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Two double sided inequalities involving sinc and hyperbolic sinc functions

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Abstract

We prove two double sided inequalities involving the cardinal sine and hyperbolic cardinal sine functions. Their interesting generalizations have been posed as open problems at the end.

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1 Introduction

The well known cardinal sine or sinc function which is defined as

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin(x)}{x} & x \neq 0, \\ 1 & x = 0 \end{cases} \tag{1.1}$$

is useful in many branches of applied mathematics. The hyperbolic counterpart of (1.1) which is known as hyperbolic cardinal sine or hyperbolic sinc is generally denoted by sinhc and is defined as follows:

$$\operatorname{sinhc}(x) = \begin{cases} \frac{\sinh(x)}{x} & x \neq 0, \\ 1 & x = 0. \end{cases} \tag{1.2}$$

For more details of hyperbolic cardinal sine, one may refer [10]. Due to their usefulness, many inequalities have been established recently involving the functions in (1.1) and (1.2). These inequalities are known by various names viz. Jordan's inequality, Wilker's inequality, Huygen's inequality, Cusa-Huygen's inequality (see for instance [1, 5, 2, 4]-[6, 11, 7, 8, 3]) etc. Motivated by these inequalities, in what follows, we establish the independent results connecting sinc and sinhc functions.

2 Double Sided Inequalities

Our main results are proposed and proved in this section.

Theorem 2.1 *If $x \in (0, \pi/2)$ then the following inequalities hold:*

$$2 + \cos^2(x) > \frac{\sin(2x)}{2x} + 2\frac{\sin(x)}{x} > 1 + 2\cos(x). \tag{2.1}$$

Proof: For the left inequality, consider the function

$$f(x) = 4x + 2x\cos^2(x) - \sin(2x) - 4\sin(x).$$

Differentiation with respect to x gives

$$f'(x) = 4 - 4x\sin(x)\cos(x) + 2\sin^2(x) - 4\cos(x).$$

Again differentiating, we have

$$\begin{aligned} f''(x) &= 4x\sin^2(x) - 4x\cos^2(x) + 4\sin(x) \\ &= 4x\sin^2(x) + 4\left[\sin(x) - x\cos^2(x)\right] > 0, \end{aligned}$$

because of the fact that $\cos^2(x) < \cos(x)$ and $\cos(x) < \sin(x)/x$ (see [4]) in $(0, \pi/2)$. Therefore $f'(x)$ is increasing in $(0, \pi/2)$ and we have $f'(x) > f'(0)$ for $x \in (0, \pi/2)$, i.e., $f'(x) > 0$. So $f(x)$ is increasing and consequently, we get

$$4x + 2x\cos^2(x) - \sin(2x) - 4\sin(x) > 0,$$

i.e.,

$$2 + \cos^2(x) > \frac{\sin(2x)}{2x} + 2\frac{\sin(x)}{x}. \tag{2.2}$$

For the right inequality, consider the function

$$g(x) = \sin(2x) + 4\sin(x) - 2x - 4x\cos(x).$$

Then we have
$$g'(x) = 4\,[x - \sin(x)]\,\sin(x),$$
which is positive since $\sin(x) < x$ and $\sin(x) > 0$ for $x \in (0, \pi/2)$. Therefore $g(x)$ is increasing for $x \in (0, \pi/2)$, implying that $g(x) > g(0) = 0$, so
$$\sin(2x) + 4\sin(x) - 2x - 4x\cos(x) > 0,$$
which is equivalent to
$$\frac{\sin(2x)}{2x} + 2\frac{\sin(x)}{x} > 1 + 2\cos(x). \tag{2.3}$$
The desired result follows by combining (2.2) and (2.3). □

Note: The left inequality in (2.1) is in fact holds for $x \in (0, \epsilon)$ where $\epsilon > 0$ and right inequality holds for $x \in (0, \pi)$.

The hyperbolic counterpart of Theorem 2.1 is proved as follows.

Theorem 2.2 *For any non-zero $x \in \mathbb{R}$ the following inequalities hold:*
$$2 + \cosh^2(x) > \frac{\sinh(2x)}{2x} + 2\,\frac{\sinh(x)}{x} > 1 + 2\cosh(x). \tag{2.4}$$

Proof: Since the functions in each term are even, the proof is restricted for $x \in (0, +\infty)$. For the left inequality, consider the function
$$\phi(x) = 4x + 2x\cosh^2(x) - \sinh(2x) - 4\sinh(x).$$
Differentiation with respect to x gives
$$\phi'(x) = 4 + 4x\sinh(x)\cosh(x) - 2\sinh^2(x) - 4\cosh(x).$$
Again differentiating, we obtain
$$\begin{aligned}\phi''(x) &= 4x\sinh^2(x) + 4x\cosh^2(x) - 4\sinh(x)\\ &= 4x\sinh^2(x) + 4\left[x\cosh^2(x) - \sinh(x)\right] > 0\end{aligned}$$
since we know that $\sinh(x)/x < \cosh(x)$ (see [4]) and $\cosh(x) < \cosh^2(x)$ for any non-zero $x \in \mathbb{R}$. Therefore $\phi'(x)$ is increasing and we have $\phi'(x) > \phi'(0) = 0$ for any $x > 0$. So $\phi(x)$ is increasing and consequently, we get
$$4x + 2x\cosh^2(x) - \sinh(2x) - 4\sinh(x) > 0,$$
i.e.,
$$2 + \cosh^2(x) > \frac{\sinh(2x)}{2x} + 2\frac{\sinh(x)}{x}. \tag{2.5}$$

For the right inequality, consider the function

$$\varphi(x) = \sinh(2x) + 4\sinh(x) - 2x - 4x\cosh(x).$$

Then we have

$$\varphi(x) = 4\left[\sinh(x) - x\right]\sinh(x),$$

which is positive since $\sinh(x) > x$ and $\sinh(x) > 0$ for $x > 0$. Therefore $\varphi(x)$ is increasing for $x > 0$, implying that $\varphi(x) > \varphi(0) = 0$, so

$$\sinh(2x) + 4\sinh(x) - 2x - 4x\cosh(x) > 0,$$

which is equivalent to

$$\frac{\sinh(2x)}{2x} + 2\frac{\sinh(x)}{x} > 1 + 2\cosh(x). \tag{2.6}$$

The desired result follows by combining (2.5) and (2.6). □

Remark 1: The double sided inequalities in (2.1) and (2.4) can be equivalently written as

$$\frac{2}{\cos(x)} + \cos(x) > \frac{\sin(x)}{x} + 2\frac{\tan(x)}{x} > \frac{1}{\cos(x)} + 2;\ x \in (0, \pi/2) \tag{2.7}$$

and

$$\frac{2}{\cosh(x)} + \cosh(x) > \frac{\sinh(x)}{x} + 2\frac{\tanh(x)}{x} > \frac{1}{\cosh(x)} + 2;\ x \neq 0.$$

The inequalities (2.7) are in the same spirit to the ones in [3, Corollary 1], i.e., for $x \in (0, \pi/2)$,

$$2 + \cos(x) > \frac{x}{\sin(x)} + 2\frac{x}{\tan(x)} > \frac{\pi}{2} + \cos(x).$$

Remark 2: Using the the right inequality in (2.7) and the inequalities $x(\cos(x))^{1/3} < \sin(x)$ (see [9]) and $\sin(x) < x$ for $x \in (0, \pi/2)$, we have the following inequalities:

$$\frac{\sin(x)}{x} + 2\frac{\tan(x)}{x} > \left(\frac{x}{\sin(x)}\right)^3 + 2 > 3.$$

3 Open Problems

We believe that the bounds in (2.1) and (2.4) can be sharpened. Beside this we also pose the following two problems:

Open problem 1: For $x \in (0, \pi/2)$ and $k \in (0, 2]$, $t \geqslant 2$, prove the following inequalities:

$$t + \cos^t(x) > \frac{\sin(tx)}{tx} + t\frac{\sin(x)}{x}$$

and

$$\frac{\sin(kx)}{kx} + k\frac{\sin(x)}{x} > 1 + k\cos(x).$$

Open problem 2: For any $x \in \mathbb{R}$ and $p \in (0, 2]$, $q \geqslant 2$ prove the following inequalities:

$$p + \cosh^p(x) > \frac{\sinh(px)}{px} + p\frac{\sinh(x)}{x}$$

and

$$\frac{\sinh(qx)}{qx} + q\frac{\sinh(x)}{x} > 1 + q\cosh(x).$$

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