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# A Two-State Multiserver Queueing System with Retrials

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**Abstract**

***In the present paper, a multiserver retrial queueing model is considered. Primary customers arrive through Poisson process. If an arriving customer finds one or more servers free then he obtains service immediately. On the other hand, if the primary customer finds all servers busy, he joins the orbit and tries to get the service after some random amount of time. Retrial times also follow Poisson distribution. Service times for each server are exponentially distributed. Time-dependent probabilities of exact number of arrivals and exact number of departures at when all servers are busy or when all servers are free for the system are obtained by solving the difference differential equations recursively. Some important performance measures and special cases also discussed.***

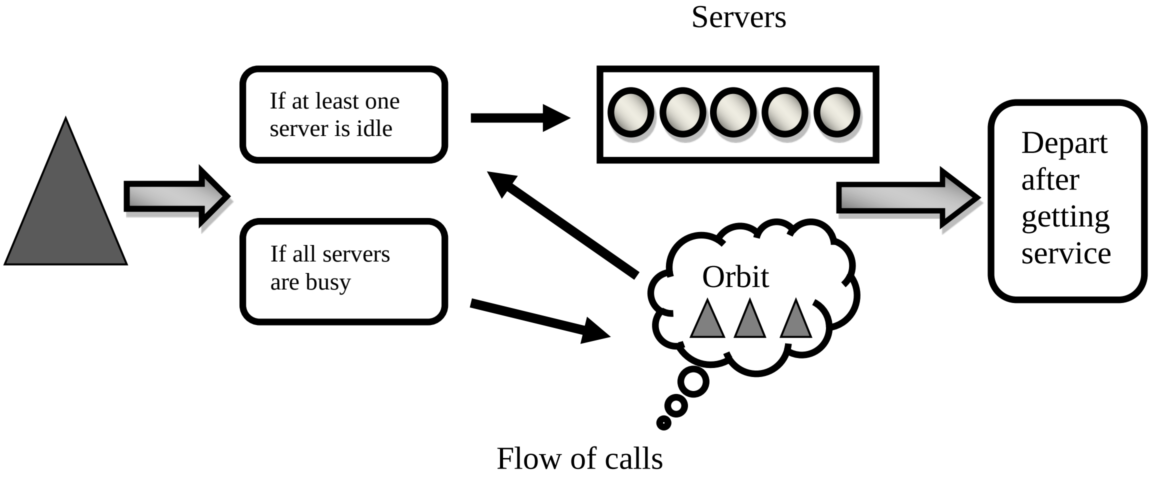
***Keywords***: *Arrivals, Departures, Multiserver, Queueing, Retrial.*

***2010 Mathematics Subject Classification***: *60K25, 90B22, 68M20.*

## 1 Introduction

In many queueing systems, it may sometimes happen that a customer that did not receive service at the first attempt from the server, then he tries again and again to avail the required service. These types of situations give rise to special types of queues known as retrial queues. Such queues are characterized by the phenomenon that a customer if can not get service immediately then he joins the virtual pool of customers called orbit from where he tries again and again until he gets the service.

The customers retrying for service are known as retrial customers. The retrial queues are visible in all day-to-day congestion situations from supermarkets to ATMs, from hospitals to admission counters, etc. wherein the customer tries again and again for service from the orbit. Queueing models with retrial customers are suitable for many service systems; e.g. in call centers, if a person is busy on another call then we often hear the message ‘the system is busy at this moment, call again after some time’. Retrial queues have been widely used in data transfer via telephone networks, LANs under RMA protocols, radio and cellular networks.




The modeling and analysis of queueing systems, especially of the retrial queueing systems, have attracted many queue theorists (logicians) since past many years. The detailed account of retrial queues along with their applications can be found in the books by Falin and Templeton [3] and Artalejo and Corral [6]. The elaborate surveys on retrial queues are found in articles by Yang and Templeton [15], Artalejo [7,8], Artalejo & Falin [9], Artalejo [10] and many more.

Multichannel retrial queueing systems serve as a suitable model in many queueing situations as in telephone exchanges, toll booths, information processing, transmission centers etc. Falin and Artalejo [4] considered a multiserver queueing system with unlimited waiting room. Neuts and Rao [11] considered the numerical solution of a multiserver retrial queueing model. Chakravorthy et. al. [14] considered a multiserver retrial queueing model in which customers arrive according to Markovian process and performed steady state analysis of the model. Pegden and Rosenshine [2], who analyzed the M/M/1 queueing system in which the state of the system is given by $i$ and $j$, where ‘$i$’ is the number of arrivals in the system and ‘$j$’ is the number of departures from the system until time $t$. This measure provides the better knowledge of a queueing system such as the probability of the exact number of units arrived by time $t$, probability of the exact number of units departed by time $t$, and also some other related information.

The present paper studies a retrial queueing model with multiserver. In this paper, we obtain the time dependent probabilities for the exact number of arrivals in the system and the exact number of departures from the system when $m$ ($m = 0,1, \dots c$) channels are free or busy.

The present work has been organized in the following manner: Section 2 describes the model and states various assumptions required to formulate the model. Section 3 defines the two dimensional state model and the difference differential equations describing the system. Section 4 deals with the time dependent solution of the model. In Section 5 some important performance measures along with some special cases are derived. The paper ends with a suitable conclusion.

# 2 Model Description

We consider a two-state multiserver retrial queueing system in which calls arrive according to Poisson process. On finding all the servers busy, calls go to some virtual place (referred as an orbit) and repeat their request for service from the orbit after some random amount of time. For distribution of arrivals, service times and retrials, we make use of the following assumptions and notations:

1) Arrivals of primary calls follow Poisson distribution with parameter $\lambda$.
2) Repeated calls follow Poisson distribution with parameter $\theta$.
3) Service times are exponentially distributed with parameter $\mu$.
4) Stochastic processes involved viz. arrivals of units, departures of units and retrials are statistically independent.

Laplace transformation $\bar{f}\,(s)$ of $f(t)$ is given by

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{dt}, \mathrm{Re(s)} > 0.$$

The Laplace inverse of

$$\frac{Q(p)}{P(p)} \text{ is } \sum_{k=1}^{n} \sum_{l=1}^{m_k} \frac{t^{m_k-l}\, e^{a_k t}}{(m_k-l)!(l-1)!} \times \frac{d^{l-1}}{dp^{l-1}} \left(\frac{Q(p)}{P(p)}\right) (p-a_k)^{m_k},$$
$$\forall\, p = a_k,\ a_i \neq a_k \text{ for } i \neq k.$$

where,

$$P(p) = (p-a_1)^{m_1} (p-a_2)^{m_2} \ldots\ldots\ldots (p-a_n)^{m_n}.$$

$Q(p)$ is a polynomial of degree < $m_1 + m_2 + m_3 + \ldots\ldots\ldots\ldots m_n - 1$.

The Laplace inverse of $\bar{N}^{a,b,c}_{n_1,n_2,n_3}(s) = \frac{1}{(s+a)^{n_1}\,(s+b)^{n_2}\,(s+c)^{n_3}}$ is

$$N^{a,b,c}_{n_1,n_2,n_3}(t) = \sum_{l=1}^{n_3} \sum_{m=1}^{l} \frac{e^{-at}\ t^{n_3-l}\,(-1)^{m+1} \binom{l-1}{m-1} \left(\prod_{g_1=0}^{l-m-1}(n_1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(n_2+g_2)\right)}{(n_3-l)!(m-1)!\,(b-a)^{n_2+m-1}\,(c-a)^{n_1+l-m}}$$
$$+\sum_{l=1}^{n_2} \sum_{m=1}^{l} \frac{e^{-bt}\ t^{n_2-l}\,(-1)^{m+1} \binom{l-1}{m-1} \left(\prod_{g_1=0}^{l-m-1}(n_1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(n_3+g_2)\right)}{(n_2-l)!(m-1)!\,(a-b)^{n_3+m-1}\,(c-b)^{n_1+l-m}}$$
$$+\sum_{l=1}^{n_1} \sum_{m=1}^{l} \frac{e^{-ct}\ t^{n_1-l}\,(-1)^{m+1} \binom{l-1}{m-1} \left(\prod_{g_1=0}^{l-m-1}(n_2+g_1)\right)\left(\prod_{g_2=0}^{m-2}(n_3+g_2)\right)}{(n_1-l)!(m-1)!\,(a-c)^{n_3+m-1}\,(b-c)^{n_2+l-m}}$$

If $L^{-1}\{f(s)\} = F(t)$ and $L^{-1}\{g(s)\} = G(t)$, then

$L^{-1}\{f(s)\, g(s)\} = \int_0^t F(\mathrm{u}) G(\mathrm{t}-\mathrm{u})\,\mathrm{du}$ = $F * G, F * G$ is called the convolution of $F$ and $G$.

# 3 The Two-Dimensional State Model

## 3.1 Definitions

$P_{i,j,0}(t)$ = The probability that there are exactly $i$ arrivals in the system and $j$ departures from the system by time $t$ when all the servers are idle.

$P_{i,j,m}(t)$ = The probability that there are exactly $i$ arrivals in the system and $j$ departures from the system by time $t$ when '$m$' servers are busy. $1 \leq m \leq c-1$.

$P_{i,j,c}(t)$ = The probability that there are exactly $i$ arrivals in the system and $j$ departures from the system by time $t$ when all the '$c$' servers are busy.
$P_{i,j}$ = The probability that there are exactly $i$ arrivals in the system and $j$ departures from the system by time $t$.

$$P_{i,j}(t) = P_{i,j,0}(t) + \sum_{m=1}^{c-1} P_{i,j,m}(t) + P_{i,j,c}(t), \ \forall \ i,j \ \& \ i \geq j$$

also

$$P_{i,j,c}(t) = 0 \ \& \ P_{i,j,m}(t) = 0 \text{ for } i \leq j, 1 \leq m \leq c-1; P_{i,j,0}(t) = 0, i < j.$$

Initially

$$P_{0,0,0}(0) = 1; \ P_{i,j,0}(0) = 0,$$
$$P_{i,j,c}(0) = 0 \ \& \ P_{i,j,m}(0) = 0, \forall \ i,j \ \neq 0 \ \& \ 1 \leq m < c.$$

## 3.2 The difference-differential equations governing the system are

$$\frac{d}{dt} P_{i,j,0}(t) = -(\lambda + (i-j)\theta)P_{i,j,0}(t) + \mu P_{i,j-1,1}(t),$$
$$i \geq j \geq 0. \qquad (1)$$

$$\frac{d}{dt} P_{i,j,m}(t) = -(\lambda + m\mu + (i-j-m)\theta)P_{i,j,m}(t) + \lambda P_{i-1,j,m-1}(t) +$$
$$(i-j-(m-1)\theta)P_{i,j,m-1}(t) + (m+1)\mu P_{i,j-1,m+1}(t),$$
$$i > j \geq 0, 1 \leq \ m < c. \qquad (2)$$

$$\frac{d}{dt} P_{i,j,c}(t) = -(\lambda + c\mu)P_{i,j,c}(t) + \lambda P_{i-1,j,c-1}(t) + \lambda P_{i-1,j,c}(t) +$$
$$(i-j-(c-1)\theta)P_{i,j,c-1}(t), i > j \geq 0. \qquad (3)$$

Using Laplace transformation $\bar{f}\ (s)$ of $f(t)$ given by

$$\bar{f}\ (s) = \int_0^\infty e^{-st} \ f(t)\mathrm{dt}, \quad Re(s) > 0,$$

in the equations (1) - (3) along with the initial conditions. We have

$$(s + \lambda + (i-j)\theta)\bar{P}_{i,j,0}(s) = \mu \bar{P}_{i,j-1,1}(s), \quad i \geq j \geq 0. \qquad (4)$$

$$(s + \lambda + m\mu + (i-j-m)\theta)\bar{P}_{i,j,m}(s) = \lambda \bar{P}_{i-1,j,m-1}(s)$$
$$+ (i-j-(m-1)\theta)\bar{P}_{i,j,m-1}(s) + (m+1)\mu \bar{P}_{i,j-1,m+1}(s),$$
$$i > j \geq 0, 1 \leq m < c. \qquad (5)$$

$$(s + \lambda + c\mu)\bar{P}_{i,j,c}(s) = \lambda \bar{P}_{i-1,j,c-1}(s) + \lambda \bar{P}_{i-1,j,c}(s)$$
$$+ (i-j-(c-1)\theta)\bar{P}_{i,j,c-1}(s), \quad i > j \geq 0 \qquad (6)$$

# 4 Solution of the Problem

Solving equations (4) to (6) recursively, we have

$$\bar{P}_{0,0,0}(s) = \frac{1}{s+\lambda}. \tag{7}$$

$$\bar{P}_{i,i,0}(s) = \frac{\mu}{(s+\lambda)}\,\bar{P}_{i,i-1,1}(s),\ i \geq 1. \tag{8}$$

$$\bar{P}_{m,0,m}(s) = \frac{\lambda}{(s+\lambda+m\mu)}\,\bar{P}_{m-1,0,m-1}(s), \quad 1 \leq m \leq c-1. \tag{9}$$

$$\bar{P}_{i,i-m,m}(s) = \frac{\lambda}{(s+\lambda+m\mu)}\bar{P}_{i-1,i-m,m-1}(s) + \frac{(m+1)\mu}{(s+\lambda+m\mu)}\bar{P}_{i,(i-m-1),m+1}(s),$$
$$m = 1 \text{ to } c-2;\ i = m+1 \text{ to } c-1. \tag{10}$$

$$\bar{P}_{c,1,c-1}(s) = \frac{\lambda}{(s+\lambda+(c-1)\mu)}\bar{P}_{c-1,1,c-2}(s) + \frac{c\mu}{(s+\lambda+(c-1)\mu)}\bar{P}_{c,0,c}(s) \tag{11}$$

$$\bar{P}_{i,1,c-1}(s) = \frac{c\mu}{(s+\lambda+(c-1)\mu+(i-j-(c-1))\theta}\ \frac{\lambda^{i-(c-1)}}{(s+\lambda+c\mu)^{i-(c-1)}}\bar{P}_{c-1,0,c-1}(s),$$
$$i > c. \tag{12}$$

$$\bar{P}_{i,0,c}(s) = \frac{\lambda^{i-(c-1)}}{(s+\lambda+c\mu)^{i-(c-1)}}\ \bar{P}_{c-1,0,c-1}(s),\ i \geq \text{c}. \tag{13}$$

$$\bar{P}_{i,j,c}(s) = \left(\sum_{k=1}^{i-j-(c-2)} \left(\frac{\lambda}{s+\lambda+c\mu}\right)^{i-j-(c-2)-k} \eta'_k(s)\bar{P}_{j+k+(c-2),j,c-1}(s)\right),$$
$$i \geq j+c, j \geq 1. \tag{14}$$

$$\text{where } \eta'_k(s) = \begin{cases} 1 & \text{for } k=1 \\ \left(1+\frac{(k-1)\theta}{s+\lambda+c\mu}\right) & \text{for } k = 2 \text{ to } i-j-c+1 \\ \frac{(k-1)\theta}{s+\lambda+c\mu} & \text{for } k = i-j-(c-2) \end{cases}$$

$$\bar{P}_{i,j,c-1}(s) = \frac{\lambda}{(s+\lambda+(c-1)\mu+(i-j-(c-1))\theta}\bar{P}_{i-1,j,((c-1)-1)}(s)$$
$$+\frac{(i-j-((c-1)-1))\theta}{(s+\lambda+(c-1)\mu+(i-j-(c-1))\theta}\bar{P}_{i,j,((c-1)-1)}(s)$$
$$+\frac{c\mu}{(s+\lambda+(c-1)\mu+(i-j-(c-1))\theta}$$
$$\left(\sum_{k=1}^{i-j-(c-3)}\left(\frac{\lambda}{s+\lambda+c\mu}\right)^{i-j-(c-3)-k} \eta'_k(s)\bar{P}_{j+k,j-1,c-1}(s)\right),$$
$$i \geq j+(c-1), j > 1. \tag{15}$$

$$\text{where } \eta'_k(s) = \begin{cases} 1 & \text{for } k=1 \\ \left(1+\frac{(k-1)\theta}{s+\lambda+c\mu}\right) & \text{for } k = 2 \text{ to } i-j-c+2 \\ \frac{(k-1)\theta}{s+\lambda+c\mu} & \text{for } k = \left(i-j-(c-3)\right) \end{cases}$$

$$\bar{P}_{i,j,m}(s) = \frac{\lambda}{(s+\lambda+m\mu+(i-j-m)\theta}\bar{P}_{i-1,j,m-1}(s) + \frac{(i-j-(m-1))\theta}{(s+\lambda+m\mu+(i-j-m)\theta}\bar{P}_{i,j,m-1}(s) + \frac{(m+1)\mu}{(s+\lambda+m\mu+(i-j-m)\theta}\left\{\frac{\lambda}{(s+\lambda+(m+1)\mu+(i-j-m)\theta}\bar{P}_{i-1,j-1,m} + \frac{(i-j-(m-1))\theta}{(s+\lambda+(m+1)\mu+(i-j-m)\theta}\bar{P}_{i,j-1,m}(s) + \frac{(m+2)\mu}{(s+\lambda+(m+1)\mu+(i-j-m)\theta}\bar{P}_{i,j-2,m+2}(s)\right\},$$

$$i \geq j+m, j \geq (c-m), m = 1,2,\ldots c-2. \quad (16)$$

$$\bar{P}_{i,j,0}(s) = \frac{(m+1)\mu}{s+\lambda+(i-j)\theta}\left[\frac{\lambda}{s+\lambda+\mu+(i-j)\theta}\bar{P}_{i-1,j-1,0}(s) + \frac{(i-j+1)\theta}{s+\lambda+\mu+(i-j)\theta}\bar{P}_{i,j-1,0}(s) + \frac{(m+2)\mu}{s+\lambda+\mu+(i-j)\theta}\left\{\frac{\lambda}{s+\lambda+2\mu+(i-j)\theta}\bar{P}_{i-1,j-2,m+1}(s) + \frac{(i-j+1)\theta}{s+\lambda+2\mu+(i-j)\theta}\bar{P}_{i,j-2,m+1}(s) + \frac{(m+3)\mu}{(s+\lambda+2\mu+(i-j)\theta}\bar{P}_{i,j-3,m+3}(s)\right\}\right],$$

$$i > j \geq c. \quad (17)$$

Taking the inverse Laplace transform of equations (7) to (17), we have

$$P_{0,0,0}(t) = e^{-\lambda t}. \quad (18)$$

$$P_{i,i,0}(t) = \mu e^{-\lambda t} * P_{i,i-1,1}(t), \ i \geq 1. \quad (19)$$

$$P_{m,0,m}(t) = \lambda e^{-(\lambda+m\mu)t} * P_{m-1,0,m-1}(t), \ 1 \leq m \leq c-1. \quad (20)$$

$$P_{i,i-m,m}(t) = \lambda e^{-(\lambda+m\mu)t} * P_{i-1,i-m,m-1}(t) + (m+1)\mu e^{-(\lambda+m\mu)t} * P_{i,(i-m-1),m+1}(t), m = 1 \text{ to } c-2; \ i = m+1 \text{ to } c-1. \quad (21)$$

$$P_{c,1,c-1}(t) = \lambda e^{-(\lambda+(c-1)\mu)t} * P_{c-1,1,c-2}(t) + c\mu e^{-(\lambda+(c-1)\mu)t} * P_{c,0,c}(t). \quad (22)$$

$$P_{i,1,c-1}(t) = c\mu\lambda^{i-(c-1)} e^{-\{\lambda+(c-1)\mu+(i-j-(c-1))\theta)t} \left\{\frac{1}{(c\mu)^{i-(c-1)}} - e^{-c\mu t}\sum_{r=0}^{i-c}\frac{(t)^r}{r!}\frac{1}{(c\mu)^{i-c-r+1}}\right\} * P_{c-1,0,c-1}(t),$$

$$i > c. \quad (23)$$

$$P_{i,0,c}(t) = \left(\lambda^{i-(c-1)}\frac{t^{i-c}}{(i-c)!}e^{-(\lambda+c\mu)t}\right) * P_{c-1,0,c-1}(t), \qquad i \geq c. \quad (24)$$

$$P_{i,j,c}(t) = \left[\left(\lambda^{i-j-(c-1)}\frac{t^{i-j-c}}{(i-j-c)!}e^{-(\lambda+c\mu)t}\right) * P_{j+c-1,j,c-1}(t) + \left\{\sum_{k=2}^{i-j-c+1}\left(\lambda^{i-j-(c-2)-k}\frac{t^{i-j-k-c+1}}{(i-j-k-c+1)!}e^{-(\lambda+c\mu)t}\right)\right\} * P_{j+k+(c-2),j,c-1}(t) +\right.$$

$$\left\{\sum_{k=2}^{i-j-c+1}\left(\lambda^{i-j-(c-2)-k}(k-1)\theta\frac{t^{i-j-k-c+2}}{(i-j-k-c+2)!}e^{-(\lambda+c\mu)t}\right)\right\} *$$

$$P_{j+k+(c-2),j,c-1}(t)+\left((i-j-c+1)\theta e^{-(\lambda+c\mu)t}\right) * P_{i,j,c-1}(t) \quad \Big],$$

$$i \geq j+c, j \geq 1. \qquad (25)$$

$$P_{i,j,c-1}(t)=\left(\lambda e^{-\{\lambda+(c-1)\mu+(i-j-(c-1))\theta\}t}\right) * P_{i-1,j,c-2}(t)$$

$$+\left(\left(i-j-(c-2)\right)\theta e^{-\{\lambda+(c-1)\mu+(i-j-(c-1))\theta\}t}\right) * P_{i,j,c-2}(t)+$$

$$\left[(c\mu)\,\lambda^{i-j-c+2}\,e^{-\{\lambda+(c-1)\mu+(i-j-(c-1))\theta\}t}\left\{\frac{1}{(c\mu)^{i-j-c+2}}-e^{-c\mu t}\sum_{r=0}^{i-j-c+1}\frac{(t)^r}{r!}\frac{1}{(c\mu)^{i-j-c-r+2}}\right\}\right]$$

$$* P_{j+1,j-1,c-1}(t)+$$

$$\left[(c\mu)\,e^{-\{\lambda+(c-1)\mu+(i-j-(c-1))\theta\}t}\left\{\begin{array}{c}\sum_{k=2}^{i-j-c+2}\lambda^{i-j-(c-3)-k}\\ \left(\frac{1}{(c\mu)^{i-j-(c-3)-k}}-e^{-c\mu t}\sum_{r=0}^{i-j-c-k+2}\frac{(t)^r}{r!}\frac{1}{(c\mu)^{i-j-(c-3)-k-r}}\right)\end{array}\right\}\right]$$

$$* P_{j+k,j-1,c-1}(t)+$$

$$\left[(c\mu)\,e^{-\{\lambda+(c-1)\mu+(i-j-(c-1))\theta\}t}\left\{\begin{array}{c}\sum_{k=2}^{i-j-c+2}\lambda^{i-j-(c-3)-k}(k-1)\theta\\ \left(\frac{1}{(c\mu)^{i-j-c-k+4}}-e^{-c\mu t}\sum_{r=0}^{i-j-c-k+3}\frac{(t)^r}{r!}\frac{1}{(c\mu)^{i-j-c-k-r+4}}\right)\end{array}\right\}\right]$$

$$* P_{j+k,j-1,c-1}(t)+\left[\begin{array}{c}(c\mu)\left((i-j-c+2)\theta\right)e^{-\{\lambda+(c-1)\mu+(i-j-(c-1))\theta\}t}\\ \left\{\frac{1}{c\mu}-\frac{e^{-c\mu t}}{c\mu}\right\}\end{array}\right] *$$

$$P_{i-c+3,j-1,c-1}(t), \qquad i \geq j+(c-1), j>1. \qquad (26)$$

$$P_{i,j,m}(t)=\lambda e^{-\{\lambda+m\mu+(i-j-m)\theta\}t} * P_{i-1,j,m-1}(t)+\left(i-j-(m-1)\right)\theta e^{-\{\lambda+m\mu+(i-j-m)\theta\}t} * P_{i,j,m-1}(t)+\lambda((m+1)\mu$$

$$e^{-\{\lambda+m\mu+(i-j-m)\theta\}t}\left\{\frac{1}{\{(m+1)\mu+(i-j-m)\theta\}}-\frac{e^{-\{(m+1)\mu+(i-j-m)\theta\}t}}{\{(m+1)\mu+(i-j-m)\theta\}}\right\} *$$

$$P_{i-1,j-1,m}(t)+\mu(m+1)(i-j-(m-1))$$

$$e^{-\{\lambda+m\mu+(i-j-m)\theta\}t}\left\{\frac{1}{\{(m+1)\mu+(i-j-m)\theta\}}-\frac{e^{-\{(m+1)\mu+(i-j-m)\theta\}t}}{\{(m+1)\mu+(i-j-m)\theta\}}\right\} *$$

$$P_{i,j-1,m}(t)+(m+1)(m+2)\mu^2 e^{-\{\lambda+m\mu+(i-j-m)\theta\}t}$$

$$\left\{\frac{1}{\{(m+1)\mu+(i-j-m)\theta\}} - \frac{e^{-\{(m+1)\mu+(i-j-m)\theta\}t}}{\{(m+1)\mu+(i-j-m)\theta\}}\right\} * P_{i,j-2,m+2}(t),$$

$$i \geq j+m, j \geq (c-m), m = 1,2,\dots c-2. \quad (27)$$

$$P_{i,j,0}(t) = \left[\lambda(m+1)\mu e^{-(\lambda+(i-j)\theta\}t}\left\{\frac{1}{\mu+(i-j)\theta\}} - \frac{e^{-\{\mu+(i-j)\theta\}t}}{\{\mu+(i-j)\theta\}}\right\}\right] * P_{i-1,j-1,0}(t) + \left[\left((i-j+1)\theta\right)(m+1)\mu e^{-(\lambda+(i-j)\theta\}t}\left\{\frac{1}{\mu+(i-j)\theta\}} - \frac{e^{-\{\mu+(i-j)\theta\}t}}{\{\mu+(i-j)\theta\}}\right\}\right] * P_{i,j-1,0}(t) + \left[(m+1)(m+2)\mu^2\lambda\left\{\frac{e^{-\{\lambda+(i-j)\theta\}t}}{2\mu^2} + \frac{e^{-\{\lambda+\mu+(i-j)\theta\}t}}{\mu^2} - \frac{e^{-\{\lambda+(i-j)\theta+2\mu\}t}}{2\mu^2}\right\}\right] * P_{i-1,j-2,m+1}(t) + \left[(m+1)(m+2)\mu^2\left((i-j+1)\theta\right)\left\{\frac{e^{-\{\lambda+(i-j)\theta\}t}}{2\mu^2} + \frac{e^{-\{\lambda+\mu+(i-j)\theta\}t}}{\mu^2} - \frac{e^{-\{\lambda+(i-j)\theta+2\mu\}t}}{2\mu^2}\right\}\right] * P_{i,j-2,m+1}(t) + \left[(m+1)(m+2)(m+3)\mu^3\left\{\frac{e^{-\{\lambda+(i-j)\theta\}t}}{2\mu^2} + \frac{e^{-\{\lambda+\mu+(i-j)\theta\}t}}{\mu^2} - \frac{e^{-\{\lambda+(i-j)\theta+2\mu\}t}}{2\mu^2}\right\}\right] * P_{i,j-3,m+3}(t), \quad i > j \geq c.$$

(28)

## 5 Performance Measures of the Model

1. The Laplace transform of the probability $P_{i.}(t)$ that exactly $i$ units arrive by time $t$ is given by:

$$\bar{P}_{i.}(s) = \sum_{j=0}^{i} \bar{P}_{i,j}(s) = \frac{\lambda^i}{(s+\lambda)^{i+1}}, \; i > 0. \quad (29)$$

And its Inverse Laplace transform is

$$P_{i.}(t) = \frac{e^{-\lambda t}(\lambda t)^i}{i!}. \quad (30)$$

The assumption on primary arrivals is that it forms a Poisson process and the above analysis of our solution also verifies the same.

2. The probability that exactly $j$ customers have been served by time $t$. $P_{.j}(t)$ in terms of $P_{i,j}(t)$ is given by:

$$P_{.j}(t) = \sum_{i=j}^{\infty} P_{i,j}(t).$$

3. From the abstract solution of our model, we verified that sum of all possible probabilities is one i.e. taking summation over $i$ and $j$ on equations (7) – (17) and adding, we get

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\left\{\bar{P}_{i,j,0}(s) + \bar{P}_{i,j,m}(s) + \bar{P}_{i,j,c}(s)\right\} = \frac{1}{s}, \; (m = 1,2,\dots c-1).$$

After taking the inverse Laplace transformation, we get

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\left\{P_{i,j,0}(t) + P_{i,j,m}(t) + P_{i,j,c}(t)\right\} = 1,$$

which is a verification of results.

4. Define $Q_{n,m}(t)$ as the probability that there are $n$ customers in the system at time $t$ and '$m$' $(m = 0,1,2,\dots c)$ servers are busy.

The probability of exactly $n$ customers in the system at time $t$ in terms $P_{i,j,m}(t)$: When $m$ servers are busy, it is defined by probability $Q_{n,m}(t)$

$$Q_{n,m}(t) = \sum_{j=0}^{\infty} P_{j+n+m,j,m}(t), \ (m = 0,1,2,\dots c).$$

The number of customers i.e. '$n$' in the orbit is obtained by using the relation: $n$ = (number of arrivals – number of departures – $m$)

Using the above relation and letting $\mu=1$ in the equations (1) to (3), the set of equations in statistical equilibrium are:

$$(\lambda + m + n\theta)Q_{n,m} = \lambda Q_{n,m-1} + (n+1)\theta Q_{n+1,m-1} + (m+1)Q_{n,m+1}, \quad 0 \le m \le c-1, n \ge 0. \qquad (31)$$

$$(\lambda + c)Q_{n,c} = \lambda Q_{n,c-1} + (n+1)\theta Q_{n+1,c-1} + \lambda Q_{n-1,c}\left(1 - \delta_{n,0}\right),$$

$$(\text{case } m = c), n \ge 0. \qquad (32)$$

where $\delta_{n,0} = \begin{cases} 1, & \text{when } n = 0 \\ 0, & \text{when } n \ge 1 \end{cases}$

Now, for generating functions

$$Q_m(z) = \sum_{n=0}^{\infty} z^n Q_{n,m}, 0 \le m \le c.$$

Letting $z = 1$, then $Q_m(1) = \sum_{n=0}^{\infty} Q_{n,m} = Q_m, \quad 0 \le m \le c.$

these equations become

$$(\lambda + m)Q_m(z) + \theta z Q'_m(z) = \lambda Q_{m-1}(z) + \theta Q'_{m-1}(z) + (m+1)Q_{m+1}(z), \quad 0 \le m \le c-1. \qquad (33)$$

$$(\lambda + c)Q_c(z) = \theta Q'_{c-1}(z) + \lambda Q_{c-1}(z) + \lambda z Q_c(z), (\text{case } m = c). \qquad (34)$$

Now introduce the bivariate generating function

$$Q(x,z) = \sum_{m=0}^{c} x^m Q_m(z).$$

Then equations (33), (34) become:

$$\lambda(1-x)Q(x,z) + \theta(z-x)Q'_z(x,z) + (x-1)Q'_x(x,z) + \lambda x^c(x-z)Q_c(z) + \mu x^c(x-z)Q'_c(z) = 0. \qquad (35)$$

Which coincide with the result (2.21) of [3].

On putting $z = 1$ in equation (33) and in equation (34), we get:

$$(\lambda + m)Q_m + \theta Q'_m = \lambda Q_{m-1} + \theta Q'_{m-1} + (m+1)Q_{m+1}, \quad 0 \le m \le c-1. \qquad (36)$$

$$(\lambda + c)Q_c = \theta Q'_{c-1} + \lambda Q_{c-1} + \lambda Q_c, \ (\text{case } m = c). \qquad (37)$$

Now consider $Q'_m = N_m$, $Q'_{m-1} = N_{m+1}$, $Q'_{c-1} = N_{c+1}$, then the equations (36) and (37) becomes

$$\lambda Q_m + \theta N_m - (m+1)Q_{m+1} = \lambda Q_{m-1} + \theta N_{m-1} - mQ_m, \quad 0 \le m \le c-1. \qquad (38)$$

$$\lambda Q_{c-1} + \theta N_{c-1} - cQ_c = 0.$$

These equations yield that

$$\lambda Q_m + \theta N_m - (m+1)Q_{m+1} = 0, \quad 0 \le m \le c-1. \qquad (39)$$

Denote the ratio $\frac{\theta N_m}{Q_m} = r_m$ , which is the rate of flow of repeated calls given that the number of busy servers equals $m$.

Then equation (39) can be rewritten as

$$Q_{m+1} = \frac{\lambda + r_m}{m+1} Q_m \qquad 0 \le m \le c-1.$$

Now solve this recursively we have:

$$Q_m = \frac{(\lambda + r_{m-1})\,(\lambda + r_{m-2})\ldots\ldots\ldots\ldots(\lambda + r_0)}{m!} Q_0, \qquad 0 \le m \le c-1 \tag{40}$$

From the normalizing condition

$$\sum_{m=0}^{c} Q_m = 1.$$

$$Q_0 = \left(\sum_{m=0}^{c} \frac{(\lambda + r_{m-1})\,(\lambda + r_{m-2})\ldots\ldots\ldots\ldots(\lambda + r_0)}{m!}\right)^{-1}. \tag{41}$$

Thus from equation (41), we can say that the steady state distribution of the number of busy servers in the retrial queue is identical to the steady state distribution of the number of busy servers in the " Erlang loss model" with steady dependent arrival rate$(\lambda + r_m)$. The extra load $r_m$ is formed by repeated calls.

## 5. Special cases

1. When we considering the units are served singly i.e. $(c = 1)$ and service time of the server is exponentially distributed with parameter $\mu$ in equations (18) to (28), then we get:

$$P_{0,0,0}(t) = e^{-\lambda t}. \tag{42}$$

$$P_{i,1,0}(t) = \mu e^{-(\lambda + (i-1)\theta)t} * P_{i,0,1}(t),\ i \ge 1. \tag{43}$$

$$P_{i,i,0}(t) = \left[(\lambda\mu)e^{-\lambda t}\left\{\frac{1}{\mu} - \frac{e^{-\mu t}}{\mu}\right\} * P_{i-1,i-1,0}(t) + (\mu\theta)e^{-\lambda t}\left\{\frac{1}{\mu} - \frac{e^{-\mu t}}{\mu}\right\} * P_{i,i-1,0}(t)\right], \quad i > 1. \tag{44}$$

$$P_{i,0,1}(t) = \lambda^i e^{-\lambda t} \times \left\{\frac{1}{(\mu)^i} - e^{-\mu t}\sum_{r=0}^{i-1} \frac{(t)^r}{r!}\frac{1}{(\mu)^{i-r}}\right\}, \quad i \ge 1. \tag{45}$$

$$P_{i,i-1,1}(t) = \left(\lambda e^{-(\lambda+\mu)t} * P_{i-1,i-1,0}(t) + \theta e^{-(\lambda+\mu)t} * P_{i,i-1,0}(t)\right), \quad i > 1. \tag{46}$$

$$P_{i,j,0}(t) = \mu\lambda^{i-j} e^{-(\lambda + (i-j)\theta)t}\left\{\frac{1}{(\mu)^{i-j}} - e^{-\mu t}\sum_{r=0}^{i-j-1}\frac{(t)^r}{r!}\frac{1}{(\mu)^{i-j-r}}\right\} * P_{j,j-1,0}(\mathrm{t}) +$$
$$e^{-(\lambda + (i-j)\theta)t}\sum_{k=2}^{i-j} \mu\lambda^{i-j-k+1}\left\{\frac{1}{(\mu)^{i-j-k+1}} - e^{-\mu t}\sum_{r=0}^{i-j-k}\frac{(t)^r}{r!}\frac{1}{(\mu)^{i-j-k-r+1}}\right\} *$$
$$P_{j+k-1,j-1,0}(\mathrm{t}) +$$
$$e^{-(\lambda + (i-j)\theta)t}\sum_{k=2}^{i-j}(\mu k\theta)\lambda^{i-j-k+1}\left\{\frac{1}{(\mu)^{i-j-k+2}} - e^{-\mu t}\sum_{r=0}^{i-j-k+1}\frac{(t)^r}{r!}\frac{1}{(\mu)^{i-j-k-r+2}}\right\} *$$
$$P_{j+k-1,j-1,0}(\mathrm{t}) + e^{-(\lambda + (i-j)\theta)t}\left\{\frac{1}{\mu} - \frac{e^{-\mu t}}{\mu}\right\}((i-j+1)\mu\theta) * P_{i,j-1,0}(\mathrm{t}) +$$

$$\mu\lambda^{i-j} e^{-(\lambda+(i-j)\theta)t}\left\{\frac{1}{(\mu)^{i-j}} - e^{-\mu t}\sum_{r=0}^{i-j-1}\frac{(t)^r}{r!}\frac{1}{(\mu)^{i-j-r}}\right\} * P_{j,j-1,1}(\mathrm{t}),$$
$$i > j > 1. \qquad (47)$$

$$P_{i,j,1}(t) = \lambda^{i-j-1} e^{-(\lambda+\mu)t}\frac{(t)^{i-j-2}}{(i-j-2)!} * P_{j+1,j,0}(\mathrm{t}) + e^{-(\lambda+\mu)t}$$
$$\sum_{k=2}^{i-j-1}\lambda^{i-j-k}\frac{(t)^{i-j-k-1}}{(i-j-k-1)!} * P_{j+k,j,0}(\mathrm{t}) +$$
$$e^{-(\lambda+\mu)t}\sum_{k=2}^{i-j-1} k\theta\lambda^{i-j-k}\frac{(t)^{i-j-k}}{(i-j-k)!} * P_{j+k,j,0}(\mathrm{t}) + (i-j)\theta e^{-(\lambda+\mu)t} *$$
$$P_{i,j,0}(\mathrm{t}) + \lambda^{i-j-1} e^{-(\lambda+\mu)t}\frac{(t)^{i-j-2}}{(i-j-2)!} * P_{j+1,j,1}(\mathrm{t}),$$
$$i \geq j+2, j \geq 1. \qquad (48)$$

and the above equations coincide with that of [12].

2. When there are two parallel servers $(c = 2)$, and service time of each server follow an exponential distribution with parameter $\mu$, then we obtained various probabilities from equations (18) to (28), and these probabilities coincide with the results of [13].

$$P_{0,0,0}(t) = e^{-\lambda t}. \qquad (49)$$

$$P_{1,1,0}(t) = \lambda\mu\,(te^{-\lambda t})e^{-(\lambda+\mu)t}. \qquad (50)$$

$$P_{i,i,0}(t) = \lambda\mu e^{-\lambda t}\left(\frac{1}{\mu} - \frac{e^{-\mu t}}{\mu}\right) * P_{i-1,i-1,0}(t) + \mu\theta e^{-\lambda t}\left(\frac{1}{\mu} - \frac{e^{-\mu t}}{\mu}\right) * P_{i,i-1,0}(t)$$
$$+ 2\mu^2 e^{-\lambda t}\left(\frac{1}{\mu} - \frac{e^{-\mu t}}{\mu}\right) * P_{i,i-2,2}(t), \qquad i > 1. \qquad (51)$$

$$P_{i,2,0}(t) = 2\mu^2 e^{-(\lambda+(i-2)\theta)t}\left(\frac{1}{(\mu+(i-2)\theta)} - \frac{e^{-(\mu+(i-2)\theta)t}}{(\mu+(i-2)\theta)}\right) * P_{i,0,2}(t),$$
$$i \geq 3. \qquad (52)$$

$$P_{1,0,1}(t) = \lambda e^{-\lambda t} \times \left(\frac{1}{\mu} - \frac{e^{-\mu t}}{\mu}\right). \qquad (53)$$

$$P_{2,1,1}(t) = \lambda e^{-(\lambda+\mu)t} * P_{1,1,0}(t) + 2\lambda\mu e^{-(\lambda+\mu)t}\left(\frac{1}{2\mu} - \frac{e^{-2\mu t}}{2\mu}\right) * P_{1,0,1}(t). \qquad (54)$$

$$P_{i,1,1}(t) = \left[2\mu\,\lambda^{i-1} e^{-(\lambda+\mu+(i-2)\theta)t}\left\{\frac{1}{(2\mu)^{i-1}} - e^{-2\mu t}\sum_{r=0}^{i-2}\frac{(t)^r}{r!}\frac{1}{(2\mu)^{i-r}}\right\}\right] *$$
$$P_{1,0,1}(t),$$
$$i > 2. \qquad (55)$$

$$P_{i,i-1,1}(t) = \lambda e^{-(\lambda+\mu)t} * P_{i-1,i-1,0}(t) + \theta e^{-(\lambda+\mu)t} * P_{i,i-1,0}(t)$$
$$+ 2\mu e^{-(\lambda+\mu)t} * P_{i,i-2,2}(t), \qquad i > 2. \qquad (56)$$

$$P_{i,0,2}(t) = \left(\lambda^{i-1}\frac{t^{i-2}}{(i-2)!} e^{-(\lambda+2\mu)t}\right) * P_{1,0,1}(t), \qquad i > 1. \qquad (57)$$

$$P_{i,j,0}(t) = \lambda\mu e^{-(\lambda+(i-j)\theta)t}\left(\frac{1}{\mu+(i-j)\theta} - \frac{e^{-(\mu+(i-j)\theta)t}}{\mu+(i-j)\theta}\right) * P_{i-1,j-1,0}(t)$$

$$+ \mu(i-j+1)\theta e^{-(\lambda+(i-j)\theta)t}$$

$$\left(\frac{1}{\mu+(i-j)\theta} - \frac{e^{-(\mu+(i-j)\theta)t}}{\mu+(i-j)\theta}\right) * P_{i,j-1,0}(t) + 2\mu^2\lambda^{i-j+1}$$

$$\left[\sum_{l=1}^{i-j+1}\sum_{m=1}^{l}\frac{e^{-(\lambda+(i-j)\theta)t}\ t^{(i-j+1)-l}\ (-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(1+g_2)\right)}{\left((i-j+1)-l\right)!(m-1)!\ (\mu)^m\ (2\mu-(i-j)\theta)^{1+l-m}} - \frac{e^{-(\lambda+\mu+(i-j)\theta)t}}{(\mu)^{(i-j+1)}(\ \mu-(i-j)\theta)} + \frac{e^{-(\lambda+2\mu)t}}{(2\mu-(i-j)\theta)^{(i-j+1)}(\ \mu-(i-j)\theta)}\right]$$

$$*\ P_{j-1,j-2,1}(t)\ +2\mu^2\sum_{k=1}^{i-j}\ \lambda^{(i-j+1)-k}$$

$$\left[\sum_{l=1}^{(i-j+1)-k}\ \sum_{m=1}^{l}\frac{e^{-(\lambda+(i-j)\theta)t}\ t^{((i-j+1)-k)-l}\ (-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(1+g_2)\right)}{(((i-j+1)-k)-l)!\,(m-1)!\ (\mu)^m\ (2\mu-(i-j)\theta)^{1+l-m}} - \frac{e^{-(\lambda+\mu+(i-j)\theta)t}}{(\mu)^{(i-j+1)-k}(\ \mu-(i-j)\theta)} + \frac{e^{-(\lambda+2\mu)t}}{(2\mu-(i-j)\theta)^{(i-j+1)-k}(\ \mu-(i-j)\theta)}\right]$$

$$*\ P_{(j-1)+k,j-2,1}(t) + 2\mu^2\sum_{k=1}^{i-j}\ \lambda^{(i-j+1)-k}\ (k\theta)$$

$$\left[\sum_{l=1}^{((i-j+1)-k)+1}\ \sum_{m=1}^{l}\frac{e^{-(\lambda+(i-j)\theta)t}\ t^{(((i-j+1)-k)+1)-l}\ (-1)^{m+1}\binom{l-1}{m-1}\left(\prod_{g_1=0}^{l-m-1}(1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(1+g_2)\right)}{\left((((i-j+1)-k)+1)-l\right)!\,(m-1)!\ (\mu)^m\ (2\mu-(i-j)\theta)^{1+l-m}} - \frac{e^{-(\lambda+\mu+(i-j)\theta)t}}{(\mu)^{((i-j+1)-k)+1}(\ \mu-(i-j)\theta)} + \frac{e^{-(\lambda+2\mu)t}}{(2\mu-(i-j)\theta)^{((i-j+1)-k)+1}(\ \mu-(i-j)\theta)}\right]$$

$$*\ P_{(j-1)+k,j-2,1}(t) + 2\mu^2\ (i-j+1)\theta$$

$$\left[\frac{e^{-(\lambda+(i-j)\theta)t}}{(\mu)(2\mu-(i-j)\theta)} - \frac{e^{-(\lambda+\mu+(i-j)\theta)t}}{(\mu)(\mu-(i-j)\theta)} + \frac{e^{-(\lambda+2\mu)t}}{(2\mu-(i-j)\theta)(\mu-(i-j)\theta)}\right]\ *\ P_{i,j-2,1}(t),$$

$$i > j \geq 3. \qquad (58)$$

$$P_{i,j,1}(t) = \lambda e^{-(\lambda+\mu+(i-j-1)\theta)t} * P_{i-1,j,0}(t) + (i-j)\theta e^{-(\lambda+\mu+(i-j-1)\theta)t} * P_{i,j,0}(t)$$
$$+\ 2\mu\,\lambda^{i-j}\ e^{-(\lambda+\mu+(i-j-1)\theta)t}\left\{\frac{1}{(2\mu)^{i-j}} - e^{-2\mu t}\sum_{r=1}^{i-j-1}\frac{(t)^r}{r!}\ \frac{1}{(2\mu)^{i-j-r}}\right\} * P_{j,j-1,1}(t)$$
$$+2\mu\, e^{-(\lambda+\mu+(i-j-1)\theta)t}\sum_{k=1}^{i-j-1}\ \lambda^{i-j-k}\ \left\{\frac{1}{(2\mu)^{i-j-k}} - e^{-2\mu t}\sum_{r=0}^{i-j-k-1}\frac{(t)^r}{r!}\ \frac{1}{(2\mu)^{i-j-k-r}}\right\}*$$
$$P_{j+k,j-1,1}(t)\ \ + 2\mu\, e^{-(\lambda+\mu+(i-j-1)\theta)t}\sum_{k=1}^{i-j-1}\ \lambda^{i-j-k}\ \ (k\theta)$$
$$\left\{\frac{1}{(2\mu)^{i-j-k+1}} - e^{-2\mu t}\sum_{r=0}^{i-j-k}\frac{(t)^r}{r!}\ \frac{1}{(2\mu)^{i-j-k+1-r}}\right\} *$$
$$P_{j+k,j-1,1}(t) + 2\mu(i-j)\theta e^{-(\lambda+\mu+(i-j-1)\theta)t} \times \left(\frac{1}{2\mu} - \frac{e^{-2\mu t}}{2\mu}\right) * P_{i,j-1,1}(t),$$

$$i \geq j+2, j \geq 2. \qquad (59)$$

$$P_{i,j,2}(t)=\left( \lambda^{i-j-1} \frac{t^{i-j-2}}{(i-j-2)!} e^{-(\lambda+2\mu)t}\right)*P_{j+1,j,1}(t)$$
$$+\sum_{k=2}^{i-j-1} \left( \lambda^{i-j-k} \frac{t^{i-j-k-1}}{(i-j-k-1)!} e^{-(\lambda+2\mu)t}\right)* \ P_{j+k,j,1}(t)$$
$$+\sum_{k=2}^{i-j-1} \left(\lambda^{i-j-k}(\mathrm{k}-1)\theta \frac{t^{i-j-k}}{(i-j-k)!} e^{-(\lambda+2\mu)t}\right) * P_{j+k,j,1}(t)+\left((i-j-1)\ \theta\ e^{-(\lambda+2\mu)t}\ \right)* P_{i,j,1}(t),$$
$$i \geq j+2, j \geq 1 \qquad (60)$$

# 6 Conclusion

Retrial queues have frequently been used for modeling of computer and telecommunication networks. This paper presents an extensive analysis of multiserver retrial queueing systems. The time dependent probabilities of exact number of arrivals and departures for the system when no, some or all servers are busy are found by solving difference differential equations recursively. Due to the two-dimensional nature of the model under study, factors are clearly understood and well quantified. Expressions for performance measures are also given. This model would be significant for modeling banking service systems, computer job processing systems, and many other such systems.

# 7 Open Problem

In this paper, we have considered primary arrivals of customers follow Poisson process. Similarly customers from the orbit i.e. secondary arrivals to the system also follow Poisson distribution but with different parameter. The service times follow an exponential distribution. The open problem here is to find time dependent probabilities while considering general distributions for primary arrivals of customers in the system, secondary arrivals of customers in the system and departures of customers from the system.

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