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Artinian and Noetherian Fuzzy Rings

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Abstract

In this paper we study rings with ascending (descending) chain conditions on their fuzzy substructures and various results are established. Also we prove some characterizations of rings with chain conditions in terms of fuzzy quotient rings and fuzzy ideals.

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1 Introduction

Since the famous paper "Fuzzy Sets" was presented in 1965 by Professor L. A. Zadeh [17], the theory of fuzzy sets has made considerable progress. It has exerted a tremendous influence on modern science and technology such as mathematics, natural science, management science, sociology, economics, philosophy, law, psychology, linguistics, medicine, control theory, signal processing, etc. The purpose of the paper is to improve the dissemination and exchange of the theory and application in the fuzzy realm. The study of the fuzzy algebraic structures started with the introduction of concepts of fuzzy subgroups and fuzzy ideals in the pioneering paper of Rosenfeld [16]. It is clear that many basic properties in ring theory are found to be carried over to fuzzy rings. Malik [3], Mukererjee and Sen [4] studied rings with chain conditions with the help of fuzzy ideals and the notions of fuzzy quotient rings were introduced by Kumar [1], Kuroaka and Kuroki [2]. In this paper we investigate

rings with chain conditions on their fuzzy substructures and prove some characterizations of rings with chain conditions in terms of fuzzy quotient rings and fuzzy ideals.

2 Preliminaries

Let R be a ring with identity. A fuzzy subset of R is a function from R to $[0,1]$. Let μ, ν be fuzzy subsets of R. We write $\mu \subseteq \nu$ if $\mu(x) \leq \nu(x)$ for all $x \in R$. We denote the image of μ by $Im(\mu)$, and for $t \in [0,1]$, let $\mu_t = \{x \in R | \mu(x) \geq t\}$, a level subset of μ. We let χ_W denote the characteristic function of a subset W of R. A fuzzy subset μ of R is a fuzzy ideal of R if for every $x, y \in R$, $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$ and $\mu(xy) \geq \max\{\mu(x), \mu(y)\}$. A ring R is left Noetherian (Artinian) if it satisfies the ascending chain condition (descending chain condition) on left ideals of R; that is, every strictly ascending (descending) chain of left ideals of R is finite.
Let X and Y be two sets and f a function of X into Y. Let μ and ν be fuzzy subsets of X and Y , respectively. Then $f(\mu)$, the image of μ under f, is a fuzzy subset of Y defined by

$$f(\mu)(y) = \begin{cases} \sup\{\mu(x) \mid x \in X, f(x) = y\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

for all $y \in Y$.
The preimage $f^{-1}(\nu)$ of ν under f is a fuzzy subset of X defined by $f^{-1}(\nu)(x) = \nu(f(x))$ for all $x \in X$.
Let μ be a fuzzy ideal of R and let $x \in R$. Then for all $r \in R$ the fuzzy subset μ_x^* of R defined by $\mu_x^*(r) = \mu(r-x)$ is termed as the fuzzy coset determined by x and μ. For all $x, y \in R$ the set of all fuzzy cosets of μ in R is a ring under the binary operations $\mu_x^* + \mu_y^* = \mu_{x+y}^*$ and $\mu_x^* \mu_y^* = \mu_{xy}^*$ and it is denoted by R_μ. We call it the fuzzy quotient ring of R induced by the fuzzy ideal μ.

3 Main results

Definition 3.1 *A ring in which every strictly descending chain of fuzzy left ideals is finite is called a fuzzy left Artinian ring. Also a fuzzy subset μ of a ring R is called fuzzy left Artinan if every strictly descending chain of fuzzy left ideals of μ is finite.*

Definition 3.2 *A ring in which every strictly ascending chain of fuzzy left ideals is finite is called a fuzzy left Noetherian ring. Also a fuzzy subset μ of a ring R is called fuzzy left Noetherian if every strictly ascending chain of fuzzy left ideals of μ is finite.*

Theorem 3.3 *Let $f : R \to S$ be a homomorphism of rings and μ be a finite-valued fuzzy left ideal of R. Then $f(\mu_t) = f(\mu)_t$.*

Proof 3.4 *Let $x \in f(\mu_t)$. Then $x = f(z)$ for some $z \in \mu_t$ and $f(\mu)(x) = sup_{f(z)=x}\mu(z) \geq t$. Thus $f(\mu_t) \subseteq f(\mu)_t$.*
Conversely if $y \in f(\mu)_t$, then $f(\mu)(y) = sup_{f(x)=y}\mu(x) \geq t$. Since μ is finite-valued there exists a $x \in f^{-1}(y)$ such that $\mu(x) \geq t$. Thus$f(\mu)_t \subseteq f(\mu_t)$.

Proposition 3.5 *If a ring R with unity is fuzzy left Artinian then every fuzzy left ideal on R has finite number of values.*

Proof 3.6 *Let R be fuzzy left Artinian. Suppose there exists a fuzzy left ideal μ of R such that $Im\mu = \{\mu(x) : x \in R\}$ is infinite. Hence there is an infinite sequence $\{t_n\}$ of elements of $Im\mu$ such that either $t_1 < t_2 < t_3...$ or $t_1 > t_2 > t_3....$*
If $t_1 < t_1 < t_1...$, then from definition $\mu_r(x) = 1-t_r, if \mu(x)_r$ we obtain astrictly descending sequence $\mu_0 \supset \mu_1 \supset \mu_2 \supset ...$ and this contradicts that R is fuzzy left Artinian.
Also if $t_1 > t_2 > t_3...$, then from definition $\mu_r(x) = t_r, if \mu(x) \geq 1-t_r$ we obtain astrictly descending sequence $\mu_0 \supset \mu_1 \supset \mu_2 \supset ...$ which is also a contradiction.

By using similar method as in the proof of Proposition 3.5 we obtain the following proposition.

Proposition 3.7 *If a ring R with unity is fuzzy left Noetherian then every fuzzy left ideal on R has finite number of values.*

Proposition 3.8 *Let R and S be two fuzzy left Artinian rings with unity 1. Then $R \times S$ is also fuzzy left Artinian.*

Proof 3.9 *Let μ be a fuzzy left ideal of $R \times S$. For all $x \in R, y \in S$ define $\mu_1(x) = \mu(x, 0)$ and $\mu_2(y) = \mu(0, y)$ be two fuzzy left idaels of R and S respectively. Let $(x, y) \in R \times S$ then $\mu(x, y) = min\{\mu_1(x), \mu_2(y)\}$ and since R and S are fuzzy Artinian rings with 1 so $Im\mu_1$ and $Im\mu_2$ are finite subsets of $[0, 1]$. Therefore $Im\mu$ is also a nite subset of $[0, 1]$.*

Proposition 3.10 *Let R be a ring. Then if R is fuzzy left Noetherian, then the set of fuzzy left ideals on R is a well ordered subset of $[0, 1]$.*

Proof 3.11 *Suppose μ is fuzzy left ideal whose set of values is not a well ordered subset of $[0, 1]$. Then there exists a strictly descending sequence $\{t_n\}$ such that $\mu(x_n) = t_n$. Define $\sigma_n : R \to [0, 1]$ as $\sigma_n(x) = t_n$, if $\mu(x) \geq t_n$ and then σ_n is a fuzzy left ideal of R. From $t_{n-1} > t_n$ we get that $\sigma_{n-1}\sigma_n$ and so $\sigma_0\sigma_1\sigma_2...$ which this contradicts that R is fuzzy left Noetherian.*

The proof the following proposition is similar as above proposition.

Proposition 3.12 *Let R be a ring. Then R is fuzzy left Noetherian if and only if the set of fuzzy left ideals on R is a well ordered subset of* $[0,1]$.

Now we prove some characterizations of rings with chain conditions in terms of fuzzy quotient rings and fuzzy ideals. The following theorem can be found in [1].

Theorem 3.13 *Let μ be a fuzzy ideal of a ring R. Then for all $x \in R$ the map $f : R \to R_\mu$ defined by $f(x) = \mu_x^*$ is a surjective homomorphism with kernel μ_t, where $t = \mu(0)$.*

The following lemma was proved in [2].

Lemma 3.14 *Let R and S be rings and $f : R \to S$ be a ring homomorphism. If f is surjective and μ is a fuzzy left ideal of R, then so is $f(\mu)$. If θ is a fuzzy left ideal of S, then so is $f^{-1}(\theta)$.*

Proposition 3.15 *A ring R is Artinian if and only if R_μ is Artinian for every fuzzy ideal μ of R.*

Proof 3.16 *Let μ be a fuzzy ideal of R and ν be any fuzzy ideal of R_μ. To show that ν is finite-valued, define a map $\theta : R \to [0,1]$ by $\theta(x) = \nu(\mu_x^*)$ for every $x \in R$. Then θ is a fuzzy ideal of R and it is finite-valued by Proposition 3.5. Also ν is also finite-valued since the set of values of θ is same to the set of values of ν , so that R_μ is Artinian.*
Conversely, let μ be a fuzzy ideal of a ring R. For all $x \in R$ we define $\nu(\mu_x^*) = \mu(x)$ such that is finite-valued and so that ν is also finite- valued. Now by Proposition 3.5, R is Artinian.

Proposition 3.17 *A ring R is Noetherian if and only if R_μ is Noetherian for every fuzzy ideal μ of R.*

Proof 3.18 *Let R be Noetherian. Then by the similar method to the proof of Proposition 3.15, we obtain that R_μ is Noetherian for every fuzzy ideal μ of R. Conversely, let μ be any fuzzy ideal of R and ν be a fuzzy ideal of R_μ. If $\nu(\mu_x^*) = \mu(x)$, then by Proposition 3.5 the set of values of ν is a well-ordered subset of* $[0,1]$. *Since the set of values of ν is same to the set of values of $im(\mu)$ is also well-ordered. Hence from Proposition 3.10 we get that R is Noetherian.*

4 Open Problem

We recall the following definitions.

Definition 4.1 *A t-norm T is a function $T : [0,1] \times [0,1] \to [0,1]$ having the following four properties:*
(T1) $T(x,1) = x$ (neutral element),
(T2) $T(x,y) \leq T(x,z)$ if $y \leq z$ (monotonicity),
(T3) $T(x,y) = T(y,x)$ (commutativity),
(T4) $T(x,T(y,z)) = T(T(x,y),z)$ (associativity),
for all $x,y,z \in [0,1]$.

It is clear that if $x_1 \geq x_2$ and $y_1 \geq y_2$, then $T(x_1,y_1) \geq T(x_2,y_2)$.

Example 4.2 *(1) Standard intersection T-norm $T_m(x,y) = \min\{x,y\}$.*
(2) Bounded sum T-norm $T_b(x,y) = \max\{0, x+y-1\}$.
(3) algebraic product T-norm $T_p(x,y) = xy$.
(4) Drastic T-norm

$$T_D(x,y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum T-norm

$$T_{nM}(x,y) = \begin{cases} \min\{x,y\} & \text{if } x+y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) Hamacher product T-norm

$$T_{H_0}(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D(x,y) \leq T(x,y) \leq T_{\min}(x,y)$ for all $x,y \in [0,1]$.

Definition 4.3 *A t-conorm C is a function $C : [0,1] \times [0,1] \to [0,1]$ having the following four properties:*
(C1) $C(x,0) = x$,
(C2) $C(x,y) \leq C(x,z)$ if $y \leq z$,
(C3) $C(x,y) = C(y,x)$,
(C4) $C(x,C(y,z)) = C(C(x,y),z)$,
for all $x,y,z \in [0,1]$.

Example 4.4 *(1) Standard union t-conorm* $C_m(x,y) = \max\{x,y\}$.
(2) Bounded sum t-conorm $C_b(x,y) = \min\{1, x+y\}$.
(3) Algebraic sum t-conorm $C_p(x,y) = x + y - xy$.
(4) Drastic T-conorm

$$C_D(x,y) = \begin{cases} y & \text{if } x = 0 \\ x & \text{if } y = 0 \\ 1 & \text{otherwise,} \end{cases}$$

dual to the drastic T-norm.
(5) Nilpotent maximum T-conorm , dual to the nilpotent minimum T-norm:

$$C_{nM}(x,y) = \begin{cases} \max\{x,y\} & \text{if } x + y < 1 \\ 1 & \text{otherwise.} \end{cases}$$

(6) Einstein sum (compare the velocity-addition formula under special relativity) $C_{H_2}(x,y) = \dfrac{x+y}{1+xy}$ *is a dual to one of the Hamacher t-norms. Note that all t-conorms are bounded by the maximum and the drastic t-conorm:* $C_{\max}(x,y) \leq C(x,y) \leq C_D(x,y)$ *for any t-conorm* C *and all* $x, y \in [0,1]$.

The open problem here is to investigate norms ovre Artinian and Noetherian fuzzy rings and to give some new results about them as author by using norms, investigated some properties of fuzzy algebra ([5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]).

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