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Some Numerical Methods to Solve a System of Volterra Integral Equations

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Abstract

In this article, the Modified of Adomian Decomposition Method (MADM) and Modified Homotopy Perturbation Method (MHPM) are used for solving the system of Volterra integral equations of the second kind. The new methods are resulted from ADM and HPM by a simple modification. The efficiency of the methods is cleared through solving some numerical examples. Comparison of the result of applying the MADM and the MHPM for revealing the new techniques are very effective and convenient.

Keywords: *System of Volterra integral equations, modified Adomian decomposition method, modified homotopy perturbation method.*
2010 Mathematics Subject Classification: 34A34, 65L05.

1 Introduction

Many physical situations are modeled by a system of linear Volterra integral equations of the second kind and occur as reformulations of other mathematical problems. Consider the system of linear Volterra integral equations of the second kind of the following form:

$$\sum_{j=1}^{n} y_j(x) = f_i(x) + \sum_{j=1}^{n} \int_a^x k_{i,j}(x,t) y_j(t) dt, \qquad i = 1, 2, \cdots, n, \tag{1}$$

where$f_i(x)$ and $k_{i,j}(x,t)$ are known functions and $y_j(x)$ are the unknown function.

In recent years, many authors focus on the development of numerical and analytical techniques for integral equations. For instance, we can mention the following works; Ray and Sahu [15] studied the numerical solutions of the system of linear Volterra integral equations of the second kind. Masouri et al. [13] used Chebyshev cardinal functions to solve a second kind Fredholm integral equations system. Abdelkawy et al. [1] implemented Jacobi spectral collocation scheme for solving Abel's integral equations. Arikoglu and Ozkol [4] based upon the differential transform method to find the solution of Fredholm integral and integro-defferential equation systems. Aboiyar and Ibrahim [2] employed the new iterative method to the approximate solution of system fractional integral equations. Sulaiman [16] used modification in the successive approximation method to solve a second kind Fredholm integral equations system with the symmetric kernel. Hamoud and Ghadle [8] studied the combined modified Laplace with the Adomian decomposition method for solving the nonlinear Volterra-Fredholm integro-differential equations. Maleknejad and Aghazadeh [14] employed the Taylor collocation method to approximate solutions of Volterra integral equations of the second kind. Maleknejad et al. [11] compared the projection method with Adomian's decomposition method for solving the system of integral equations. Maleknejad et al. [12] discussed the convergence of Legender polynomial for solving Fredholm's integral equation of the second kind. Al-Saar et al. [3] used the approximate Solutions of Fredholm integral equations by the Adomian decomposition method and its modification. Hamoud and Ghadle [6] introduced the modified Adomian decomposition method for solving fuzzy Volterra-Fredholm integral equations. Hamoud and Ghadle [7] presented recent advances on reliable methods for solving Volterra-Fredholm integral and integro-differential equations.

In this article, a simple modification on ADM and HPM will be studied and will be applied to solve the systems of Volterra integral equations of the second kind. Section 2. describes the solution of the linear Volterra integral equations system by using ADM and its modification. HPM and its modification are described for solving a system of linear Volterra integral equations in section 3. Section 4. reports our numerical findings and demonstrates the accuracy of the proposed methods by considering numerical examples. Section 5 ends this article with a brief conclusion.

2 Modified Adomian Decomposition Method (MADM)

In this section, we apply the ADM and MADM for the system of linear Volterra integral equations of the second kind. To this end, Consider the i-th equation of (1) as:

$$y_i(x) = f_i(x) + \int_a^x \sum_{j=1}^{n} k_{i,j}(x,t)y_j(t)dt, \qquad i = 1, 2, ..., n. \tag{2}$$

To solve (1) by the ADM , let

$$y_i(x) = \sum_{m=0}^{\infty} y_{i,m}(x), \qquad i = 1, 2, ..., n. \tag{3}$$

Substituting (3) into (2), we obtain

$$\sum_{m=0}^{\infty} y_{i,m}(x) = f_i(x) + \int_a^x \sum_{j=1}^{n} k_{i,j}(x,t)(\sum_{m=0}^{\infty} y_{j,m}(t))dt, \qquad i = 1, 2, ..., n. \tag{4}$$

By using the recursion scheme of the ADM, we define

$$\begin{gathered} y_{i,0}(x) = f_i(x), \qquad i = 1, 2, ..., n, \\ y_{i,k+1}(x) = \int_a^x \sum_{j=1}^{n} k_{i,j}(x,t)y_{j,k}(t)dt, \\ i = 1, 2, ..., n, \quad k = 0, 1, ... \end{gathered} \tag{5}$$

Observe that the better approximations of the solution can be obtained by the following truncated series [9, 10]:

$$\varphi_{i,k}(x) = \sum_{m=0}^{k-1} y_{i,m}(x), \qquad where \quad \lim_{p\to 1} \varphi_{i,k}(x) = y_i(x). \tag{6}$$

Note here that the convergence question of this technique has been formally proved and justified by [5] .

Now we propose a simple modification to accelerate convergence rate of the ADM applied to systems of linear Volterra integral equations of the second kind. In the ADM, the first terms of $y_i(x)$ is defined the known function $f_i(x)$. In the modification technique, we define the first terms of (3) as follows

$$\begin{aligned} y_{1,0}(x) &= f_1(x) \\ y_{i,0}(x) &= f_i(x) + \int_a^x \sum_{j=1}^{i-1} k_{i,j}(x,t) y_{j,0}(t)dt, \qquad i = 2, ..., n. \end{aligned} \tag{7}$$

In other words, the values of $y_{i,0}(x), for \quad i = 2, ..., n$ are modified by previous values and the other terms of (3) are defined as follows

$$\begin{aligned} y_{1,k+1}(x) &= \int_a^x \sum_{j=1}^{n} k_{i,j}(x,t) y_{j,k}(t)dt, \\ y_{i,k+1}(x) &= \int_a^x (\sum_{j=1}^{i-1} k_{i,j}(x,t) y_{j,k+1}(t) \\ &+ \sum_{j=i}^{n} k_{i,j}(x,t) y_{j,k}(t))dt, \quad i = 2, ..., n. \quad k = 0, 1, ... \end{aligned} \tag{8}$$

For solving Volterra integral equation system of the second kind we use this method with $n = 2$. Consider the following second kind Volterra integral equation system

$$\begin{aligned} y_1(x) &= f_1(x) + \int_0^x (k_{1,1}(x,t) y_1(t) + k_{1,2}(x,t) y_2(t))dt, \\ y_2(x) &= f_2(x) + \int_0^x (k_{2,1}(x,t) y_1(t) + k_{2,2}(x,t) y_2(t))dt, \end{aligned} \tag{9}$$

For solving (9) by the ADM, let

$$\begin{aligned} y_1(x) &= \sum_{m=0}^{\infty} y_{1,m}(x), \\ y_2(x) &= \sum_{m=0}^{\infty} y_{2,m}(x), \end{aligned} \tag{10}$$

Substituting (9) into (10), we obtain

$$\begin{aligned}
&\sum_{m=0}^{\infty} y_{1,m}(x) = f_1(x) + \int_0^x (k_{1,1}(x,t)\sum_{m=0}^{\infty} y_{1,m}(t) + k_{1,2}(x,t)\sum_{m=0}^{\infty} y_{2,m}(t))dt,\\
&\sum_{m=0}^{\infty} y_{2,m}(x) = f_2(x) + \int_0^x (k_{2,1}(x,t)\sum_{m=0}^{\infty} y_{1,m}(t) \qquad (11)\\
&+k_{2,2}(x,t)\sum_{m=0}^{\infty} y_{2,m}(t))dt,
\end{aligned}$$

The recursion scheme of the MADM is given as

$$\begin{aligned}
y_{1,0}(x) &= f_1(x),\\
y_{2,0}(x) &= f_2(x) + \int_0^x (k_{2,1}(x,t)y_{1,0}(t), \qquad (12)
\end{aligned}$$

and

$$\begin{aligned}
y_{1,1}(x) &= \int_0^x (k_{1,1}(x,t)y_{1,0}(t) + k_{1,2}(x,t)y_{2,0}(t))dt,\\
y_{2,1}(x) &= \int_0^x (k_{2,1}(x,t)y_{1,1}(t) + k_{2,2}(x,t)y_{2,0}(t))dt, \qquad (13)
\end{aligned}$$

the values of$y_{2,0}(x)$ and $y_{2,1}(x)$ are modified by previous values and the other terms of (9) are defined as follows

$$\begin{aligned}
y_{1,k+1}(x) &= \int_0^x (k_{1,1}(x,t)y_{1,k}(t) + k_{1,2}(x,t)y_{2,k}(t))dt,\\
y_{2,k+1}(x) &= \int_0^x (k_{2,1}(x,t)y_{1,k+1}(t) + k_{2,2}(x,t)y_{2,k}(t))dt, \qquad (14)\\
k &= 1,2,...
\end{aligned}$$

Eq (14) is used to calculate each $y_{2,k+1}$, for $k = 1,2,...$

3 Modified Homotopy Perturbation Method (MHPM)

In this section, we apply the HPM and MHPM for the system of linear Volterra integral equations of the second kind.System (1) can be presented as the eq.(2). For solving system (2), by homotopy perturbation method we construct the following homotopy

$$y_i(x) = f_i(x) + p\int_a^x \sum_{j=1}^{n} k_{i,j}(x,t)y_j(t)dt, \qquad i = 1, 2, ..., n. \tag{15}$$

Suppose the solutions of system (15) have the following form

$$y_i(x) = y_{i,0}(x) + py_{i,1}(x) + p^2 y_{i,2}(x) + \dots, \qquad i = 1, 2, ..., n. \tag{16}$$

where $y_{i,j}(x), i = 1, 2, \dots$ are functions which should be determined.

Substituting (16) into (15) and equating the coefficients of p with the same power leads to

$$\begin{aligned}
&p^0 : y_{i,0}(x) = f_1(x), \quad i = 1, 2, \dots, n,\\
&p^1 : y_{i,1}(x) = \int_a^x \sum_{j=1}^{n} k_{i,j}(x,t)y_{j,0}(t)dt, \qquad i = 2, ..., n,\\
&p^2 : y_{i,2}(x) = \int_a^x \sum_{j=1}^{n} k_{i,j}(x,t)y_{j,1}(t)dt, \qquad i = 2, ..., n,\\
&p^3 : y_{i,3}(x) = \int_a^x \sum_{j=1}^{n} k_{i,j}(x,t)y_{j,2}(t)dt, \qquad i = 2, ..., n,
\end{aligned}$$

The approximated solutions of (2), therefore, can be obtained by setting $p = 1$

$$y_i(x) = \sum_{i=0}^{\infty} y_{i,j}(x) = \lim_{p\to 1} y_i(x), \qquad i = 1, 2, \dots, n. \tag{17}$$

Now we propose a simple modification to accelerate the convergence of HPM, when it is used for systems of linear Volterra integral equations of the second kind.The modified idea is based on the replacement of the functions

$k_{i,j,1}(t)$, $k_{i,j,2}(tx)$ and $f_i(x)$ by their Taylor expansions ,where the the kernels $k(x,t)$ are separable, and $f_i(x)$ are analytic.

$$\begin{aligned} f_i(x) &= \sum_{l=0}^{\infty} f_{i,l}(x) \\ k_{i,j}(x,t) &= k_{i,j,1}(t)k_{i,j,2}(x) = \sum_{l=0}^{\infty} k_{i,j,1,l}(t) \sum_{l=0}^{\infty} k_{i,j,2,l}(x) \end{aligned} \tag{18}$$

where $f_{i,j}(x) = \frac{f_i(x_l)(x-x_l)}{l}$, $k_{i,j,1,l}(x) = \frac{k_{i,j,1}(x_l)(x-x_l)}{l}$ and

$k_{i,j,1,l}(t) = \frac{k_{i,j,1}(t_l)(t-t_l)}{l}$, respectively.

Substitution Eqs. (18) into Eq. (2) results in

$L(y_i) = y_i(x) - \sum_{l=0}^{\infty} f_{i,l}(x) - \int_0^x \sum_{l=0}^{\infty} k_{i,j,1,l}(t) \sum_{l=0}^{\infty} k_{i,j,2,l}(x) y_j(t)dt = 0,$

$j = 1, 2, \dots, n.$

The following homotopy can be constructed

$$\begin{aligned} &H(y_j, p) = y_j(x) - \sum_{l=0}^{\infty} f_{i,l}(x)p^l - \\ &p\int_0^1 \sum_{l=0}^{\infty} k_{i,j,1,l}(t)p^l \sum_{l=0}^{\infty} k_{i,j,2,l}(x)p^l y_j(t)dt = 0, \qquad j = 1, 2, \dots, n. \end{aligned} \tag{19}$$

Substituting (16) into (19), and equating the coefficients of the terms with identical powers of p, components of the series solution (17) will be obtained. This modification is simple and very effective tool which usually leads to the exact solutions. This method can be used for problems in which the homotopy perturbation method does not work.

4 Numerical Results

Numerical example is investigated in this section by the proposed methods.

Example : Consider the linear system of Volterra integral equation of the second kind:

$$y_1(x) = 1 + \int_0^x (\frac{1}{2(t+1)} y_1(t) - 2ty_2(t))dt, \tag{20}$$
$$y_2(x) = \int_0^x (\frac{1}{2(t+1)} y_2(t) + 2ty_1(t))dt,$$

which have the exact solutoin $y_1(x) = \sqrt{1+x}cos(x^2)$, $y_2(x) = \sqrt{1+x}sin(x^2)$

a) Applying the MADM we have

$y_{1,0}(x) = 1,$

$y_{2,0}(x) = 0 + \int_0^x 2ty_{1,0}(t)dt = x^2,$

and

$y_{1,m+1}(x) = \int_0^x (\frac{1}{2(t+1)} y_{1,m}(t) - 2ty_{2,m}(t))dt,$

$y_{2,m+1}(x) = \int_0^x (\frac{1}{2(t+1)} y_{2,m}(t) - 2ty_{1,m+1}(t))dt, \qquad m = 0, 1, \ldots$

b) Applying the MHPM we have

The Taylor series of the function $\frac{1}{2(t+1)}$, can be presented as follows:

$\frac{1}{2(t+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{2}$

by substitution of these series into Eq. (20) the following homotopy can be constructed.

$$y_1(x) = 1 + p \int_0^x (\sum_{n=0}^{\infty} \frac{(-1)^n t^n}{2} p^n y_1(t) - 2ty_2(t))dt, \tag{21}$$
$$y_2(x) = p \int_0^x (\sum_{n=0}^{\infty} \frac{(-1)^n t^n}{2} p^n y_2(t) + 2ty_1(t))dt,$$

Substituting (16) into (21), and equating the coefficients of the terms with identical powers of p, reads to

$p^0 : y_{1,0}(x) = 1, \qquad y_{2,0}(x) = 0$
$p^1 : y_{1,1}(x) = \frac{x}{2}, \qquad y_{2,1}(x) = x^2$
$p^{j+1} : y_{1,j+1}(x) = \int_0^x (\sum_{k=0}^{j} (\frac{(-1)^k t^k}{2} y_{1,j-k}(t)) - 2ty_{2,j}(t))dt,$

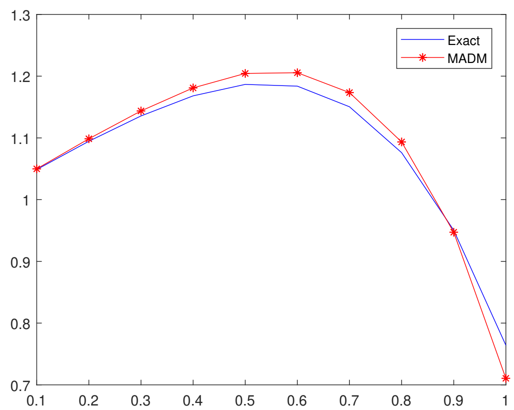

Figure 1: Comparison Between MADM and Exact Solutions $y_1(x)$

$$y_{2,j+1}(x) = \int_0^x (\sum_{k=0}^{j} (\frac{(-1)^k t^k}{2} y_{2,j-k}(t)) + 2t y_{1,j}(t)) dt, ,$$

Fig.1 and Fig. 2 show the comparison between the exact solution and the approximate solution obtained by the MADM and MHPM respectively for $y_1(x)$ and Fig.3 and Fig.4 show the comparison between the exact solution and the approximate solution obtained by the MADM and MHPM respectively for $y_2(x)$. It is seen from figures the solution obtained by the proposed methods nearly identical to the exact solution. In this example, the simplicity and accuracy of the proposed methods are illustrated by computing the absolute error of $E_3(x)$ for Example 1. The accuracy of the result can be improved by introducing more terms of the approximate solutions. In table 1 and table 2, MADM and MHPM solutions are compared with the exact solution of the Volterra integral Eq. (20). There is good agreement between the exact and approximate solution obtained by the proposed methods. The tables also show the absolute error between the exact and approximate solutions.

5 Conclusion

In this article, a modified form of ADM and HPM, for solving the system of linear Volterra integral equations of the second kind are studied successfully. The modified methods are better than ADM and HPM in the sense of accuracy and applicability. The MHPM and MADM are very powerful and efficient techniques for finding approximate solution for a wide class of problems. The MHPM is easy to apply and also requires less computations than MADM. Illustrative example that is presented clearly support this claim. The results have been approved the efficiency of these methods for solving the system of

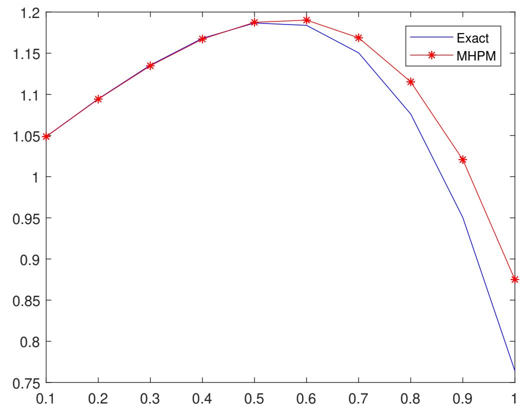

Figure 2: Comparison Between MHPM and Exact Solutions $y_1(x)$

Table 1: Comparison between exact solutions $y_1(x)$ and approximate solutions

x	$y_{Exact}(x)$	$y_{MADM}(x)$	$y_{MHPM}(x)$	$E_3 y_{MADM}(x)$	$E_3 y_{MHPM}(x)$
0.1	1.048756408	1.049873022	1.0487	1.1166×10^{-3}	5.6408×10^{-5}
0.2	1.094568876	1.098577447	1.0942	4.0085×10^{-3}	3.6887×10^{-4}
0.3	1.135560831	1.143660889	1.1347	8.1000×10^{-3}	8.6083×10^{-4}
0.4	1.168103075	1.180981693	1.1672	12.8786×10^{-3}	9.03075×10^{-4}
0.5	1.186670519	1.2044261	1.1875	17.7555×10^{-3}	8.2948×10^{-4}
0.6	1.183826247	1.205598349	1.1902	21.7721×10^{-3}	6.3737×10^{-3}
0.7	1.150421299	1.173407781	1.1687	22.9864×10^{-3}	18.2787×10^{-3}
0.8	1.076124383	1.093466763	1.1152	17.3423×10^{-3}	39.0756×10^{-3}
0.9	0.950408001	0.947200247	1.0207	3.2077×10^{-3}	70.2919×10^{-3}
1.0	0.764102848	0.710553188	0.8750	53.5496×10^{-3}	110.8971×10^{-3}

Table 2: Comparison between exact solutions $y_2(x)$ and approximate solutions

x	$y_{Exact}(x)$	$y_{MADM}(x)$	$y_{MHPM}(x)$	$E_3 y_{MADM}(x)$	$E_3 y_{MHPM}(x)$
0.1	0.010487913	0.00688301	0.0105	3.6049×10^{-3}	1.2087×10^{-5}
0.2	0.04380612	0.03013173	0.0440	13.6743×10^{-3}	1.9388×10^{-5}
0.3	0.102477313	0.073767507	0.1035	28.7098×10^{-3}	1.0226×10^{-3}
0.4	0.188507844	0.141813908	0.1920	46.6939×10^{-3}	3.4921×10^{-3}
0.5	0.30300673	0.237922467	0.3125	65.0842×10^{-3}	9.4932×10^{-3}
0.6	0.445595575	0.364785732	0.4680	80.8098×10^{-3}	22.4044×10^{-3}
0.7	0.613621084	0.523348922	0.6615	90.2721×10^{-3}	47.8789×10^{-3}
0.8	0.801221761	0.711858669	0.8960	89.3630×10^{-3}	94.7782×10^{-3}
0.9	0.998360972	0.924827783	1.1745	73.5331×10^{-3}	176.1390×10^{-3}
1.0	1.190019679	1.152051581	1.5000	37.9680×10^{-3}	309.9803×10^{-3}

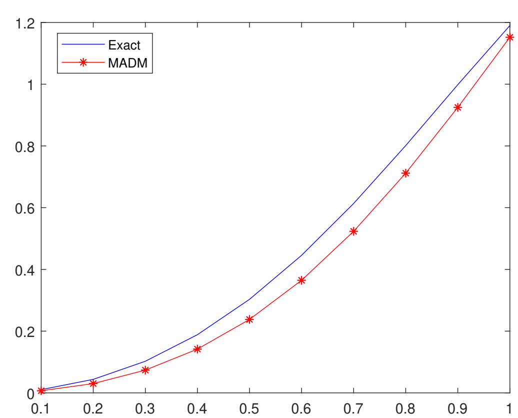

Figure 3: Comparison Between MADM and Exact Solutions $y_2(x)$

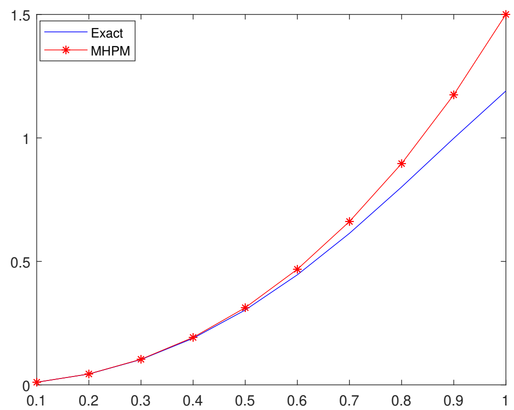

Figure 4: Comparison Between MHPM and Exact Solutions $y_2(x)$

Volterra integral equations of the second kind.

6 Open Problem

Study the variational iteration method (VIM) for solving linear Volterra integral equation of the second kind of the form:

$$y(x) = f(x) + \lambda \int_a^x k(x,t)y(t)dt, \tag{22}$$

and nonlinear Volterra integral equation of of the second kind of the form:

$$y(x) = f(x) + \lambda \int_a^x k(x,t)F(y(t))dt, \tag{23}$$

where the kernel $K(x,t)$ and the function $f(x)$ are given real valued functions, λ is a parameter and $F(y(x))$ is a nonlinear function of $y(x)$ and the unknown function $y(x)$ appears inside and outside the integral sign.

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References

[1] M.A. Abdelkawy, S.S. Ezz-Eldien and A.Z.Amin, *A Jacobi spectral collocation scheme for solving Abel's integral equations*, Progr. Fract. Differ. Appl., 1(3) (2015), 1–14.

[2] T. Aboiyar and H.Ibrahim, *Approximation of systems of Volterra integral equations of the second kind using the new iterative method*, Internat. J. Appl. Sci. Comput. Math. Theory, 1(4) (2015), 12–27.

[3] F.M. Al-Saar, K.P. Ghadle and P. A.Pathade, *The approximate solutions of fredholm integral equations by Adomian decomposition method and its modification*, Int. J. Math. Appl., 6 (2018), 327–336.

[4] A. Arikoglu and I. Ozkol, *Solutions of integral and integro-differential equation systems by using differential transform method*, Comput. Math. Appl., 56(9) (2008), 2411–2417.

[5] E. Babolian and J. Biazar, *Solution of a system of linear Volterra equations by Adomian decomposition method*, Far East J. Math. Sci., 2 (2002), 935–945.

[6] A. A. Hamoud and K.P. Ghadle,*Modified Adomian decomposition method for solving fuzzy Volterra-Fredholm integral equations*, J. Indian Math. Soc., 85(1-2) (2018) 53–69.

[7] A. A. Hamoud and K. P. Ghadle, *Recent advances on reliable methods for solving Volterra-Fredholm integral and integro-differential equations*, Asian J. Math. Comput. Res., 24 (2018), 128–157.

[8] A. A. Hamoud and K.P. Ghadle, *The combined modified Laplace with Adomian decomposition method for solving the nonlinear Volterra-Fredholm integro-differential equations*, J. Korean Soc. Ind. Appl. Math., 21 (2017), 17–28.

[9] A. A. Hamoud and K.P. Ghadle, *The approximate solutions of fractional Volterra-Fredholm integro-differential equations by using analytical techniques*, Probl. Anal. Issues Anal., 25(1) (2018), 41–58.

[10] A. A. Hamoud, A. D. Azeez and K. P.Ghadle, *A study of some iterative methods for solving fuzzy Volterra-Fredholm integral equations*, Indonesian J. Elec. Eng. & Comp. Sci., 11(3) (2018), 857–867.

[11] K. Maleknejad, K. Nouri and L.Torkzadeh, *Comparison projection method with Adomian's decomposition method for solving system of integral equations*, Bull. Malays. Math. Sci. Soc., 34(2) (2011), 379–388.

[12] K. Maleknejad, K. Nouri and M. Yousefi, *Discussion on convergence of Legender polynomial for numerical solution of integral equations*, Appl. Math. Comput., 193(2) (2007), 335–339.

[13] Z. Masouri, S. Hatamzadeh-Varmazyar and E. Babolian, *Numerical method for solving system of Fredhlom integral equations using Chebyshev cardinal functions*, Adv. Tech. Eng. Electromagn.,(2014), 1–13.

[14] K. Maleknejad and N. Aghazadeh, *Numerical solution of Volterra integral equations of the second kind with convolution kernel by using Taylor-series expansion method*,Appl. Math. Comput., 161 (2005), 914–922.

[15] S. S. Ray and P. K. Sahu, *Numerical methods for solving Fredholm integral equations of second kind*, Abstr. Appl. Anal., (2013), 1–17.

[16] N. A. Sulaiman, *Some numerical methods to solve a system of Fredholm integral equations of the 2nd kind with symmetric kernel*, Kirkuk Univ. J. Sci. Stud., 4(2) (2009), 108–116.