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On $log-(m, h_1, h_2)-$convex functions and related integral inequalities

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Abstract

In this work a new definition of generalized convexity called $log-(m, h_1, h_2)-$convexity is introduced, and some properties and applications to special integral inequalities are given. From the obtained results some others inequalities for several classical definitions of generalized convex functions are deduced.

Keywords: *$log-(m, h_1, h_2)-$convex functions, integral Inequalities, Hermite - Hadamard Inequality, log-convex functions.*
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1 Introduction

In the last decades, Convex Analysis has taken place in the field of Mathematics, as an interesting and useful branch in order to study different problems related to pure and applied sciences, and has been extended and generalized in several directions using different new techniques. Several inequalities, for example, Hermite-Hadamard and Ostrowski inequalities using convex functions and their variant forms had been developed and studied, see [4, 9, 12, 13, 15, 18]. The first of them, Hermite-Hadamard inequality, is one the most important inequalities related to convex function and in recent years, a lot of attention has received to derive Hermite-Hadamard type inequalities for various types of convex functions.

Also, the concept of convexity has had a great evolution due to its wide application in various fields of science. In the last years generalizations of the convexity have arisen, like: log-convexity [3],$s-$convexity in the first and second sense [2, 6, 17, 23], $m-$convexity [5, 22], $P-$convexity [1], $MT-$convexity [19], $\varphi-$convexity [16], $(k, h_1, h_2)-$convexity [7, 8], $(m, h_1, h_2)-$convexity [25], and others.

This work is motivated by the results obtained by S.S. Dragomir [14], S.S. Dragomir and B. Bond [10] , M. A. Noor, F. Qi and M. U. Awan [21], and M. Tunç and A. Açikel [26] in the works whose studies are about the Hermite-Hadamard inequality using $\log-$convex functions. Here, a new class of generalized convex functions, called $\log-(m, h_1, h_2)-$convex function, are introduced, in addition, some properties of this class are proved and some Hermite Hadamard type inequalities,using this new definition, are established.

2 Preliminaries

This section contains some definitions concerning the area of generalized convexity, in particular those that involve the concept of logarithmically convex functions or briefly $\log-$convex functions.

As C. Nicolescu and L.E. Persson appointed in [20], a function $f : I \to \mathbb{R}_+$, where I is an interval, is called $\log-$convex if the inequality

$$f\left(tx + (1-t)y\right) \leq [f(x)]^t\,[f(y)]^{1-t} \tag{1}$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Since the logarithmic function is increasing in $(0, \infty)$ the inequality (1) can be written as

$$\log f\left(tx + (1-t)y\right) \leq t \log f(x) + (1-t) \log f(y)$$

being it equivalent to (1).

The work of M.A. Noor, F. Qi and M.U. Awan [21], using their owns words, was inspired in others, and in particular to that wrote by S. Varošanec [28]. They established that given a non-negative function $h : J \to \mathbb{R}$ with $[0, 1] \subset J$ and $h \not\equiv 0$ and a function $f : I \to R_+$, it is said that f is a $\log-h-$convex if the inequality

$$f\left(tx + (1-t)y\right) \leq [f(x)]^{h(t)}\,[f(y)]^{h(1-t)} \tag{2}$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

In the same way M. Tunç and E. Yüksel in [27] introduced the following definition: A function $f : [0, b] \to \mathbb{R}_+$ is said to be logarithmic $m-$convex or briefly log −m−convex if the inequality

$$f\left(tx + m(1-t)y\right) \leq \left[f(x)\right]^t \left[f(y)\right]^{m(1-t)} \quad (3)$$

holds for some fixed $m \in (0,1]$, and for all $x, y \in [0,b]$ and $t \in [0,1]$.

Obviously, if $m = 1$ in (3) then f is just a logarithmic convex function.

Using the concepts of s−convex function in the second sense and log−convex function, B-Y. XI and F. Qi in [24] introduced the concept of $log - s$−convex functions in the second sense: A function $f : I \to R_+$ is called log −s−convex function in the second sense for some fixed $s \in (0,1]$ if the inequality

$$f\left(tx + (1-t)y\right) \leq \left[f(x)\right]^{t^s} \left[f(y)\right]^{(1-t)^s} \quad (4)$$

holds for all $x, y \in I$ and $t \in [0,1]$. If $s = 1$ in (4) then this definition coincides with the inequality (1).

Also, M.A. Noor, F. Qi, and M.U. Awan in [21] introduced the following concepts. First: A function $f : I \to R_+$ is called log −P−convex function if the inequality

$$f\left(tx + (1-t)y\right) \leq f(x)f(y) \quad (5)$$

holds for all $x, y \in I$ and $t \in [0,1]$. And secondly: A function $f : I \to R_+$ is called log −Q−convex function if the inequality

$$f\left(tx + (1-t)y\right) \leq \left[f(x)\right]^{1/t} \left[f(y)\right]^{1/(1-t)} \quad (6)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Following the previous constructions of generalized convexity, M. Tunç and A. Açikel in [26] introduced the concept of (α, β)−logarithmically convexity in the first and second sense as follows: A function $f : I \subset \mathbb{R}_0 \to \mathbb{R}_+$ is said to be (α, β)−logarithmically convex in the first sense if the inequality

$$f\left(tx + (1-t)y\right) \leq \left[f(x)\right]^{t^\alpha} \left[f(y)\right]^{\left(1-t^\beta\right)} \quad (7)$$

holds for some $\alpha, \beta \in (0,1]$ and for all $x, y \in I$ and $t \in [0,1]$, and a function $f : I \subset \mathbb{R}_0 \to \mathbb{R}_+$ is said to be (α, β)−logarithmically convex in the second sense if the inequality

$$f\left(tx + (1-t)y\right) \leq \left[f(x)\right]^{t^\alpha} \left[f(y)\right]^{(1-t)^\beta} \quad (8)$$

holds for some $\alpha, \beta \in (0,1]$ and for all $x, y \in I$ and $t \in [0,1]$.

For the development of this work it will be necessary the following Lemma proved by S. S. Dragomir and R.P. Agarwal in [11].

Lemma 2.1 *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $f' \in L\left([a,b]\right)$ the the following inequality holds*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx$$

$$= \frac{b-a}{2}\int_0^1\int_0^1 \left(f'\left(ta+(1-t)b\right) - f'\left(sa+(1-s)b\right)\right)(s-t)dtds$$

3 Main results

This section is subdivided into two subsections. In the first of them the basic definition of a new class of log-convex functions depending on a certain parameter and some pair of non-negative functions is introduced, also some properties related are proved. In the second subsection some integral inequalities are established.

3.1 The class of $\log$ -(m, h_1, h_2) -convex functions

Definition 3.1 *Let $h_1, h_2 : [0,1] \to \mathbb{R}$ non-negative functions and $m \in (0,1]$. A function $f : I \subset \mathbb{R} \to \mathbb{R}_+$, where I is an interval, is called* log -(m, h_1, h_2) *-convex if the inequality*

$$f(tx + m(1-t)y) \leq [f(x)]^{h_1(t)} [f(y)]^{mh_2(t)} \tag{9}$$

or

$$\log f(tx + m(1-t)y) \leq h_1(t) \log f(x) + mh_2(t) \log f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

For some values of m, and particular functions h_1 and h_2 it is had the following:

1. If $m = 1, h_1(t) = t$ and $h_2(t) = 1-t$ for all $t \in [0,1]$ then the inequality (9) becomes in inequality (1) valid for $\log-$convex functions.
2. If $m = 1$, $h_1(t) = h(t)$ and $h_2(t) = h(1-t)$ for $t \in [0,1]$ and some non-negative function $h : J \to \mathbb{R}$ with $[0,1] \subset J$ and $h \not\equiv 0$, then (9) becomes in inequality (2) for $\log-h-$convex functions.
3. If $h_1(t) = t$ and $h_2(t) = 1-t$ for all $t \in [0,1]$ the inequality (9) becomes in inequality (3) for $\log-m-$convex functions.
4. If $m = 1, h_1(t) = t^s$ and $h_2(t) = (1-t)^s$ for all $t \in [0,1]$, for some fixed $s \in (0,1]$ then the inequality (9) becomes in inequality (4) for $\log-s-$convex functions in the second sense.
5. If $m = 1, h_1(t) = h_2(t) = 1$ for all $t \in [0,1]$ then the inequality (9) becomes in inequality (5) for $\log-P-$convex functions.
6. If $m = 1$ and $.h_1(t) = 1/t$ and $h_2(t) = 1/(1-t)$ for all $t \in (0,1)$ then the inequality (9) becomes in inequality (6) for $\log-Q-$convex functions.
7. If $m = 1$ and $h_1(t) = t^\alpha$ and $h_2(t) = (1-t)^\beta$ for all $t \in [0,1]$, for some fixed $\alpha, \beta \in (0,1]$ then the inequality (9) becomes in inequality (8) for $\log-(\alpha,\beta)-$convex functions in the second sense.

8. If $m = 1$ and $h_1(t) = t^{\alpha}$ and $h_2(t) = 1 - t^{\beta}$ for all $t \in [0,1]$, for some fixed $\alpha, \beta \in (0,1]$ then the inequality (9) becomes in inequality (7) for $\log - (\alpha, \beta)$ −convex functions in the first sense.

Proposition 3.2 *Let $h_1, h_2 : [0,1] \to \mathbb{R}$ be two non-negative functions and $m \in (0,1]$. Let $f, g : I \subset \mathbb{R} \to \mathbb{R}_+$ be two* log *-(m, h_1, h_2) -convex functions and $a, b \in I$ with $a < b$. If $\alpha \in R_+$ then (fg) and $(\alpha f)$is a $\log - (m, h_1, h_2)$ −convex function.*

Proof. Let $f, g : I \subset \mathbb{R} \to \mathbb{R}_+$ be two $\log - (m, h_1, h_2)$ −convex functions and $a, b \in I$ with $a < b$. Then

$$\begin{aligned}\log(fg)(ta + m(1-t)b) &= \log f(ta + (1-t)b) + \log g(ta + (1-t)b) \\ &\leq h_1(t)\log f(a) + mh_2(t)\log f(b) \\ &\qquad\qquad + h_1(t)\log g(a) + mh_2(t)\log g(b) \\ &= h_1(t)(\log(fg)(a)) + mh_2(t)(\log(fg)(b)).\end{aligned}$$

If $\alpha \in R_+$ it is had that

$$\begin{aligned}\log(\alpha f)(ta + m(1-t)b) &= \log\alpha \log f(ta + (1-t)b) \\ &\leq \log\alpha\,(h_1(t)\log f(a) + mh_2(t)\log f(b)) \\ &= h_1(t)\log\alpha f(a) + mh_2(t)\log\alpha f(b).\end{aligned}$$

The proof is complete. ■

Proposition 3.3 *Let $h_1, h_2 : [0,1] \to \mathbb{R}$ be two non-negative functions and $m \in (0,1]$. Let $f_i : I \subset \mathbb{R} \to \mathbb{R}_+$ be $\log - (m, h_1, h_2)$ −convex functions for $i = 1, 2, ..., n$, and $a, b \in I$ with $a < b$. Then $\prod_{i=1}^{n} f_i$ is a $\log - (m, h_1, h_2)$ −convex function.*

Proof. Using induction and the Proposition 3.2 , suppose that for $i = n-1$ the affirmation holds then

$$\log\left(\prod_{i=1}^{n} f_i\right)(ta + m(1-t)b)$$

$$
\begin{aligned}
&= \log\left(f_n\prod_{i=1}^{n-1} f_i\right)(ta+m(1-t)b) \\
&= \log f_n\left(ta+m(1-t)b\right)+\log\left(\prod_{i=1}^{n-1} f_i\right)(ta+m(1-t)b) \\
&\leq h_1(t)\log f_n(a)+mh_2(t)\log f_n(b) \\
&\qquad +h_1(t)\log\left(\prod_{i=1}^{n-1} f_i\right)(a)+mh_2(t)\log\left(\prod_{i=1}^{n-1} f_i\right)(b) \\
&= h_1(t)\log\left(\prod_{i=1}^{n} f_i\right)(a)+mh_2(t)\log\left(\prod_{i=1}^{n} f_i\right)(b).
\end{aligned}
$$

The proof is complete. ■

Proposition 3.4 *Let $h_1, h_2 : [0,1] \to \mathbb{R}$ be two non-negative functions and $m \in (0,1]$. Let $\{f_i\}_{i=1}^{\infty}$ a sequence of $\log-(m, h_1, h_2)-$convex functions each of one defined in an interval $I \subset \mathbb{R}$ and $a, b \in I$ with $a < b$. If $f = \lim_{n\to\infty} f_i$ pointwise then f is a $\log-(m, h_1, h_2)-$convex function.*

Proof. Let $t \in [0,1]$ and $a, b \in I$ with $a < b$ then

$$
\begin{aligned}
f\left(ta+m(1-t)b\right) &= \lim_{i\to\infty} f_i\left(ta+m(1-t)b\right) \\
&\leq \lim_{i\to\infty}\left([f_i(a)]^{h_1(t)}\,[f_i(b)]^{mh_2(t)}\right) \\
&= \left(\lim_{i\to\infty}[f_i(a)]^{h_1(t)}\right)\left(\lim_{i\to\infty}[f_i(b)]^{mh_2(t)}\right) \\
&= [f(a)]^{h_1(t)}\,[f(b)]^{mh_2(t)}.
\end{aligned}
$$

The proof is complete. ■

3.2 Some inequalities of Hermite-Hadamard Type

Theorem 3.5 *Let $h_1, h_2 : [0,1] \to \mathbb{R}$ be two non-negative functions and $m \in (0,1]$. Let $f : [0, b/m] \to \mathbb{R}_+$ be a $\log-(m, h_1, h_2)-$convex function and $a < b$. If $\log f$ is integrable on $[0, b/m)$ then*

$$
f\left(\frac{a+b}{2}\right) \leq \left[\exp\left(\frac{1}{b-a}\int_a^b \log f(u)du\right)\right]^{h_1\left(\frac{1}{2}\right)}
\times\left[\exp\left(\frac{1}{b-a}\int_{a/m}^{b/m} \log f(u)du\right)\right]^{m^2h_2\left(\frac{1}{2}\right)} \tag{10}
$$

and

$$\exp\left(\frac{1}{b-a}\int_a^b \log f(u)du\right) \leq [f(a)]^{\int_0^1 h_1(t)dt}\left[f\left(\frac{b}{m}\right)\right]^{m\int_0^1 h_2(t)dt}. \tag{11}$$

Proof. From Definition 3.1 it is had that

$$\log f\left(\frac{x+y}{2}\right) \leq h_1\left(\frac{1}{2}\right)\log f(x) + mh_2\left(\frac{1}{2}\right)\log f\left(\frac{y}{m}\right).$$

Letting

$$x = ta + (1-t)b \text{ and } y = (1-t)a + tb$$

it is obtained that

$$\log f\left(\frac{a+b}{2}\right) \leq h_1\left(\frac{1}{2}\right)\log f\left(ta+(1-t)b\ \right)+mh_2\left(\frac{1}{2}\right)\log f\left(\frac{(1-t)a+tb}{m}\right)$$

for all $t \in [0,1]$.
Integrating over $t \in [0,1]$ it follows that

$$\log f\left(\frac{a+b}{2}\right) \leq h_1\left(\frac{1}{2}\right)\int_0^1 \log f\left(ta+(1+t)b\right)dt$$
$$+mh_2\left(\frac{1}{2}\right)\int_0^1 \log f\left(\frac{(1-t)a+tb}{m}\right)dt. \tag{12}$$

With the change of variable $u = ta + (1+t)b$ in the first integral and $u = ((1-t)a+tb)/m$ in the second integral of (12) then

$$\log f\left(\frac{a+b}{2}\right) \leq h_1\left(\frac{1}{2}\right)\frac{1}{b-a}\int_a^b \log f(u)du + h_2\left(\frac{1}{2}\right)\frac{m^2}{b-a}\int_{a/m}^{b/m}\log f(u)d_qu,$$

and taking exponential it is obtained that

$$f\left(\frac{a+b}{2}\right) \leq \left[\exp\left(\frac{1}{b-a}\int_a^b \log f(u)du\right)\right]^{h_1\left(\frac{1}{2}\right)}$$
$$\times\left[\exp\left(\frac{1}{b-a}\int_{a/m}^{b/m}\log f(u)d_qu\right)\right]^{m^2h_2\left(\frac{1}{2}\right)}.$$

For the second inequality it is had that

$$\log f\left(ta+(1+t)b\ \right) \leq h_1(t)\log f(a) + mh_2(t)\log f\left(\frac{b}{m}\right).$$

Integrating over $t \in [0,1]$

$$\int_0^1 \log f\left(ta+(1+t)b\ \right)dt \le \log f(a)\int_0^1 h_1\left(t\right)dt + m\log f\left(\frac{b}{m}\right)\int_0^1 h_2\left(t\right)dt,$$

and with the change of variable $u = ta + (1-t)b$ it follows that

$$\begin{aligned}\frac{1}{b-a}\int_a^b \log f(u)du &\le \log f(a)\int_0^1 h_1\left(t\right)dt + m\log f\left(\frac{b}{m}\right)\int_0^1 h_2\left(t\right)dt\\ &= \log\left([f(a)]^{\int_0^1 h_1(t)dt}\left[f\left(\frac{b}{m}\right)\right]^{m\int_0^1 h_2(t)dt}\right),\end{aligned}$$

and taking exponential we obtain the desired result

$$\exp\left(\frac{1}{b-a}\int_a^b \log f(u)du\right) \le [f(a)]^{\int_0^1 h_1(t)dt}\left[f\left(\frac{b}{m}\right)\right]^{m\int_0^1 h_2(t)dt}.$$

The proof is complete. ■

Corollary 3.6 *Let* $f:[a,b]\to\mathbb{R}_+$ *be a* $\log-$*convex function. If* $\log f$ *is integrable on* $[a,b]$ *then*

$$f\left(\frac{a+b}{2}\right) \le \exp\left(\frac{1}{(b-a)}\int_a^b \log f(u)du\right) \le \sqrt{[f(a)]\left[f\left(b\right)\right]}.$$

Proof. Letting $m=1$ and $h_1(t)=t, h_2(t)=1-t$ for $t\in[0,1]$ in Theorem 3.5 it is obtained that

$$h_1(1/2) = h_2(1/2) = \frac{1}{2}$$

and

$$\int_0^1 h_2\left(t\right)dt = \int_0^1 h_1\left(t\right)dt = \frac{1}{2}.$$

Replacing these values in the inequalities (10) and (11) it follows the desired result. ■

Corollary 3.6 coincides with that proved by S.S. Dragomir in [14].

Corollary 3.7 *Let* $f:[a,b]\to\mathbb{R}$ *be a log-*$s-$*convex function in the second sense. If* $\log f$ *is integrable on* $[a,b]$ *then*

$$\left[f\left(\frac{a+b}{2}\right)\right]^{2^{s-1}} \le \exp\left(\frac{1}{(b-a)}\int_a^b \log f(u)du\right) \le \left([f(a)]\left[f\left(b\right)\right]\right)^{\frac{1}{s+1}}.$$

Proof. Letting $m = 1$ and $h_1 = t^s, h_2 = (1-t)^s$ for $t \in [0,1]$ in Theorem 3.5 it is had that

$$h_1(1/2) = h_2(1/2) = \frac{1}{2^s}$$

and

$$\int_0^1 h_1(t)\,dt = \int_0^1 h_2(t)\,dt = \frac{1}{s+1}.$$

Replacing these values in the inequalities (10) and (11) it follows the desired result. ■

Corollary 3.7 coincides with Corollary 3.3 proved by M.A. Noor, F. Qi, and M.U. Awan in [21].

Corollary 3.8 *Let $f : [a,b] \to \mathbb{R}$ be a $\log -h-$convex function. If $\log f$ is integrable on $[a,b]$, then the following inequality holds*

$$\left[f\left(\frac{a+b}{2}\right)\right]^{1/2h(1/2)} \leq \exp\left(\frac{1}{b-a}\int_a^b \log f(u)du\right) \leq \left([f(a)]\,[f(b)]\right)^{\int_0^1 h(t)dt}.$$

Proof. If in Theorem 3.5 it is chosen $m = 1, h_1(t) = h(t), h_2(t) = h(1-t)$ for $t \in [0,1]$, then result follows. ■

Corollary 3.8 coincides with the Theorem 3.1 proved by M.A. Noor, F. Qi, and M.U. Awan in [21] .

Corollary 3.9 *Let $f : [a,b] \to \mathbb{R}_+$ be a log $-$ $P-$convex function. If $\log f$ is integrable on $[a,b]$, then the following inequality holds*

$$f\left(\frac{a+b}{2}\right) \leq \left[\exp\left(\frac{2}{b-a}\int_a^b \log f(u)du\right)\right] \leq [f(a)]\,[f(b)].$$

Proof. Letting $m = 1, h_1(t) = h_2(t) = 1$ for $t \in [0,1]$ in Theorem 3.5 then we have

$$h_1(1/2) = h_2(1/2) = 1$$

and

$$\int_0^1 h_1(t)\,dt = \int_0^1 h_2(t)\,dt = 1$$

so, replacing in inequalities (10) and (11) we obtain the desired result. ■

Corollary 3.9 coincides with Corollary 3.4 proved by M.A. Noor, F. Qi, and M.U. Awan in [21].

Corollary 3.10 *Let $f : [a,b] \to \mathbb{R}$ be a log $-$ $Q-$convex function. If $\log f$ is integrable on $[a,b]$, then the following inequality holds*

$$\left[f\left(\frac{a+b}{2}\right)\right]^{1/4} \leq \exp\left(\frac{1}{b-a}\int_a^b \log f(u)du\right).$$

Proof. Letting $m=1, h_1(t)=1/t$ and $h_2(t)=1/(1-t)$ in Theorem 3.5 we have

$$h_1\left(1/2\right)=h_2(1/2)=2.$$

Replacing these values in inequality (10) we find the looked inequality. ■

Corollary 3.10 coincide with the Corollary 3.5 proved by M.A. Noor, F. Qi, and M.U. Awan in .[21]

Corollary 3.11 *Let $f: I\subset R_0\to\mathbb{R}_+$ be a $(\alpha,\beta)-$logarithmically convex function. If $\log f$ is integrable on $[a,b]$, then the following inequality holds*

$$\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{2^{\alpha+\beta}}{2^{\beta}+2^{\alpha+\beta}-2^{\alpha}}}\leq\exp\left(\frac{1}{b-a}\int_a^b\log f(u)du\right)\leq\left[f(a)\right]^{\frac{1}{\alpha+1}}\left[f\left(b\right)\right]^{\frac{\beta}{\beta+1}}.$$

Proof. Letting $m=1, h_1(t)=t^{\alpha}$ and $h_2(t)=1-t^{\beta}$ in Theorem 3.5 it is had that

$$h_1\left(1/2\right)=\frac{1}{2^{\alpha}}, h_2(1/2)=1-\frac{1}{2^{\beta}},$$

$$\int_0^1 h_1\left(t\right)dt=\frac{1}{\alpha+1}\text{ and }\int_0^1 h_2\left(t\right)dt=1-\frac{1}{\beta+1}.$$

Replacing in inequalities (10) and (11) we achieve the desired result. ■

Corollary 3.12 *Let $f:[0,b/mI\subset R_0\to\mathbb{R}_+$ be a $(\alpha,\beta)-$logarithmically convex function. If $\log f$ is integrable on $[a,b]$, then the following inequality holds*

$$\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{2^{\alpha+\beta}}{2^{\alpha}+2^{\beta}}}\leq\exp\left(\frac{1}{b-a}\int_a^b\log f(u)du\right)\leq\left[f(a)\right]^{\frac{1}{\alpha+1}}\left[f\left(b\right)\right]^{\frac{1}{\beta+1}}.$$

Proof. Letting $m=1, h_1(t)=t^{\alpha}$ and $h_2(t)=(1-t)^{\beta}$ in Theorem 3.5 we have

$$h_1\left(1/2\right)=\frac{1}{2^{\alpha}}, h_2(1/2)=\frac{1}{2^{\beta}},$$

$$\int_0^1 h_1\left(t\right)dt=\frac{1}{\alpha+1}\text{ and }\int_0^1 h_2\left(t\right)dt=\frac{1}{\beta+1}$$

Replacing in inequalities (10) and (11) it follows that

$$\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{2^{\alpha+\beta}}{2^{\alpha}+2^{\beta}}}\leq\exp\left(\frac{1}{b-a}\int_a^b\log f(u)du\right)\leq\left[f(a)\right]^{\frac{1}{\alpha+1}}\left[f\left(b\right)\right]^{\frac{1}{\beta+1}}$$

The proof is complete. ■

Theorem 3.13 *Let* $h_1,h_2:[0,1]\to\mathbb{R}$ *be two non-negative functions and* $m\in(0,1]$. *Let* $f:[0,b/m]\to\mathbb{R}_+$ *be a* $\log-(m,h_1,h_2)-$*convex function and* $a<b$. *Then the inequalities*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b[f(x)]^{h_1(t)}\left[f\left(\frac{(x-mb)(b-ma)}{(a-mb)}+ma\right)\right]^{mh_2(t)}dx \tag{13}$$

and

$$\frac{1}{b-a}\int_a^b\sqrt{f(x)f\left(\frac{(x-mb)(b-ma)}{(a-mb)}+ma\right)}dt$$
$$\leq\int_0^1\sqrt{[f(a)]^{h_1(t)+mh_2(t)}[f(b)]^{h_1(t)+mh_2(t)}}\,dt \tag{14}$$

hold.

Proof. For the inequality (14) we use the fact that f is *log*$-(m,h_1,h_2)-$convex, henceforth it follows that

$$f(ta+(1-t)b)\leq[f(a)]^{h_1(t)}[f(b/m)]^{mh_2(t)}$$

and

$$f(tb+(1-t)a)\leq[f(b)]^{h_1(t)}[f(a/m)]^{mh_2(t)},$$

multiplying both inequalities it follows that

$$f(ta+(1-t)b)f(tb+(1-t)a)\leq[f(a)]^{h_1(t)}[f(b/m)]^{mh_2(t)}\;[f(b)]^{h_1(t)}[f(a/m)]^{mh_2(t)},$$

and taking square roots we have

$$\sqrt{f(ta+m(1-t)b)f(tb+m(1-t)a)}\leq\sqrt{[f(a)]^{h_1(t)+mh_2(t)}[f(b)]^{h_1(t)+mh_2(t)}}.$$

Integrating both members over $t\in[0,1]$

$$\int_0^1\sqrt{f(ta+m(1-t)b)f(tb+m(1-t)a)}dt$$
$$\leq\int_0^1\sqrt{[f(a)]^{h_1(t)+mh_2(t)}[f(b)]^{h_1(t)+mh_2(t)}}dt.$$

With the change of variables $x=ta+m(1-t)b$ then

$$\frac{1}{b-a}\int_a^b\sqrt{f(x)f\left(\frac{(x-mb)(b-ma)}{(a-mb)}+ma\right)}dt$$
$$\leq\int_0^1\sqrt{[f(a)]^{h_1(t)+^{mh_2(t)}}\;[f(b)]\,h_1(t)^{mh_2(t)}}\,dt.$$

For the inequality (13) it must be observed that

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{ta+(1-t)b+tb+(1-t)a}{2}\right) \\ &\leq [f(ta+(1-t)b)]^{h_1(t)}\left[f\left(\frac{tb+(1-t)a}{m}\right)\right]^{mh_2(t)}. \end{aligned}$$

So, integrating over $t\in[0,1]$ and using the change of variable $x=ta+(1-t)b$ it follows that

$$f\left(\frac{a+b}{2}\right)\leq\int_0^1 [f(x)]^{h_1(t)}\left[f\left(\frac{(x-mb)(b-ma)}{(a-mb)}+ma\right)\right]^{mh_2(t)}dx.$$

The proof is complete. ■

Remark 3.14 *Letting $m=1, h_1(t)=h_2(t)=1/2$ for all $t\in[0,1]$ in Theorem 3.13 it is had the following inequality for log-convex functions*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b\sqrt{f(x)f(a+b-x)}dx\leq\sqrt{[f(a)][f(b)]}$$

and this make coincidence with the integral inequality proved in Theorem 2.1 in [10].

Theorem 3.15 *Let $h_1,h_2:[0,1]\to\mathbb{R}$ be two non-negative functions and $m\in(0,1]$. Let $f:[0,b/m]\to\mathbb{R}_+$ be a $\log-(m,h_1,h_2)-$convex function and $a<b$. Then the following inequality holds*

$$\begin{aligned} &\exp\left(\frac{1}{b-a}\int_a^b\log(f(x)f(a+b-x))\,dx\right) \\ &\qquad\leq[f(a)f(b)]^{\int_0^1 h_1(t)dt}[f(a/m)f(b/m)]^{m\int_0^1 h_2(t)dt}. \qquad (15) \end{aligned}$$

Proof. From Definition 3.1 we have

$$\log f(ta+(1-t)b)\leq h_1(t)\log f(a)+mh_2(t)\log f(b/m)$$

and

$$\log f(tb+(1-t)a)\leq h_1(t)\log f(b)+mh_2(t)\log f(a/m).$$

Adding these last inequalities

$$\begin{aligned} \int_0^1 \log f(ta+(1-t)b) &+ \log f(tb+(1-t)a)\,dt \\ &\leq (\log f(a)+\log f(b))\int_0^1 h_1(t)dt \\ &\qquad +m(\log f(a/m)+\log f(b/m))\int_0^1 h_2(t). \end{aligned}$$

With a change of variable $x = ta + (1 - t)b$ then we can write

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b \log\left(f(x)f\left(a+b-x\right)\right)dx \\
&\qquad\leq\quad \left(\log f(a)+\log f(b)\right)\int_0^1 h_1(t)dt \\
&\qquad\qquad +m\left(\log f(a/m)+\log f(b/m)\right)\int_0^1 h_2(t),
\end{aligned}$$

and taking exponential it follows the desired result

$$\begin{aligned}
&\exp\left(\frac{1}{b-a}\int_a^b \log\left(f(x)f(a+b-x)\right)dx\right) \\
&\qquad\leq [f(a)f(b)]^{\int_0^1 h_1(t)dt}\,[f(a/m)f(b/m)]^{m\int_0^1 h_2(t)dt}
\end{aligned}$$

The proof is complete. ■

Remark 3.16 *If $m = 1$ and $h_1(t) = t$ and $h_2(t) = 1 - t$ for all $t \in [0, 1]$ it is had the following inequality for log-convex functions*

$$\int_0^1 h_1(t)dt = \int_0^1 h_1(t)dt = \frac{1}{2},$$

replacing these values in inequality (15) *it is obtained*

$$\exp\left(\frac{1}{b-a}\int_a^b \log\left(f(x)f(a+b-x)\right)dx\right) \leq [f(a)f(b)]\,.$$

Theorem 3.17 *Let $h_1, h_2 : [0, 1] \to \mathbb{R}$ be two non-negative functions and $m \in (0, 1]$. Let $f : [0, b/m] \to \mathbb{R}_+$ be a $\log - (m, h_1, h_2)$ −convex function and $a < b$. If f is integrable on $[a, b]$ then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\qquad\leq 2\left(b-a\right)\int_0^1 \left|f'(a)\right|^{h_1(t)}\left|f'\left(b/m\right)\right|^{mh_2(t)}\left(\frac{1}{2}-t+t^2\right)dt
\end{aligned}$$

Proof. Using Lemma 2.1 and the triangular inequality we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_0^1 f(x)dx\right|$$

$$\leq \frac{b-a}{2}\int_0^1\int_0^1 |s-t|\left(|f'(ta+(1-t)b)|+|f'(sa+(1-s)b)|\right)dtds$$
$$= \frac{b-a}{2}\left(\int_0^1\int_0^1 |s-t|\,|f'(a)|^{h_1(t)}\,|f'(b/m)|^{mh_2(t)}\,dtds\right.$$
$$\left.+\int_0^1\int_0^1 |s-t|\,|f'(a)|^{h_1(s)}\,|f'(b/m)|^{mh_2(s)}\,dtds\right)$$
$$= \frac{b-a}{2}\left(\int_0^1 |f'(a)|^{h_1(t)}\,|f'(b/m)|^{mh_2(t)}\left(\int_0^1|s-t|\,ds\right)dt\right.$$
$$\left.+\int_0^1 |f'(a)|^{h_1(s)}\,|f'(b/m)|^{mh_2(s)}\left(\int_0^1|s-t|\,dt\right)ds\right)$$
$$= 2(b-a)\int_0^1 |f'(a)|^{h_1(t)}\,|f'(b/m)|^{mh_2(t)}\left(\frac{1}{2}-t+t^2\right)dt.$$

The proof is complete. ■

Remark 3.18 *Letting* $h_1(t)=t^\alpha$ *and* $h_2(t)=1-t^\alpha$ *for some fixed* $m,\alpha\in(0,1]$ *in Theorem 3.17 it is obtained Theorem 2.2 in [27], and in the particular case when* $\alpha=1$ *and* $m=1$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_0^1 f(x)dx\right|$$
$$\leq (b-a)\int_0^1 |f'(a)|^t\,|f'(b)|^{(1-t)}\left(\frac{1}{2}-t+t^2\right)dt$$
$$= (b-a)\,|f'(b)|\int_0^1 \eta^t\left(\frac{1}{2}-t+t^2\right)dt$$

where $\eta=|f'a|\,/\,|f'b|$. *If* $\eta=1$ *then the integral*

$$\int_0^1 \eta^t\left(\frac{1}{2}-t+t^2\right)dt=\int_0^1\left(\frac{1}{2}-t+t^2\right)dt=\frac{1}{3},$$

so, we obtain that

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_0^1 f(x)dx\right|\leq\frac{(b-a)\,|f'(b)|}{3}. \tag{16}$$

In the case of $\eta<1$ *it follows*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_0^1 f(x)dx\right|$$
$$\leq \frac{(b-a)\,|f'(b)|}{2}\,\frac{(\eta-1)\ln^2\eta-2(\eta+1)\ln\eta+4(\eta-1)}{\ln^3\eta}. \tag{17}$$

4 Conclusion

In this work a new definition of a class of generalized convex functions called log $-(m, h_1, h_2)-$convex functions was introduced and some properties associated with class were proved. Furthermore, some integrals inequalities were established, in particular those related with the Hermite-Hadamard Inequality and from these results similar inequalities for other type of generalized log-convex functions were deduced.

5 Open Problem

It is a well-known fact the validity of the inequality of Hermite-Hadamard for convex functions, and indeed, there are many works carried out that prove the aforementioned inequality for generalizations of convex functions. Therefore, there are some questions related to this topic:

1. Is it possible to prove the Hermite-Hadamard inequality for log-(m, h_1, h_2)-convex functions whose domains are convex sets in $\mathbb{R}^2$?
2. Is it possible to prove the Hermite-Hadamard inequality for log-(m, h_1, h_2)-convex functions whose domains are convex sets in normed spaces?

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