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A commutativity criterion for groups

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Abstract

Let G be a group. In this short note, we give a criterion of commutativity of G based on the commutativity of the subsets of G with a certain number of elements.

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1 Introduction

The commutativity is one of the most important properties of algebraic structures. There are many criterions that imply the commutativity of a group (see e.g. [2, 5]). Yet another such criterion is presented in the following.

Let G be a group and k be a positive integer. A subset A of G is called a *k-subset* if $|A| = k$. Given two subsets A and B of G, we will denote by AB their product, i.e. $AB = \{ab \,|\, a \in A, b \in B\}$. Also, we will say that *A and B commute* if $AB = BA$. Our main result is stated as follows.

Theorem 1.1. *For $k \in \{1, 2, 3\}$, the group G is commutative if and only if every two of its k-subsets commute.*

Note that the above equivalence does not hold for $k \geq 4$. To prove this, it suffices to show that every two 4-subsets of the symmetric group S_3 commute. This is obtained by the following lemma.

Lemma 1.2. *Let G be a finite group and A, B be two subsets of G satisfying $|A| + |B| > |G|$. Then $AB = G$.*

Finally, we remark that Lemma 1.2 assures the commutativity of *every* two subsets A and B of G satisfying $|A| + |B| > |G|$. Also, we remark that its conclusion does not hold if A and B satisfy the condition $|A| + |B| = |G|$, as shows the next example.

Example. In S_3, let $A = B = A_3$, the alternating group of degree 3. Then $AB = A_3 \neq S_3$. More generally, if G is a group containing a subgroup H of index 2, then by taking $A = B = H$ we get $AB = H \neq G$.

2 Proofs of the main results

2.1 Proof of Theorem 1.1

Obviously, if the group G is commutative, then every two of its k-subsets commute.

Conversely, we have to prove that $xy = yx$ for all $x, y \in G$. In the case $k = 1$ this follows immediately from the commutativity of the 1-subsets $A = \{x\}$ and $B = \{y\}$. For $k = 2$, we observe that we can assume $x \neq 1$ and $y \neq 1$ (for $x = 1$ or $y = 1$ the equality $xy = yx$ is clearly satisfied). Let the 2-subsets $A = \{1, x\}$ and $B = \{1, y\}$. Then

$$AB = \{1, x, y, xy\} \text{ and } BA = \{1, x, y, yx\},$$

and thus $AB = BA$ implies $xy = yx$. For $k = 3$, let us assume that G is not commutative. Then there is $x \in G$ such that

$$(*) \qquad x \neq x^{-1} \text{ and } x \notin Z(G),$$

where $Z(G)$ is the center of G. Indeed, if for all $x \in G$ we have $x = x^{-1}$ or $x \in Z(G)$, then let a and b be two arbitrary elements of G. If at least one of them is contained in $Z(G)$ we get $ab = ba$, while if $a = a^{-1}$ and $b = b^{-1}$ we get

$$ab = (ab)^{-1} = b^{-1}a^{-1} = ba \text{ in the case } ab = (ab)^{-1},$$

respectively

$$ab = ab^3 = [(ab)b]b = [b(ab)]b = [(ba)b]b = (ba)b^2 = ba \text{ in the case } ab \in Z(G).$$

Therefore G is commutative, a contradiction. Pick $x \in G$ satisfying the properties $(*)$ and $y \in G$ such that $xy \neq yx$. By taking the 3-subsets

$$A = \{1, x, y\} \text{ and } B = \{1, x^{-1}, y\},$$

it follows that

$$AB = \{1, x, y, x^{-1}, y^2, xy, yx^{-1}\} \text{ and } BA = \{1, x, y, x^{-1}, y^2, yx, x^{-1}y\},$$

and thus $AB = BA$ implies $xy \in BA$. This easily leads to a contradiction. Hence the group G is commutative. ∎

2.2 Proof of Lemma 1.2

We have to prove that every element $x \in G$ can be written as $x = ab$, where $a \in A$ and $b \in B$. Let $x \in G$ and consider the subset $xB^{-1} = \{xb^{-1} \mid b \in B\}$ of G. Then $|xB^{-1}| = |B|$, and so $|A| + |xB^{-1}| > |G|$. Suppose that $A \cap xB^{-1} = \emptyset$. By using the Inclusion-Exclusion Principle, we obtain

$$|G| \geq |A \cup xB^{-1}| = |A| + |xB^{-1}| > |G|,$$

a contradiction. Consequently, there exists $a \in A \cap xB^{-1}$, implying that $a = xb^{-1}$ for some $b \in B$. Then $x = ab$, completing the proof. ∎

3 Open problems

Problem 1. Let $k \geq 4$ be an integer and let G be a finite group of order at least $2k$. Assume that every two k-subsets of G commute. Is it true that G is commutative?

Problem 2. The *commutativity degree* of a finite group G is defined by

$$d(G) = \frac{|\{(x, y) \in G^2 \mid xy = yx\}|}{|G|^2}$$

and measures the probability that two elements of G commute. Its origin lies in a paper by P. Erdös and P. Turán (see [1]) published in 1968, and also in the Ph. D thesis of K.S. Joseph [3] submitted in 1969. Since then, many generalizations of $d(G)$ have been studied (see e.g. [4]). Inspired by the above result, we are able to indicate yet another one, namely:

Let G be a finite group of order n. For every positive integer k we denote by $\mathcal{P}_k(G)$ the set of all k-subsets of G. Clearly, we have $|\mathcal{P}_k(G)| = \binom{n}{k}$. Define

$$d_{\mathcal{P}_k(G)} = \frac{|\{(A, B) \in \mathcal{P}_k(G)^2 \mid AB = BA\}|}{\binom{n}{k}^2}.$$

Remark that $d_{\mathcal{P}_1(G)} = d(G)$, and therefore a new generalization of the commutativity degree of G is obtained. Also, for $k = 2$ and $k = 3$ we have $d_{\mathcal{P}_k(G)} = 1$ if and only if G is commutative by Theorem 1.1. Study the quantity $d_{\mathcal{P}_k(G)}$.

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