}] - p_1H[f,p_2...p_n] - (p_n - p_1)H[f,p_i] = (\alpha p_n - \beta p_1)p_i$$

$$p_nH[f,p_1...p_{n-1}] - p_1H[f,p_2...p_n]{\equiv}(x_1 - x_n)H[f,p_i] \quad [p_i]$$

but

$$\frac{p_nH[f,p_1...p_{n-1}] - p_1H[f,p_2...p_n]}{(x_1 - x_n)}{\equiv}H[f,p_i] \quad [p_i]$$

$\frac{p_nH[f,p_1...p_{n-1}] - p_1H[f,p_2...p_n]}{(x_1 - x_n)}$ *is a polynomial of degree not greater than* $\deg p - 1. = nd - 1$ *it is check the conditions for congruence of* $H[f,p_1...p_n]$.

therefore

$$\frac{p_nH[f,p_1...p_{n-1}] - p_1H[f,p_2...p_n]}{(x_1 - x_n)} = H[f,p_1...p_n]$$

■

Lemma 4.4

$$f[p_1,...,p_{n-1}] - f[p_2,...,p_n] = (x_1 - x_n)f[p_1,...,p_n] \tag{4.3}$$

Proof. *Bye lemma(Neville -Aitken type formula)*

$$p_n H[f, p_1...p_{n-1}] - p_1 H[f, p_2...p_n] = (x_1 - x_n)H[f, p_1...p_n]$$

$$p_n H[f, p_1...p_{n-1}] = p_n(f[p_1,...,p_{n-1}]p_1...p_{n-2} + R_1; \deg R_1 < (n-2)d - 1$$

$$p_1 H[f, p_2...p_n] = p_1(f[p_2,...,p_n]p_2...p_{n-1} + R_2; \deg R_2 < (n-2)d - 1$$

Then by writing

$$p_n = p_{n-1} + (x_{n-1} - x_n)$$

We obtain

$$p_n H[f, p_1...p_{n-1}] = f[p_1,...,p_{n-1}]p_1...p_{n-1} + (x_{n-1} - x_n)(f[p_1,...,p_{n-1}]p_1...p_{n-2} + R_1) + p_{n-1}R_1$$

$$p_1 H[f, p_2...p_n] = f[p_2,...,p_n]p_1...p_{n-1} + p_1 R_2$$

$$p_n H[f, p_1...p_{n-1}] - p_1 H[f, p_2...p_n] =$$

$$f[p_1,...,p_{n-1}]p_1...p_{n-1} - f[p_2,...,p_n]p_1...p_{n-1} + (x_{n-1} - x_n)(f[p_1,...,p_{n-1}]p_1...p_{n-2} + R_1) + p_{n-1}R_1 - p_1$$

$$\deg((x_{n-1} - x_n)(f[p_1,...,p_{n-1}]p_1...p_{n-2} + R_1) + p_{n-1}R_1 - p_1) < (n-1)d - 1$$

Then

$$f[p_1,...,p_{n-1}] - f[p_2,...,p_n] = (x_1 - x_n)f[p_1,...,p_n]$$

Lemma 4.5

$$f[p_1,...,p_n] = \sum_{i=1}^{n} \frac{H[f, p_i]}{\prod\limits_{i \neq j}(x_i - x_j)} \tag{4.4}$$

Proof. *by recursion, assume that*

$$f[p_1,...,p_{n-1}] = \sum_{i=1}^{n-1} \frac{H[f, p_i]}{\prod\limits_{i \neq j}(x_i - x_j)}$$

show that

$$f[p_1,...,p_n] = \sum_{i=1}^{n} \frac{H[f, p_i]}{\prod\limits_{i \neq j}(x_i - x_j)}$$

$$\frac{f[p_1,...,p_{n-1}] - f[p_2,...,p_n]}{(x_1 - x_n)} = f[p_1,...,p_n]$$

$$\frac{f[p_1,...,p_{n-1}] - f[p_2,...,p_n]}{(x_1 - x_n)} = \frac{1}{(x_1 - x_n)} \sum_{i=1}^{n-1} \frac{H[f, p_i]}{\prod\limits_{i \neq j}(x_i - x_j)} - \frac{1}{(x_1 - x_n)} \sum_{i=2}^{n} \frac{H[f, p_i]}{\prod\limits_{i \neq j}(x_i - x_j)}$$

$$= \frac{H[f,p_1]}{\prod_{j=1}^{n}(x_1-x_j)} + \frac{1}{(x_1-x_n)}\sum_{i=2}^{n-1}\frac{H[f,p_i]}{\prod_{i\neq j}^{n-1}(x_i-x_j)} + \frac{H[f,p_n]}{\prod_{j=1}^{n}(x_n-x_j)} - \frac{1}{(x_1-x_n)}\sum_{i=2}^{n-1}\frac{H[f,p_i]}{\prod_{j=2}^{n}(x_i-x_j)}$$

$$= \frac{H[f,p_1]}{\prod_{j=2}^{n}(x_1-x_j)} + \frac{1}{(x_1-x_n)}\sum_{i=2}^{n-1}\frac{H[f,p_i]}{(x_i-x_1)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_{n-1})}$$

$$+\frac{H[f,p_n]}{\prod_{j=1}^{n-1}(x_n-x_j)} - \frac{1}{(x_1-x_n)}\sum_{i=2}^{n-1}\frac{H[f,p_i]}{\prod_{j=2}^{n}(x_i-x_2)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_n)}$$

$$\frac{f[p_1,...,p_{n-1}]-f[p_2,...,p_n]}{(x_1-x_n)} = \frac{H[f,p_1]}{\prod_{j=1}^{n}(x_1-x_j)} + \frac{H[f,p_n]}{\prod_{j=1}^{n-1}(x_n-x_j)} +$$

$$\frac{1}{(x_1-x_n)}\sum_{i=2}^{n-1}\frac{(x_1-x_n)H[f,p_i]}{\prod_{j=2}^{n}(x_i-x_2)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_n)}$$

$$f[p_1,...,p_n] = \frac{f[p_1,...,p_{n-1}]-f[p_2,...,p_n]}{(x_1-x_n)} = \sum_{i=1}^{n}\frac{H[f,p_i]}{\prod_{i\neq j}(x_i-x_j)}$$

■

■

Lemma 4.6

$$p(\xi)-p(z) = (\xi-z)\sum_{p=1}^{d}\prod_{i>p}(\xi-x_n^i)\prod_{i<p}(z-x_n^i) \qquad (4.5)$$

Proof. *Definie the sequence* $N_j(z)$ *by* $N_1(z)=1$ *and* $N_{j+1}(z)=(z-x_n^j)N_j(z)$,*so that* $N_{d+1}(z)=p_n(z)$, *and :*

$$\prod_{i<p}(z-x_n^i) = N_p(z)$$

$$\prod_{i>p}(\xi-x_n^i) = \frac{\prod_{i>1}(\xi-x_n^i)}{\prod_{i<p+1}(\xi-x_n^i)}$$

$$(\xi - z)\sum_{p=1}^{d} N_p(z)\frac{N_{d+1}(\xi)}{N_{p+1}(z)} = \sum_{p=1}^{d}(\xi - x_n^p + x_n^p - z)\frac{N_p(z)}{N_{p+1}(\xi)}N_{d+1}(\xi)$$

$$(\xi - z)\sum_{p=1}^{d} N_p(z)\frac{N_{d+1}(\xi)}{N_{p+1}(\xi)} = \sum_{p=1}^{d}(-(z - x_n^p)\frac{N_p(z)}{N_{p+1}(\xi)} + \frac{\xi - x_n^p}{N_{p+1}(\xi)})$$

$$(\xi - z)\sum_{p=1}^{d} N_p(z)\frac{N_{d+1}(\xi)}{N_{p+1}(\xi)} = \sum_{p=1}^{d}(\frac{-N_{p+1}(z)}{N_{p+1}(\xi)} + \frac{N_p(z)}{N_p(\xi)})N_{d+1}(\xi)$$

$$when : \sum_{p=1}^{d} u_p - u_{p+1} = u_1 - u_{d+1}$$

$$then : (\xi - z)\sum_{p=1}^{d} N_p(z)\frac{N_{d+1}(\xi)}{N_{p+1}(\xi)} = (\frac{N_1(z)}{N_1(\xi)} - \frac{N_{d+1}(z)}{N_{d+1}(\xi)})N_{d+1}(\xi)$$

$$(\xi - z)\sum_{p=1}^{d} N_p(z)\frac{N_{d+1}(\xi)}{N_{p+1}(\xi)} = (1 - \frac{N_{d+1}(z)}{N_{d+1}(\xi)})N_{d+1}(\xi)$$

$$(\xi - z)\sum_{p=1}^{d} N_p(z)\frac{N_{d+1}(\xi)}{N_{p+1}(\xi)} = p_n(\xi) - p_n(z) = p(\xi) - p(z)$$

This result, in a little bit different form ,may be found in a paper L. Verde-Star [12](propositon 2.1,p,217).We include his proof for the convenience of the reader.We note that the definition1 and the lemme1 remain true when the p_n *are any(not even distinct) polynomials.*

We conclude this section in proving a convergence lemma extending a classical theorem of Guelfond.Its proof is a simple adaptation of the classical $(p(X) = X)$ *one.*

Lemma 4.7 *[7] Suppose that* P *is monic of degree* d *and that* (x_n) *is a sequence of complexe numbers satisfaying* $|x_{n-1}| \leq |x_n| \leq \lambda n^{\eta}$ *for* n *large enough.Set* $P_n(X) = P(X) - x_n$*.If further* f *is an entire function of ordre at most* $\frac{d}{\mu}$ *and type at most* σ *then the sequence of polynomials* $L_n(z) = H[f, p_1...p_n](z)$ *converges uniformly on every compact set to* f *provided that:*

$$\sigma(2\lambda)^{\frac{1}{\mu}} < 2\int_0^1 \frac{t^{\frac{1}{\mu}-1}}{2-t} \tag{4.6}$$

■

4.2 PSEUDO ARITHMETIC AND GEOMETRIC ENTIRE FUNCTIONS

$p(X)$ is always a non constant monic complex polynomial of degree $d \geq 1$, f an entire function on $\mathbb{C}$.

Definition 4.8 *f is said to be $p-$pseudo arithmetic with respect to (an arithmetic sequence)$(r_n)_{n\geq 0}$, if for each n, $H[f, p_n] \in \mathbb{Z}[X]$ where $p_n(X) = p(X) - r_n$.*

Similarly f is said to be $p-$pseudo geometric with respect to (a geometric sequence) g_n, if for each n, $H[f, p_n] \in \mathbb{Z}[X]$ where $p_n(X) = p(X) - g_n$.

Theorem 4.9 *If f is $p-$pseudo arithmetic with respect to $r_n = rn + r_0$ and if further f is of order at most d and type less $\log \frac{2}{r}$ then f is a polynomial.*

The condition on the growth cannot be relaxed.

Proof. We shall assume, without loss of generality that $r_0 = 0$ for otherwise we can change the constant term of p .

first by appliying lemme (4.7) (suppose that p is monic of degree d and that (x_n) is a sequence of complex numbers satisfaying $| x_{n-1} | \leq | x_n | \leq \lambda n^\eta$ for n large enough).

set $p_n(X) = p(X) - x_n$, if further f is an entire function of order at most $\frac{d}{\eta}$ and type at most σ then the sequence of polynomial $L_N(z) = H[f, p_0, ..., p_n](z)$ converges uniformly on every compact set to f provided that $\sigma(2\lambda)^{\frac{1}{\eta}} < 2\int_0^1 \frac{t^{\frac{1}{\eta}-1}}{2-t} dt$ with $\lambda = r$ and $\eta = 1$ we see that for any entire function f of order at most d and type less than $\log \frac{2}{r}$ we have

$$f(z) = \sum_{n=1}^{\infty} f[p_1, p_2, ..., p_n](z) p_1(z) p_2(z) ... p_{n-1}(z) \tag{4.7}$$

uniformly on every compact sets of $\mathbb{C}$.

To prove the theorem it is therefore enough to show that for n large the polynomials $f[p_1, p_2, ..., p_n](z)$ are the zero polynomial.

But by lemma (4.4)

$$f[p_1, p_2, ..., p_n] = \sum_{i=1}^{n} \frac{H[f, pi]}{\prod\limits_{I \neq J} (x_i - x_j)}$$

we get:

$$f[p_1, p_2, ..., p_n] = \frac{1}{r^n (n-1)!} \sum_{i=1}^{n} (-1)^{n-1} \begin{pmatrix} n-1 \\ i-1 \end{pmatrix} H[f, pi]$$

$$f\left[p_1,p_2,...,p_n\right] \;=\frac{1}{r^n(n-1)!}q_n \tag{4.8}$$

where q_n is a polynomial with integral coefficients.

Now we have

$$p(z)=(z-a_1)^{r_1}(z-a_2)^{r_2}....(z-a_s)^{r_s},\;\; with\; a_i\neq a_j\;,(i\neq j),\sum r_i=d \tag{4.9}$$

we put $z=a_i$ in (4.7) and find

$$f(a_i)=\frac{1}{r}\sum_{n=1}^{\infty}q_n(a_i)(-1)^{n+1} \tag{4.10}$$

$$\frac{1}{r}\sum_{n=1}^{\infty}q_n(a_i)(-1)^{n+1}\;\rightarrow\; f(a_i),\; if\;\; n\rightarrow\infty$$

from which we deduce that $\lim_{n\to\infty}q_n(a_i)=0$. Further we know that when a series of holomorphic functions converges uniformly to a to f then the differentiated series converges similarly to f'.

Here we thus have

$$f'(z)=\sum_{n=1}^{\infty}\frac{1}{r^n(n-1)!}\left[\frac{dq_n}{dz}(z)p_1(z)p_2(z).....p_{n-1}(z)+q_n(z)\frac{d}{dz}\left[p_1p_2.....p_n\right](z)\right] \tag{4.11}$$

and if we put again $z=a_i$, the second term in the sum vanishes for $p'(a_i)=0$. So that we obtain

$$f'(a_i)=\sum_{n=1}^{\infty}\frac{1}{r^n(n-1)!}q_n^{'}(a_i)p_1(a_i)p_2(a_i)....p_{n-1}(a_i) \tag{4.12}$$

$$f'(a_i)=\sum_{n=1}^{\infty}\frac{1}{r^n(n-1)!}q_n^{'}(a_i)(-1)^{n-1}r^{n-1}(n-1)!$$

wich gives $\lim_{n\to\infty}q_n'(a_i)=0$. We go on and finally get that

$$\lim_{n\to\infty}q_n^{(j)}(a_i)=0\;,for\;1\leq i\leq s,1\leq j\leq r_i-1 \tag{4.13}$$

writing

$$q_n(z)=\alpha_n^{d-1}z^{d-1}+\alpha_n^{d-2}z^{d-2}+....+\alpha_n^{0}$$

$$\alpha_n=(\alpha_n^{d-1},\alpha_n^{d-2},....,\alpha_n^{0})\in\mathbb{Z}^d\subset\mathbb{C}^d$$

and, with $g_j(z)=z^j$.

$$A = \begin{pmatrix} g_0(a_1) & . & . & . & g_d(a_1) \\ g\prime_0(a_1) & . & . & . & g'_d(a_1) \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ g_0^{r_1-1}(a_1) & . & . & . & g_d^{(r_1-1)} \\ . & . & . & . & . \\ . & . & . & . & . \\ g_0(a_s) & . & . & . & g_d(a_s) \\ g\prime_0(a_s) & . & . & . & g'_d(a_s) \\ . & . & . & . & . \\ . & . & . & . & . \\ g_0^{r_s-1}(a_s) & . & . & . & g_d^{r_s-1}(a_s) \end{pmatrix}$$

$$A \begin{pmatrix} \alpha_n^0 \\ . \\ . \\ . \\ . \\ . \\ \alpha_n^{d-2} \\ \alpha_n^{d-1} \end{pmatrix} = \begin{pmatrix} \alpha_n^0 + \alpha_n^1 a_1 + ... + \alpha_n^{d-1} a_1 \\ . \\ . \\ . \\ \alpha_n^0 + \alpha_n^1 a_s + ... + \alpha_n^{d-1} a_s. \\ . \\ . \\ . \end{pmatrix} = \begin{pmatrix} q_n(a_1) \\ q'_n(a_1) \\ . \\ . \\ q_n^{r_1-1}(a_1) \\ . \\ q_n^{r_s-1}(a_s) \end{pmatrix}$$

We see that the conditions (4.7) mean $\lim_{n\to\infty} A\alpha_n = 0$ but it is easly seen that the matrix A is inversible(it is the vandermondian matrix) we meet in Hermite interpolation theory.

Consequently we have

$$0 = A^{-1}(0) = A^{-1}(\lim_{n\to\infty} A(\alpha_n)) = \lim_{n\to\infty} \alpha_n. \tag{4.14}$$

Then

$$\lim_{n\to\infty} q_n = 0$$

Since the coordinates of α_n are intergral (remenber$q_n \in Z[X]$) (4.14) implies that for n large $\alpha_n = 0$ and therefore $q'_n = 0$ which give, as indicated above,that f is a polynomial.

Finally we see that the hypothesis cannot be relaxed by considering $f(z) = \exp(\frac{\log 2}{r}(p(z) - r_0))$ wihich is of ordre d and type $\frac{\log 2}{r}$ while for each n , $H[f, p_n] = 2^n \in Z[X]$ .The theorem is proved. ∎

Theorem 4.10 *Let $g_n = r\beta^n, \beta = \exp\lambda$, $\lambda > 0$. We set $p_n = p - g_n$, if f is an entire function such that for all $n > 0$*

$$H[f, p_n] \in \mathbb{Z}[X],$$

and if further f satisfies for R large.

$$\log M(R) \leq \frac{d^2}{4\log\beta}\log^2(R) - (\frac{d}{2} - \frac{d\log|r|}{2\log\beta})\log R - \omega(R) \tag{4.15}$$

where $\omega(R)$ is any function tending to ∞ as $R \to \infty$ then f is a polynomial.

Lemma 4.11 *Let $g_n = r\beta^n$, $\beta = \exp\lambda$, $\lambda > 0$. We set $p_n = p - g_n$. If f is an entire function which satisfies for R large.*

$$\log M(R) \leq \frac{d^2}{2\lambda}\log^2(R) + \alpha d \log R \quad \textit{where}: \quad \alpha < \frac{-\log|r|}{\lambda} - \frac{1}{2} \tag{4.16}$$

thene the sequence of polynomial $L_n(z) = H[p_1, ..., p_n](z)$ converges to f uniformly on every compact of $\mathbb{C}$.

Proof. We fix ϵ small enough to verify

$$\alpha < \frac{-\log|r|}{\lambda} - \frac{1}{2} + \frac{\log(1-\epsilon)}{\lambda}$$

We write $R_n(z) = f(z) - L_n(z)$, by the Harmite formula if $|\ z\ | \leq \rho$, $|\xi| = R$, $R > \rho$.

$$R_n(z) = \frac{1}{2\pi i}\int_{|\xi|=R} \frac{p_1(z)...p_n(z)}{p_1(\xi)...p_n(\xi)}\frac{f(\xi)}{\xi - z}d\xi$$

with $|\ x_k\ | = r\beta^k$ we get for $|\ z\ | \leq \rho < R$, R large enough,

$$\log(R_n(z)) \leq \log M(R) - \log(1 - \frac{\rho}{R}) + \sum_{j=1}^{n}\log(\frac{k + |r|\beta^j}{(1-\epsilon)R^d - r\beta^j}) \tag{4.17}$$

we take $R = R(n)$ defined by $(1-\epsilon)R^d = 2|r|\beta^k$ whence

$$\log(R_n(z)) \leq \log M(R(n)) - \log(1 - \frac{\rho}{R(n)}) + \sum_{j=1}^{n}\log(\frac{|r|\beta^j(1 + k|r|^{-1}\beta{-}^j)}{2|r|\beta^n(1 - \frac{|r|}{2|r|\beta^{j-n}})})$$

$$\log(R_n(z)) \leq \log M(R(n)) - \log(1 - \frac{\rho}{R(n)}) + \sum_{j=1}^{n}\log(\frac{|r|\beta^j(1 + k|r|^{-1}\beta{-}^j)}{2|r|\beta^n(1 - \frac{|r|}{2|r|\beta^{j-n}})})$$

$$\log(R_n(z)) \leq \log M(R(n)) - \log(1-\frac{\rho}{R(n)}) + \sum_{j=1}^{n}\log(\frac{|\, r\, |\, (\exp(\lambda))^k(1+k\,|r|^{-1}\,(\exp(\lambda))-^j)}{2\,|\, r\,|\,(\exp(\lambda))^n(1-\frac{|r|}{2|r|(\exp(\lambda))^{j-n}})})$$

$$\log(R_n(z)) \leq \log M(R(n)) - \log(1-\frac{\rho}{R(n)}) - \sum_{j=1}^{n}\lambda(n-j) - \sum_{j=1}^{n}\log 2 + \sum_{j=1}^{n}\log((1+k\,|r|^{-1}\,(\exp(\lambda))^{-j}) - \log(1-\frac{1}{2(\exp(\lambda))^{j-n}})$$

the series

$$\sum_{j=1}^{n}\log((1+k\,|r|^{-1}\,(\exp(\lambda))-^j) \sim \sum_{j=1}^{n}k\,|r|^{-1}\,(\exp(\lambda))^{-j},$$

and

$$\sum_{j=1}^{n}\log(1-\frac{1}{2(\exp(\lambda))^{j-n}}) \sim \sum_{j=1}^{n}(-\frac{1}{2(\exp(\lambda))^{j-n}})$$

with

$$\frac{1}{\beta} = \frac{1}{e^\lambda} < 1$$

then

$$\sum_{j=1}^{n}\log((1+k\,|r|^{-1}\,(\exp(\lambda))^{-j}) - \log(1-\frac{1}{2(\exp(\lambda))^{j-n}}) = O(1)$$

$$\log(R_n(z)) \leq \log M(R(n)) - \sum_{k=1}^{n}\lambda(n-k) - \sum_{k=1}^{n}\log 2 + O(1)$$

$$\log M(R_n(z)) \leq \log M(R(n)) - \sum_{k=1}^{n}\lambda(n-k) - \sum_{k=1}^{n}\log 2 + O(1)$$

$$\log(R_n(z)) \leq \log M(R(n)) - \frac{n(n-1)}{2}\lambda - n\log 2 + O(1). \tag{4.18}$$

Now by (4.16) the right hand member of (4.18) is not greater than

$$\log(R_n(z)) \leq \frac{1}{2\lambda}\log^2\frac{(2\,|\,r\,|\,\beta^n)}{(1-\epsilon)} + \alpha\log\frac{(2\,|\,r\,|\,\beta^n)}{(1-\epsilon)} - \frac{n(n-1)}{2}\lambda - n\log 2 + O(1). \tag{4.19}$$

A simple calculation shows that (4.18) is equal to

$$\log(R_n(z)) \le \frac{1}{2\lambda}[(\log(2|r|) - n\lambda - \log(1-\epsilon)]^2 + \alpha[\log(2|r|) - n\lambda - \log(1-\epsilon)] \\ -\frac{n(n-1)}{2} - n\log 2) + O(1)$$

$$\begin{aligned} \log(R_n(z)) &\le C_1 n^2 + C_2 n + C_3 \\ \textit{with} \quad C_1 &= 0, C_2 = \log(2|r| - \log(1-\epsilon) + \alpha\lambda + \frac{\lambda}{2} - \log 2 \\ \textit{and} \qquad \alpha &< \frac{-\log|r|}{\lambda} - \frac{1}{2} + \frac{\log(1-\epsilon)}{\lambda}. \end{aligned}$$

Finally $\log(R_n(z))$ *tends to* $-\infty$ *as* $n \to \infty$, $R_n(z) \to 0$ *as* $n \to \infty$.

$L_n(z)$ therefore converges to f on $|z| \le \rho$. Since ρ is a arbitary the lemme is proved. ■

Proof. (Of theorem)

it is easily seen that the condition (4.15) implies (4.16) hence $L_N(Z) := H[p_1, ..., p_n](Z)$ converge to f uniformly on every compact set. We are again going to prove that for n large we have:

$$A_n(z) := f[p_1, ..., p_n] = 0$$

Now one hand, by lemme (4.3) we get:

$$A_n(z) = \sum_{i=1}^{n} \frac{H[f, p_i]}{\prod_{i \ne j}(r\beta^i - r\beta^j)}$$

$$A_n(z) = \sum_{i=1}^{n} \frac{H[f, p_i]}{\prod_{i \ne j} r(\beta^i - 1 + 1 - \beta^j)}$$

$$A_n(z) = \sum_{i=1}^{n} H[f, p_i] \frac{1}{r^{n-1}\beta^{1+2+3+...+i-1}(\beta^{i-1}-1)(\beta^{i-2}-1)(\beta^{i-3}-1).....(\beta-1)}$$

$$A_n(z) = \sum_{i=1}^{n} H[f, p_i] r^{-(n-1)}\beta^{-(1+2+3+...+i-1)} \left[(\beta^{i-1}-1)(\beta^{i-2}-1)(\beta^{i-3}-1)....(\beta-1)\right]^{-1}$$

$$A_n(z) = \sum_{i=1}^{n} H[f, p_i] r^{-(n-1)}\beta^{-\frac{i(i-1)}{2}} \left[(\beta^{i-1}-1)(\beta^{i-2}-1)(\beta^{i-3}-1)....(\beta-1)\right]^{-1}, \tag{4.20}$$

then one can prove (see $[7, 179 - 180]$) that $T_n A_n(z) \in Z[X]$ where:

$$T_n := r^{(n-1)} \beta^{\frac{n(n-1)}{2}} (\beta^{n-1} - 1)(\beta^{n-2} - 1)(\beta^{n-3} - 1)....(\beta - 1) \qquad (4.21)$$

and since $T_n A_n$ is a polynomial of degree not greater than d with integral coefficients. The theorem is proved. ■

5 Open Problem

Let P_n be a sequence of distinct polynomials (of bounded degree) with some holomorphic properties, what can be said of an entire function f the Lagrange interpolation polynomial of with respect to roots of P_n is in $\mathbb{Z}[X]$?

References

[1] J.P.Bezivin. Une generalisation a plusieurs variables d'un resultat de Guelfond.Analysis. 4, 125-141, 1984.

[2] J.P.Bezivin. Sur les points ou une fonction analytique prend des valeurs entires.Ann.Inst.Fourier,Grenoble. 40, 785-809, 4(1990).

[3] B.Djebbar. Approximation et croissance des fonctions séparément harmoniques entiéres et ses fonctions pluriharmoniques entiéres.These doctorat d'Etat (2003).

[4] J.P.Demailly. Analyse numérique et équations différentielles, Presses universitaires de Grenoble, 1996.

[5] J.P.Demailly. Analyse numérique et équations différentielles ,Broché – 2006.

[6] C. Harrat and B. Djebbar. Holomorphic functions of several complex variables. Jordan Journal of Mathematics and Statistics 10 (4), 2017, pp 281-295.

[7] A.O. Guelfond. Calcul des différences finies.1963.

[8] F. Gramain. Entire function of one or several variables taking integer values in a geometrical progression.1988.

[9] T. RivoalI et M. Welter. Sur les fonctions arithmétiques non entiéres. p.155-179,2009.

[10] M. Pommiez. Sur les différences divisées successives et les restes des séries de Newton généralisées.tome28(1964), p.101-110.

[11] G. Valirond. Sur les fonctions entières d'ordre nul et les équations différentielles. tome 53(1925), p.34-42.

[12] G. Valirond. Sur les fonctions entières vérifiant une classe d'équations différentielles., tome 51, p.33-45, 1923.

[13] L. Verde-Star. Divided differences and combinatorial identities,Stud. Appl. Math,85:215–242, 1991.

[14] S. Lang. Complex Analysis, 4e éd., Springer, 1999 (ISBN 0-387-98592-1).

[15] L.I.Ronkin. Introduction to the theory of entire functions of several variables, Amer. Math. Soc. Providence (1974).