Int. J. Open Problems Compt. Math., Vol. 12, No. 4, December 2019
ISSN 1998-6262; Copyright ©ICSRS Publication, 2019
www.i-csrs.org

Dualistic structure on Sasakian manifolds

BELDJILALI Gherici

Laboratory of Quantum Physics and Mathematical Modeling (LPQ3M),
University of Mascara, Algeria
e-mail: gherici.beldjilali@univ-mascara.dz

Received 23 October 2019; Accepted 2 December 2019

(Communicated by Iqbal H. Jebril)

Abstract

Affine connections compatible with a contact structure are defined and conditions for two compatible connections on a Sasaki manifold to form a dualistic structure are given.

2010 Mathematics Subject Classification: 53C05, 53D17, 70G45.
Keywords: *conjugate connections, dualistic structure, Sasakian manifold, statistical manifold.*

1 Introduction

A fundamental theorem of Riemannian geometry states that given a metric, there is a unique connection (among the class of torsion-free connections) that "preserves" the metric, i.e., the following condition is satisfied

$$X.g(Y,Z) = g(\widehat{\nabla}_X Y, Z) + g(Y, \widehat{\nabla}_X Z) \tag{1}$$

for all vectors fields X, Y, Z on M.
Such a connection, denoted as $\widehat{\nabla}$, is called the Levi-Civita connection. It is compatible with the metric.
It turns out that one can define a kind of "compatibility" relation more general than expressed by (1), by introducing the notion of "conjugacy" (definoted by $*$) between two connections.

Geometry of conjugate connections is a natural generalization of geometry of Levi-Civita connections from Riemannian manifolds theory. Since conjugate connections arise from affine differential geometry and from geometric theory of statistical inferences [3].
Let (M, g) be a Riemannian manifold and ∇ an affine connection on M. A connection ∇^* is called conjugate connection of ∇ with respect to the metric g if

$$X.g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) \tag{2}$$

for arbitrary vectors fields X, Y, Z on M. The triple of a Riemannian metric and a pair of conjugates connection (g, ∇, ∇^*) satisfying (2) is called a dualistic structure on M.

Clearly, $(\nabla^*)^* = \nabla$. Moreover, $\widehat{\nabla}$, which satifies (1) , is special in the sense that it is self-conjugate $\widehat{\nabla}^* = \widehat{\nabla}$.

In the present paper, we shall introduce the notion of affine connections compatible with a contact structure and provide conditions for two such compatible connections to form a dualistic structure on Sasaki manifold.

2 Preliminaries

For more background on almost contact metric manifolds, we recommend the reference [5, 7].
An odd-dimensional Riemannian manifold (M, g) is said to be an almost contact metric manifold if there exist on M a $(1,1)$ tensor field φ, a vector field ξ (called the structure vector field) and a 1-form η such that $\eta(\xi) = 1$, $\phi^2(X) = -X + \eta(X)\xi$ and $g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$, for any vector fields X,Y on M. In particular, in an almost contact metric manifold we also have $\varphi\xi = 0$ and $\eta \circ \varphi = 0$.

Such a manifold is said to be a contact metric manifold if $d\eta = \phi$, where $\phi(X,Y) = g(X, \varphi Y)$ is called the fundamental 2-form of M. If, in addition, ξ is a Killing vector field, then M is said to be a K-contact manifold. It is well-known that a contact metric manifold is a K-contact manifold if and only if $\nabla_X \xi = -\varphi X$, for any vector field X on M. In a K-contact manifold, we have

$$(\nabla_X \varphi)Y = R(\xi, X)Y$$

for any X,Y ([5], pp. 92-94).
On the other hand, the almost contact metric structure of M is said to be normal if $[\varphi, \varphi](X,Y) = -2d\eta\ (X,Y)\xi$, for any X, Y, where $[\varphi, \varphi]$ denotes the Nijenhuis torsion of φ, given by

$$[\varphi, \varphi](X,Y) = \phi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$$

A normal contact metric manifold is called a Sasakian manifold. It can be proved that a Sasakian manifold is K-contact, and that an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X.$$

3 Main results

Let $(M^{2n+1}, \varphi, \xi, \eta, \phi, g)$ be a contact metric manifold where

$$\phi(X,Y) = d\eta(X,Y) = g(X, \varphi Y)$$

the 2-form on M for all X, Y vectors field on M .

We say that a pair of affine connections ∇ and ∇' are compatible with the contact structure ϕ if

$$X.\phi(Y,Z) = \phi(\nabla_X Y, Z) + \phi(Y, \nabla'_X Z) + \eta(Y)g(X,Z) - \eta(Z)g(X,Y). \tag{3}$$

In what follows, we shall prove some preliminary results needed to determine a dualistic structure on a Sasakian manifold (see Theorem 3.11).

Proposition 3.1 *Let ∇ and ∇' be an affine connections on the smooth manifold M, compatible with the contact structure ϕ and T, T' their torsion tensors respectively. Then:*

(a)

$$\begin{aligned} \phi\big(T(X,Y) - T'(X,Y), Z)\big) &= (\nabla'\phi)(X,Y,Z) - (\nabla'\phi)(Y,X,Z) \\ &+ \eta(X)g(Y,Z) - \eta(Y)g(X,Z), \end{aligned}$$

(b)

$$(\nabla\phi)(Y,X,Z) - (\nabla\phi)(X,Y,Z) = \eta(X)g(Y,Z) - \eta(Y)g(X,Z)$$

if and only if

$$(\nabla'\phi)(Y,X,Z) - (\nabla'\phi)(X,Y,Z) = \eta(X)g(Y,Z) - \eta(Y)g(X,Z),$$

for all X, Y, Z vectors fields on M.

Proof 3.2 *Using the compatibility condition, we get*

(a)

$$\begin{aligned}\phi\big(T(X,Y)-T'(X,Y),Z\big) &:= \phi\big(\nabla_X Y-\nabla_Y X-\nabla'_X Y+\nabla'_Y X,Z\big)\\ &= X.\phi(Y,Z)-\phi(Y,\nabla'_X Z)-\eta(Y)g(X,Z)\\ &+ \eta(Z)g(X,Y)-Y.\phi(X,Z)+\phi(X,\nabla'_Y Z)\\ &+ \eta(X)g(Y,Z)-\eta(Z)g(X,Y)-\phi(\nabla'_X Y,Z)\\ &+ \phi(\nabla'_Y X,Z)\\ &= (\nabla'\phi)(X,Y,Z)-(\nabla'\phi)(Y,X,Z)\\ &+ \eta(X)g(Y,Z)-\eta(Y)g(X,Z).\end{aligned}$$

(b) We have

$$(\nabla'\phi)(X,Y,Z) := X.\phi(Y,Z)-\phi(\nabla'_X Y,Z)-\phi(Y,\nabla'_X Z),$$

and we knowing that

$$\begin{aligned}\phi(\nabla'_X Y,Z) &= -\ \phi(Z,\nabla'_X Y)\\ &= -X.\phi(Z,Y)+\phi(\nabla_X Z,Y)-\eta(Y)g(X,Z)+\eta(Z)g(X,Y)\\ &= X.\phi(Y,Z)-\phi(Y,\nabla_X Z)-\eta(Y)g(X,Z)+\eta(Z)g(X,Y),\end{aligned}$$

then

$$\begin{aligned}(\nabla'\phi)(X,Y,Z) &:= -X.\phi(Y,Z)+\phi(\nabla_X Y,Z)+\phi(Y,\nabla_X Z)\\ &+ 2\eta(Y)g(X,Z)-2\eta(Z)g(X,Y)\\ &= -(\nabla\phi)(X,Y,Z)+2\eta(Y)g(X,Z)-2\eta(Z)g(X,Y).\end{aligned}$$

Now suppose that for all X,Y,Z vectors fields on M,

$$(\nabla'\phi)(Y,X,Z)-(\nabla'\phi)(X,Y,Z)=\eta(X)g(Y,Z)-\eta(Y)g(X,Z),$$

using the above equation, we obtain

$$(\nabla\phi)(Y,X,Z)-(\nabla\phi)(X,Y,Z)=\eta(X)g(Y,Z)-\eta(Y)g(X,Z),$$

and conversely.

Corollary 3.3 *Let ∇ and ∇' be an affine connections on the smooth manifold M, compatible with the contact structure ϕ and T,T' their torsion tensors respectively. Then:*

(a) If $(\nabla'\phi)(Y,X,Z)-(\nabla'\phi)(X,Y,Z)=\eta(X)g(Y,Z)-\eta(Y)g(X,Z)$,
then $T'=T$.

(b) If $(\nabla'\phi)(Y,X,Z) - (\nabla'\phi)(X,Y,Z) = \eta(X)g(Y,Z) - \eta(Y)g(X,Z)$, *and* $T' = 0$,
then $(\nabla\phi)(Y,X,Z) - (\nabla\phi)(X,Y,Z) = \eta(X)g(Y,Z) - \eta(Y)g(X,Z)$ *and* $T = 0$, *too.*

Proof 3.4 *Follows from Proposition 3.1*

Proposition 3.5 *Let* ∇ *be an affine connection on the Sasakian manifold* M *such that*

$$(\nabla_X\varphi)Y = g(X,Y)\xi - \eta(Y)X, \qquad (*)$$

for all X, Y *vectors fields on* M.
If ∇ *and* ∇' *are compatible with the contact structure* ϕ, *then*

$$\nabla'\varphi = \nabla\varphi.$$

Proof 3.6 *By* $(*)$, *we have*

$$\begin{aligned}\phi\big(Z, (\nabla'_X\varphi)Y - (\nabla_X\varphi)Y\big) &= \phi\big(Z, (\nabla'_X\varphi)Y\big) - \phi\big(Z, (\nabla_X\varphi)Y\big)\\ &= \phi\big(Z, \nabla'_X\varphi Y\big) + \phi\big(\varphi Z, \nabla'_X Y\big) + \eta(Y)\phi(Z,X),\end{aligned}$$

using the compatibility condition, we obtain

$$\begin{aligned}\phi\big(Z, (\nabla'_X\varphi)Y - (\nabla_X\varphi)Y\big) &= X.\phi(Z,\varphi Y) - \phi(\nabla_X Z, \varphi Y) - \eta(Z)g(X,\varphi Y)\\ &+ X.\phi(\varphi Z, Y) - \phi(\nabla_X\varphi Z, Y)\\ &= -\eta(Z)g(X,\varphi Y) - \phi\big((\nabla_X\varphi)Z, Y\big),\end{aligned}$$

replacing $(\nabla_X\varphi)Z = g(X,Z)\xi - \eta(Z)X$, *we obtain*

$$\phi\big(Z, (\nabla'_X\varphi)Y - (\nabla_X\varphi)Y\big) = 0 \qquad \Leftrightarrow \qquad \nabla'\varphi = \nabla\varphi.$$

Lemma 3.7 *Let* ∇ *be an affine connection on the Sasakian manifold* $(M, \varphi, \xi, \eta, g)$. *If*

$$(\nabla_X\varphi)Y = g(X,Y)\xi - \eta(Y)X.$$

for all X, Y *vectors fields on* M *then,*

1. $\nabla_X\xi = -\varphi X + \eta(\nabla_X\xi)\xi$,
2. $\eta(\nabla_\xi\varphi X) = \eta(\nabla_{\varphi X}\xi) = 0$,
3. $\eta(\nabla_X\xi) = g(X, \nabla_\xi\xi) = \eta(X)\eta(\nabla_\xi\xi)$.

Proof 3.8 *Suppose that for all* X, Y *vectors fields on* M,

$$(\nabla_X\varphi)Y = g(X,Y)\xi - \eta(Y)X.$$

1. *We can see that* $\nabla_\xi \varphi = 0$ *and for* $Y = \xi$ *we have*

$$\begin{aligned}(\nabla_X \varphi)\xi &= \eta(X)\xi - X \Leftrightarrow -\varphi\nabla_X\xi = \eta(X)\xi - X\\ &\Leftrightarrow \varphi^2\nabla_X\xi = \varphi X\\ &\Leftrightarrow \nabla_X\xi = -\varphi X + \eta(\nabla_X\xi)\xi,\end{aligned}$$

2. *We have*

$$2d\eta(X,Y) = X\eta(Y) - Y\eta(X) - \eta([X,Y]),$$

for $Y = \xi$ *we obtain*

$$d\eta(X,\xi) = -\xi\eta(X) - \eta([X,\xi]),$$

knowing that $d\eta(X,Y) = \phi(X,Y) = g(X,\varphi Y)$ *then,*

$$\xi\eta(X) = \eta([\xi,X]),$$

replacing X *by* φX *we obtain*

$$\begin{aligned}\eta([\xi,\varphi X]) = 0 \Leftrightarrow \eta(\nabla_{\varphi X}\xi) &= \eta(\nabla_\xi\varphi X)\\ &= \eta\big((\nabla_\xi\varphi)X\big) + \eta(\varphi\nabla_\xi X)\\ &= 0.\end{aligned}$$

3. *We have* $X = -\varphi^2 X + \eta(X)\xi$ *so,*

$$\eta(\nabla_X\xi) = -\eta(\nabla_{\varphi^2 X}\xi) + \eta(X)\eta(\nabla_\xi\xi),$$

using lemma (3.7, 2) we obtain

$$\eta(\nabla_X\xi) = \eta(X)\eta(\nabla_\xi\xi).$$

On the other hand,

$$\begin{aligned}g(X,\nabla_\xi\xi) &= g\big(-\varphi^2 X + \eta(X)\xi, \nabla_\xi\xi\big)\\ &= g(\varphi X, \varphi\nabla_\xi\xi) + \eta(X)\eta(\nabla_\xi\xi),\end{aligned}$$

we can see that $\varphi\nabla_\xi\xi = 0$ *so,*

$$g(X,\nabla_\xi\xi) = \eta(X)\eta(\nabla_\xi\xi).$$

which prove our assertions.

Proposition 3.9 *Let ∇ be an affine connection on the Sasakian manifold $(M, \varphi, \xi, \eta, g)$ such that for all X, Y vectors fields on M,*

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X.$$

If

$$(\nabla\phi)(Y, X, Z) - (\nabla\phi)(X, Y, Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z),$$

for all X, Y, Z vectors fields on M then ∇g is symmetric.

Proof 3.10 *For all X, Y, Z vectors fields on M, we have*

$$\begin{aligned}(\nabla g)(X, Y, \varphi Z) &:= X.g(Y, \varphi Z) - g(\nabla_X Y, \varphi Z) - g\big(Y, \nabla_X(\varphi Z)\big)\\ &= X.\phi(X, Z) - \phi(\nabla_X Y, Z) - g\big(Y, (\nabla_X \varphi)Z + \varphi\nabla_X Z\big)\\ &= X.\phi(X, Z) - \phi(\nabla_X Y, Z) - \phi(Y, \nabla_X Z) - g\big(Y, (\nabla_X \varphi)Z\big),\end{aligned}$$

using again $(\nabla_X \varphi)Z = g(X, Z)\xi - \eta(Z)X$, we get

$$(\nabla g)(X, Y, \varphi Z) = (\nabla\phi)(X, Y, Z) - \eta(Y)g(X, Z) + \eta(Z)g(Y, X), \tag{4}$$

and we have also

$$(\nabla g)(Y, X, \varphi Z) = (\nabla\phi)(Y, X, Z) - \eta(X)g(Y, Z) + \eta(Z)g(X, Y), \tag{5}$$

differentiating 4 from 5 we obtain

$$\begin{aligned}(\nabla g)(Y, X, \varphi Z) - (\nabla g)(X, X, \varphi Z) &= (\nabla\phi)(Y, X, Z) - (\nabla\phi)(X, Y, Z)\\ &- \eta(X)g(Y, Z) + \eta(Y)g(X, Z).\end{aligned}$$

so, if

$$(\nabla\phi)(Y, X, Z) - (\nabla\phi)(X, Y, Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z),$$

then

$$(\nabla g)(X, Y, \varphi Z) = (\nabla g)(Y, X, \varphi Z).$$

On the other hand, we have

$$(\nabla g)(X, Y, \xi) = X\eta(Y) - \eta(\nabla_X Y) - g(Y, \nabla_X \xi),$$

using lemma (3.7,1) then lemma (3.7,3) we obtain,

$$(\nabla g)(X, Y, \xi) = X\eta(Y) - \eta(\nabla_X Y) + d\eta(Y, X) - \eta(Y)\eta(X)\eta(\nabla_\xi \xi).$$

In the same way, we have

$$(\nabla g)(Y, X, \xi) = Y\eta(X) - \eta(\nabla_Y X) + d\eta(X, Y) - \eta(X)\eta(Y)\eta(\nabla_\xi \xi).$$

differentiating the two above equations we obtain

$$(\nabla g)(X, Y, \xi) - (\nabla g)(X, Y, \xi) = 0.$$

This completes the proof of the proposition.

Recall that (M, ∇, g) is statistical manifold if ∇ is a torsion free affine connection and ∇g is symmetric, for g a pseudo-Riemannian metric on M. If ∇' is another affine connection on M such that ∇ and ∇' are compatible with g, then ∇' is also torsion free and $\nabla' g$ is symmetric. In this case, (M, ∇', g) is also statistical manifold called the dual statistical manifold of (M, ∇, g) and we say that (∇, ∇', g) is a dualistic structure on M [2].

Theorem 3.11 *Let $(M, \varphi, \xi, \eta, \phi, g)$ be a Sasakian manifold and let ∇' be a torsion free affine connection on M such that*

$$(\nabla'_X \varphi)Y = g(X, Y)\xi - \eta(Y)X,$$

$$(\nabla'\phi)(Y, X, Z) - (\nabla'\phi)(X, Y, Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z),$$

for all X, Y, Z vectors field on M and ∇ and ∇' are compatible with ϕ. Then (M, ∇', g) and (M, ∇, g) are statistical manifolds and (∇, ∇', g) is a dualistic structure on M.

Proof 3.12 *Follows from propositions (3.1),(3.5) and (3.9).*

4 Open Problem

By considering propositions 4 and 7 in [4], it is proved that there are Kähler and Sasakian manifolds of any dimensions. We start with $\mathbb{R}^2$ and its natural Kähler structure. There is a 3-dimension Sasakian manifold $M = \mathbb{R} \times \mathbb{R}^2$. Then, by the method of Oubinã [6] and consequently we have a Kählerian manifold of dimension 4. Continuing the current method, we produce Kähler and Sasaki manifolds of any dimensions $n \geq 2$.

regarding this close relation, one ask if it is possible to construct a dualistic structure on a Kählerian manifold starting from a dualistic structure on a Sasakian manifold and vice versa, based on this paper and work [2] presented by M. Blaga.

References

[1] Abdou salam D., Dualistic structures on doubly warped product manifolds, *Int. Electron. J. Geom.* Vol. 6, No.1, (2013), pp. 41-45 .

[2] Adara, M. Blaga, Dualistic structures on Kähler manifolds, *Revista de la Union Mat, Argentina.*, Vol 53, N. 1, (2012), 55-60.

[3] Amari, S. and Nagaoka, H., Methods of Information geometry, AMS, *Oxford University Press*, Vol. 191, (2000).

[4] G. Beldjilali, A. Mohamed cherif, K. Zegga, From a single Sasakian manifold to a family of Sasakian manifolds, *Beitr Algebra Geom* Vol. 60, No 3, (2018), 445-458.

[5] Blair, D. E., Riemannian Geometry of Contact and Symplectic Manifolds, *Progress in Mathematics 203*, Birkhiiuser, (2002).

[6] J. A. Oubiña, New classes of almost contact metric structures. *Publ. Math.* Debrecen, **32** (1985), 187-193.

[7] Yano, K. and Kon, M., Structures on Manifolds, *Series in Pure Math.*, Vol 3, World Sci.,1984.