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Proof of Some New Conjectures on Algebraic and Algebraic-Trigonometric Inequalities

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Abstract

In the paper, we prove some conjectural new algebraic-trigonometric inequalities of Laub-Ilani type with n terms. The inequalities were posted as the conjecture 2.12 and 2.13 in the paper A.,Y., Özban, "New Algebraic-Trigonometric Inequalities of Laub-Ilani type", Bull. Aust. Math. Soc., 2017, doi 10.1017/S0004972 717000156. We also find optimal bounds for one power-exponential function of two variables and prove the conjecture 2.10 proposed by Y. Nishizava in the paper: "Symmetric Inequalities with Power-Exponential Functions", Indian J. Pure Appl. Math., 48(3): 335-344, 2017, doi. 10.10 07/S13226-017-0230-y.

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1 Introduction

The Luab-Ilani inequality was introduced in the "Problems and Solutions" section of the American Mathematical Monthly as Problem E3116 [3] by Laub. Several years later a solution of the problem was done. The Laub-Ilani inequality states:

if $x, y > 0$, then

$$x^x + y^y \geq x^y + y^x. \tag{1}$$

The more general form of the inequality states:

if $\{x_1, x_2, ..., x_n\}$ is a sequence of non negative real numbers and $\{y_1, y_2, ..., y_n\}$ is any permutation of this sequence, then

$$x_1^{x_1} + x_2^{x_2} + ... + x_n^{x_n} \geq x_1^{y_1} + x_2^{y_2} + ... + x_n^{y_n}. \tag{2}$$

The inequalities were proved in the papers [3],[4].

Many other similar inequalities were proved by Cirtoaje, Matejíčka, Miyagi and Nishizawa,... (see [1], [5], [6], [8], [9], [10]).

The goal of the paper is to prove the following conjectures.

Conjecture 1.1. *(2.12, [10]) If $x_1, x_2, ..., x_n \in (0, 1]$ and if $\{y_1, y_2, ..., y_n\}$ is any permutation of the finite sequence $\{x_1, x_2, ..., x_n\}$, then*

a)

$$\sin(x_1^{x_1}) + \sin(x_2^{x_2}) + ... + \sin(x_n^{x_n}) > \sin(x_1^{y_1}) + \sin(x_2^{y_2}) + ... + \sin(x_n^{y_n}),$$

b)

$$\cos^{x_1} x_1 + \cos^{x_2} x_2 + ... + \cos^{x_n} x_n < \cos^{y_1} x_1 + \cos^{y_2} x_2 + ... + \cos^{y_n} x_n,$$

c)

$$\cos^{r_1} x_1 + \cos^{r_2} x_2 + ... + \cos^{r_n} x_n < \cos^{t_1} x_1 + \cos^{t_2} x_2 + ... + \cos^{t_n} x_n,$$

where $r_i = \sin x_i$ and $t_i = \sin y_i$ for $i = 1, 2..., n$.

Conjecture 1.2. *(2.13, [10]) If $x_1, x_2, ..., x_n \in (0, \pi/2]$ and if $\{y_1, y_2, ..., y_n\}$ is any permutation of the finite sequence $\{x_1, x_2, ..., x_n\}$, then*

a)

$$\cos(x_1^{x_1}) + \cos(x_2^{x_2}) + ... + \cos(x_n^{x_n}) < \cos(x_1^{y_1}) + \cos(x_2^{y_2}) + ... + \cos(x_n^{y_n}),$$

b)

$$\sin^{x_1} x_1 + \sin^{x_2} x_2 + ... + \sin^{x_n} x_n > \sin^{y_1} x_1 + \sin^{y_2} x_2 + ... + \sin^{y_n} x_n,$$

c)

$${x_1}^{\sin x_1} + {x_2}^{\sin x_2} + ... + {x_n}^{\sin x_n} > {x_1}^{\sin y_1} + {x_2}^{\sin y_2} + ... + {x_n}^{\sin y_n}.$$

Conjecture 1.3. *(2.10, [9]) If a and b are non negative real numbers with $a+b=1/2$, then the inequality*

$$\frac{1}{2} \le a^{(2b)^k} + b^{(2a)^k} \le 1,$$

holds for $0 \le k \le 1$.

We also prove the following assertion:

Let $a, b > 0$ such that $a+b=1/m$, $m \ge e^{\sqrt{2}}$, then the inequality

$$\sqrt{\frac{2}{m}} \le a^{mb} + b^{ma} \le 1$$

holds, where $\sqrt{\frac{2}{m}}$, 1 are the best possible constants.

2 Main results

In this section we prove the conjectures 2.12 (see [10]), 2.10 (see [9]), Lemmas 2.1, 2.2 and partially the conjecture 2.13 (see [10]).

2.1 Lemmas and theorems

Lemma 2.1. *Let $a, b > 0$ such that $a+b=1/m$, $m \ge e^{\sqrt{2}}$, then the inequality*

$$\sqrt{\frac{2}{m}} \le a^{mb} + b^{ma} \le 1 \tag{3}$$

holds, where $\sqrt{\frac{2}{m}}$, 1 are the best possible constants.

Proof. Let m be fixed such that $m \ge e^{\sqrt{2}}$. Put

$$F(a,m) = a^{1-ma} + \left(\frac{1}{m} - a\right)^{ma} = \lambda(a,m) + \lambda(1/m - a, m)$$

where $\lambda(a,m) = a^{1-ma}$. We get

$$\begin{aligned} F'_a(a,m) &= a^{1-ma}\left(-m\ln a + \frac{1}{a} - m\right) + \\ & \quad m\left(\frac{1}{m} - a\right)^{ma}\left(\ln\left(\frac{1}{m} - a\right) - \frac{ma}{1-ma}\right). \end{aligned}$$

It is evident that

$$F'_a\left(\frac{1}{2m},m\right)=0,\quad \lim_{a\to 0^+}F'_a(a,m)=1-m\ln m<0,$$
$$F'_a\left(\frac{1}{m},m\right)=m\ln m-1>0,\quad F\left(\frac{1}{2m},m\right)=\sqrt{\frac{2}{m}},$$
$$\lim_{a\to 0^+}F(a,m)=1,\quad F\left(\frac{1}{m},m\right)=1.$$

If we show that $F''_{aa}>0$ for $0<a\le\frac{1}{m}$ the proof will be done. We have

$$\begin{aligned}F''_{aa}(a,m) &= a^{1-ma}\left(-m\ln a+\frac{1}{a}-m\right)^2+a^{1-ma}\left(-\frac{m}{a}-\frac{1}{a^2}\right)+\\ &\quad m^2\left(\frac{1}{m}-a\right)^{ma}\left(\ln\left(\frac{1}{m}-a\right)-\frac{ma}{1-ma}\right)^2+\\ &\quad m\left(\frac{1}{m}-a\right)^{ma}\left(-\frac{m}{1-ma}-\frac{m}{(1-ma)^2}\right).\end{aligned}$$

It can be rewriting as

$$F''_{aa}(a,m)=a^{-1-ma}v(a,m)+\left(\frac{1}{m}-a\right)^{ma}w(a,m),$$

where

$$\begin{aligned}v(a,m) &= (-ma\ln a+1-ma)^2-am-1,\\ w(a,m) &= \left(m\ln\left(\frac{1}{m}-a\right)-\frac{m^2a}{1-ma}\right)^2-\frac{m^2}{1-ma}-\frac{m^2}{(1-ma)^2}.\end{aligned}$$

To prove $F''_{aa}>0$ it suffices to shove $v(a,m)>0$. It follows from $F''_{aa}(a,m)=\lambda''_{aa}(a,m)+\lambda''_{aa}(1/m-a,m)$. Put $t=ma$, then $0<t\le 1$ and

$$\alpha(t)=v(a,m)=(1-t\ln t+t\ln m-t)^2-t-1.$$

Let $\ln m\ge\sqrt{2}$. Then, it suffices to prove

$$\alpha^*(t)=\left(1+(\sqrt{2}-1)t-t\ln t\right)^2-t-1>0.$$

From $\alpha^*(0)=0$, $\alpha^*(1)=0$, it suffices to show $\alpha^{*''}(t)\le 0$.
We have

$$\alpha^{*''}(t)=2\left(2-\sqrt{2}+\ln t\right)^2+2\frac{t-\sqrt{2}t+t\ln t-1}{t}.$$

$\alpha^{*''}(t) \le 0$ is equivalent to

$$s(t) = 2\ln^2 t + (10 - 4\sqrt{2})\ln t + 2(7 - 5\sqrt{2}) - \frac{2}{t} < 0.$$

We have $s(0) = -\infty$, $s(1) = -2.1421$. So it suffices to prove

$$s'(t) = \frac{1}{t}p(t) = 4\frac{\ln t}{t} + (10 - 4\sqrt{2})\frac{1}{t} + \frac{2}{t^2} > 0.$$

$p(t) > 0$ is equivalent to

$$q(t) = 4t\ln t + (10 - 4\sqrt{2})t + 2 > 0.$$

But, it follows from $q(0) = 2$, $q(1) = 12 - 4\sqrt{2} > 0$, $q(t_0) = 1.5031624$, where $t_0 = e^{\sqrt{2}-3.5} = 0.12420939$ because of

$$q'(t) = 4(\ln t + 1) + 10 - 4\sqrt{2} = 0$$

only for t_0. The proof is complete. □

Lemma 2.2. *Let $a, b > 0$ such that $a + b = c$ where $5e^{-5} \le c \le e^{-1}$ and $m \ge e^5/5$, then the inequality*

$$2\left(\frac{c}{2}\right)^{\frac{mc}{2}} \le a^{mb} + b^{ma} \le 1 \tag{4}$$

holds, where $2\left(\frac{c}{2}\right)^{\frac{mc}{2}}$, 1 are the best possible constants.

Proof. Let m be fixed such that $m \ge e^5/5$. Put

$$F(a, m) = a^{m(c-a)} + (c - a)^{ma} = \lambda(a, m) + \lambda(c - a, m).$$

Then

$$F'_a(a, m) = a^{m(c-a)}\left(-m\ln a + \frac{m(c-a)}{a}\right) + (c - a)^{ma}\left(m\ln(c - a) - \frac{ma}{c - a}\right).$$

It is evident that $F(0, m) = 1$, $F(c, m) = 1$, $F'_a(c/2, m) = 0$. If we show that $F''_{aa}(a, m) \ge 0$ the proof will be done. To prove $F''_{aa}(a, m) \ge 0$ it suffices to show $\lambda''_{aa}(a, m) \ge 0$ which is equivalent to $v(a, m) > 0$ for $5e^{-5} \le c \le e^{-1}$ and $m \ge e^5/5$, $0 < a \le c$. It follows from

$$F''_{aa}(a, m) = \lambda''_{aa}(a, m) + \lambda''_{aa}(c - a, m) = a^{m(c-a)}v(a, m) + (c - a)^{ma}v(c - a, m),$$

where

$$v(a, m) = m\left(a\ln a - c + a\right)^2 - a - c.$$

To prove $v(a,m) > 0$ it suffices to show $v''_{aa}(a,m) \le 0$ because of

$$v(0,m) = c(mc-1) \ge 0,$$

$$v(c,m) = c(mc\ln^2 c - 2) \ge c(e^5 c\ln^2 c/5 - 2) = ck(c) \ge 0.$$

It follows from $k(5e^{-5}) = 9.4959$, $k(e^{-2}) = 14.0684$, $k(e^{-1}) = 8.9196$ and from

$$k'(c) = \frac{e^5}{5}\ln c(\ln c + 2)) = 0$$

if $c = 1$ or $c = e^{-2}$.

Now we show $v''_{aa}(a,m) \le 0$. This inequality is equivalent to

$$K(a,c) = 5a + 5a\ln a + a\ln^2 a - c \le 0.$$

It is evident that extreme points of K lie on the boundary of $M = \{(a,c); 0 < a \le c, 5e^{-5} \le c \le e^{-1}\}$ because of $K'_c(a,c) = -1$. Next we have $K(0,c) = -c \le 0$.

Easy computations give

$$K(a,e^{-1}) = 5a + 5a\ln a + a\ln^2 a - e^{-1}$$

and

$$K'_a(a,e^{-1}) = (\ln a + 2)(\ln a + 5).$$

From $K(e^{-5},e^{-1}) = -0.3342$, $K(e^{-2},e^{-1}) = -0.5032$, $K(e^{-1},e^{-1}) = 0$, $K(0,e^{-1}) = -e^{-1}$ we obtain $K(a,e^{-1}) \le 0$. Similarly we get $K(e^{-5},5e^{-5}) = -1.3878 * 10^{-17}$, $K(5e^{-5},5e^{-5}) = -0.0491$, $K(e^{-2},5e^{-5}) = -0.1690$, $K(0,5e^{-5}) = -5e^{-5}$. So $K(a,5e^{-5}) \le 0$.

Next

$$K(c,c) = 4c + 5c\ln c + c\ln^2 c = c(\ln c + 1)(\ln c + 4).$$

Because of $5e^{-5} > e^{-4}$ and $K(e^{-1},e^{-1}) = 0$ we obtain $K(c,c) \le 0$ for $c \in (5e^{-5},e^{-1})$. The proof is complete. □

Now we prove the conjecture 2.10 (see [8]).

Theorem 2.1. *Let a and b be non negative real numbers, $0 \le k \le 1$. Then*

α)

$$\frac{1}{2} \le a^{(2b)^k} + b^{(2a)^k} \le 1 \quad \textit{for} \quad a + b = 1/2,$$

β)

$$\frac{1}{2} \le a^{(mb)^k} + b^{(ma)^k} \le 1 \quad \textit{for} \quad a + b = 1/m, \quad m \ge e^{\sqrt{2}}.$$

$1, 1/2$ *are the best possible constants.*

Proof. First we show α). Let a and b be fixed non negative real numbers such that $a + b = 1/2$. Put
$$F(k) = a^{(2b)^k} + b^{(2a)^k}.$$
We get
$$F'(k) = a^{(2b)^k}(2b)^k \ln a \ln(2b) + b^{(2a)^k}(2a)^k \ln b \ln(2a) > 0.$$
So
$$F(0) = \frac{1}{2} = a + b \le a^{(2b)^k} + b^{(2a)^k} \le F(1) = a^{(2b)} + b^{(2a)} \le 1.$$
The inequality $a^{(2b)} + b^{(2a)} \le 1$ for $a + b = 1/2$ was proved by Nishizava in [9].

Proof of β) follows from Lemma 2.1 and from
$$G'(k) = a^{(mb)^k}(mb)^k \ln a \ln(mb) + b^{(ma)^k}(ma)^k \ln b \ln(ma) > 0,$$
where
$$G(k) = a^{(mb)^k} + b^{(ma)^k}.$$
□

Remark 2.2. *We note that Lemma 2.2 implies: if a and b are non negative real numbers with $a + b = c$, where $5e^{-5} \le c \le e^{-1}$ and $m \ge e^5/5$, then the inequality*
$$c \le a^{(2b)^k} + b^{(2a)^k} \le 1$$
holds for $0 \le k \le 1$.

Now we prove the conjecture 2.12 (see [10]).

Theorem 2.3. *Let $x_1, x_2, ..., x_n \in (0, 1]$ and let $\{y_1, y_2, ..., y_n\}$ be any permutation of the finite sequence $\{x_1, x_2, ..., x_n\}$ where $n \ge 1$, $n \in N$, then*

a)
$$\sin(x_1^{x_1}) + \sin(x_2^{x_2}) + ... + \sin(x_n^{x_n}) > \sin(x_1^{y_1}) + \sin(x_2^{y_2}) + ... + \sin(x_n^{y_n}),$$

b)
$$\cos^{x_1} x_1 + \cos^{x_2} x_2 + ... + \cos^{x_n} x_n < \cos^{y_1} x_1 + \cos^{y_2} x_2 + ... + \cos^{y_n} x_n,$$

c)
$$\cos^{r_1} x_1 + \cos^{r_2} x_2 + ... + \cos^{r_n} x_n < \cos^{t_1} x_1 + \cos^{t_2} x_2 + ... + \cos^{t_n} x_n,$$
where $r_i = \sin x_i$ and $t_i = \sin y_i$ for $i = 1, 2..., n$.

Proof. We use mathematical induction. The inequalities a), b), c) were proved for $n = 2$ (see [10]). Let $n \geq 3$. Without lost of generality suppose $0 < x_1 < \min\{x_2, ..., x_n\} < 1$. Let $\{y_1, y_2, ..., y_n\}$ be a permutation of the finite sequence $\{x_1, x_2, ..., x_n\}$. We can suppose $y_1 = x_i$, $y_j = x_1$ where $i \neq j \neq 1$. (It is evident that if $j = 1$ or if $i = j$ or $i = 1$ the proof follows from an inductive hypothesis.) Let a) be valid for all $2 \leq k \leq n - 1$. Put

$$F(t) = \sin(t^t) + \sin(x_2^{x_2}) + ... + \sin(x_n^{x_n}) - \sin(t^{x_i}) - \sin(x_j^t) - \sum_{l=2, l\neq j, l\neq 1}^{n} \sin(x_l^{y_l})$$

for $0 < t \leq \max\{x_i, x_j\}$. From the inductive hypothesis it follows

$$F(x_i) = \sum_{l=2}^{n} \sin(x_l^{x_l}) - \sin(x_j^{x_i}) - \sum_{l=2, l\neq j}^{n} \sin(x_l^{y_l}) > 0.$$

Similarly, the inductive hypothesis gives

$$F(x_j) = \sum_{l=2}^{n} \sin(x_l^{x_l}) - \sin(x_j^{x_i}) - \sum_{l=2, l\neq j}^{n} \sin(x_l^{y_l}) > 0.$$

The proof of a) will be done if we show $F'_t < 0$ for $0 < t \leq \min\{x_i, x_j\}$.

Simple computation gives $F'_t = v(t) + w(t)$ where

$$v(t) = \cos(t^t)t^t - \cos(t^{x_i})x_i t^{x_i - 1}$$

and

$$w(t) = \cos(t^t)t^t \ln t - \cos(x_j^t){x_j}^t \ln x_j.$$

In the paper [10], page 90, it was shown that $v(t) < 0$ for $0 < t \leq x_i \leq 1$ and $w(t) < 0$ for $0 < t \leq x_j \leq 1$. So, the proof of a) is complete.

Similarly, we prove b) and c). Really, in the case b)
let

$$G(t) = \cos^t t + \cos^{x_2} x_2 + ... + \cos^{x_n} x_n - \cos^{x_i} t - \cos^t x_j - \sum_{l=2, l\neq j, l\neq 1}^{n} \cos^{y_l} x_l$$

for $0 < t \leq \max\{x_i, x_j\}$. Easy to see

$$G(x_i) = \sum_{l=2}^{n} \cos^{x_l} x_l - \cos^{x_i} x_j - \sum_{l=2, l\neq j}^{n} \cos^{y_l} x_l < 0.$$

It follows from the inductive hypothesis. Similarly, the inductive hypothesis gives

$$G(x_j) = \sum_{l=2}^{n} \cos^{x_l} x_l - \cos^{x_i} x_j - \sum_{l=2, l\neq j}^{n} \cos^{y_l} x_l < 0.$$

The proof of b) will be done if we show $G'_t > 0$ for $0 < t \le \min\{x_i, x_j\}$.

Differentiation of $G(t)$ yields $G'_t = \alpha(t) + \tan(t)\beta(t)$ where

$$\beta(t) = \tan(t)\left(\cos^{x_i}(t)x_i - \cos^{t}(t)t\right),$$

and

$$\alpha(t) = \cos^t(t)\ln(\cos t) - \cos^t(x_j)\ln(\cos x_j).$$

In the paper [10], page 92, it was shown that $\alpha(t) > 0$ for $0 < t \le x_j \le 1$ and $\beta(t) > 0$ for $0 < t \le x_i \le 1$ which completes the proof of b).

The case c).

Let

$$\begin{aligned} H(t) &= \cos^{\sin t} t + \cos^{\sin x_2} x_2 + ... + \cos^{\sin x_n} x_n - \cos^{\sin x_i} t - \\ &\quad \cos^{\sin t} x_j - \sum_{l=2, l\neq j, l\neq 1}^{n} \cos^{\sin y_l} x_1 \end{aligned}$$

for $0 < t \le \max\{x_i, x_j\}$. We see that

$$H(x_i) = \sum_{l=2}^{n} \cos^{\sin x_l} x_l - \cos^{\sin x_i} x_j - \sum_{l=2, l\neq j}^{n} \cos^{\sin y_l} x_l < 0.$$

It follows from the inductive hypothesis. Similarly, the inductive hypothesis gives

$$H(x_j) = \sum_{l=2}^{n} \cos^{\sin x_l} x_l - \cos^{\sin x_i} x_j - \sum_{l=2, l\neq j}^{n} \cos^{\sin y_l} x_l < 0.$$

The proof of c) will be done if we show $H'_t > 0$ for $0 < t \le \min\{x_i, x_j\}$.

Differentiation of $H(t)$ yields $H'_t = \cos(t)\gamma(t) - \tan(t)\delta(t)$ where

$$\delta(t) = \cos^{\sin t}(t)\sin t - \cos^{\sin x_i}(t)\sin x_i,$$

and

$$\gamma(t) = \cos^{\sin t}(t)\ln(\cos t) - \cos^{\sin t}(x_j)\ln(\cos x_j).$$

In the paper [10], page 91, it was shown that $\gamma(t) > 0$ for $0 < t \le x_j \le 1$ and $\delta(t) < 0$ for $0 < t \le x_i \le 1$ which completes the proof of c). $\square$

Now we partially prove the conjecture 2.13 (see [10]).

Theorem 2.4. *Let* $x_1, x_2, ..., x_n \in (0, 1]$ *or* $x_1, x_2, ..., x_n \in (1, \pi/2]$ *and let* $\{y_1, y_2, ..., y_n\}$ *be any permutation of the finite sequence* $\{x_1, x_2, ..., x_n\}$ *where* $n \ge 1$, $n \in N$, *then*

b)

$$\sin^{x_1} x_1 + \sin^{x_2} x_2 + ... + \sin^{x_n} x_n > \sin^{y_1} x_1 + \sin^{y_2} x_2 + ... + \sin^{y_n} x_n,$$

c)

$${x_1}^{\sin x_1} + {x_2}^{\sin x_2} + ... + {x_n}^{\sin x_n} > {x_1}^{\sin y_1} + {x_2}^{\sin y_2} + ... + {x_n}^{\sin y_n}.$$

Proof. The proofs of b) and c) are the same as in the theorem 2.3. It suffices to use mathematical induction, the same method of calculations and inequalities from the paper [10], pages 94–95. □

Remark 2.5. *We note, that in the paper, the software MATLAB was used for some computations.*

3 Open Problem

The conjecture 2.12, [10] is still open for the case a). The cases b) and c) are only partially proved. To prove the conjecture 2.12, [10] completely it is necessary to show that:

Let $x_1, x_2, ..., x_n \in (0, \pi/2]$ and let $\{y_1, y_2, ..., y_n\}$ be any permutation of the finite sequence $\{x_1, x_2, ..., x_n\}$ where $n \geq 1$, $n \in N$, then

(a)

$$\cos(x_1^{x_1}) + \cos(x_2^{x_2}) + ... + \cos(x_n^{x_n}) < \cos(x_1^{y_1}) + \cos(x_2^{y_2}) + ... + \cos(x_n^{y_n}).$$

Let $x_1, x_2, ..., x_n \in (0, \pi/2]$ such that there are $i, j \in \{1, 2, ..., n\}$, $i \neq j$, $0 < x_i \leq 1 < x_j \leq \pi/2$ and let $\{y_1, y_2, ..., y_n\}$ be any permutation of the finite sequence $\{x_1, x_2, ..., x_n\}$ where $n \geq 1$, $n \in N$, then

(b)

$$\sin^{x_1} x_1 + \sin^{x_2} x_2 + ... + \sin^{x_n} x_n > \sin^{y_1} x_1 + \sin^{y_2} x_2 + ... + \sin^{y_n} x_n,$$

(c)

$${x_1}^{\sin x_1} + {x_2}^{\sin x_2} + ... + {x_n}^{\sin x_n} > {x_1}^{\sin y_1} + {x_2}^{\sin y_2} + ... + {x_n}^{\sin y_n}.$$

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