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# A New Generalized Sequence Space of Interval Numbers Defined by Orlicz Function

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**Abstract**

***In this study, we introduce a new generalized sequence space of interval numbers using by Orlicz function and examine some properties of resulting sequence classes of interval numbers.***

**Keywords:** *Keywords Interval number, Orlicz function, complete space.*
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## 1 Introduction

The topic of interval analysis has been studied for a long time. For a detailed discussion we may suggest refer to some books, for example Moore [20]. The main issue is to regard to closed intervals as a kind of "points". Hereafter we will called them "interval numbers".

Interval arithmetic was first suggested by Dwyer [12] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [20] in 1959 and Moore and Yang [14] in 1962. Furthermore, Moore [20], Dwyer [13] and Markov [15] have developed applications to differential equations.

Chiao in [11] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Şengönül and Eryilmaz in [16] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete metric space. Recently, Esi in [1, 2, 3, 4, 5, 6], Esi and Braha [7], Esi and Esi [8], Esi and Hazarika [9] defined and studied different properties of interval numbers.

# 2 Preliminaries

We denote the set of all real valued closed intervals by I$R$. Any elements of I$R$ is called interval number and denoted by $\overline{x} = [x_l, x_r]$. Let $x_l$ and $x_r$ be first and last points of $x$ interval number, respectively. For $\overline{x}_1, \overline{x}_2 \in$I$R$, we have $\overline{x}_1 = \overline{x}_2 \Leftrightarrow x_{1_l} = x_{2_l}$, $x_{1_r} = x_{2_r}$. $\overline{x}_1 + \overline{x}_2 = \{x \in R\text{: } x_{1_l} + x_{2_l} \leq x \leq x_{1_r} + x_{2_r}\}$, and if $\alpha \geq 0$, then $\alpha\overline{x} = \{x \in R\text{: } \alpha x_{1_l} \leq x \leq \alpha x_{1_r}\}$ and if $\alpha < 0$, then $\alpha\overline{x} = \{x \in R : \alpha x_{1_r} \leq x \leq \alpha x_{1_l}\}$,

$$\overline{x}_1.\overline{x}_2 = \{x \in R : \min\{x_{1_l}.\ x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}\} \leq x \leq \max\{x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l},\ x_{1_r}.x_{2_r}\}.$$

The set of all interval numbers $IR$ is a complete metric space defined by

$$d(\overline{x}_1, \overline{x}_2) = \max\{|x_{1_l} - x_{2_l}|, |x_{1_r} - x_{2_r}|\} \text{ [11]}.$$

In the special case $\overline{x}_1 = [a, a]$ and $\overline{x}_2 = [b, b]$, we obtain usual metric of $R$. Let us define transformation $f : N \to R$ by $k \to f(k) = \overline{x}_k$, $\overline{x} = (\overline{x}_k)$. Then $\overline{x} = (\overline{x}_k)$ is called sequence of interval numbers. The $\overline{x}_k$ is called $k^{th}$ term of sequence $\overline{x} = (\overline{x}_k)$. Let $w^i$denotes the set of all interval numbers with real terms and the algebraic properties of $w^i$ can be found in [15].

Now we recall the definition of convergence of interval numbers:

**Definition 2.1.** *[11] A sequence $\overline{x} = (\overline{x}_k)$ of interval numbers is said to be convergent to the interval number $\overline{x}_o$ if for each $\varepsilon > 0$ there exists a positive integer $k_o$ such that $d(\overline{x}_k, \overline{x}_o) < \varepsilon$ for all $k \geq k_o$ and we denote it by $\lim_k \overline{x}_k = \overline{x}_o$.*

Thus, $\lim_k \overline{x}_k = \overline{x}_o \Longleftrightarrow \lim_k x_{k_l} = x_{o_l}$ and $\lim_k x_{k_r} = x_{o_r}$ .

The set of all closed intervals in I$R$ is not real vector space. The main reason is that there will be no additive inverse element for each interval numbers. In this work we wish to present some special classes of interval numbers on the interval valued metric space.

Let $p = (p_k)$ be a bounded sequence of strictly positive real numbers. If $H = \sup_k p_k$, then for any two complex numbers $a_k$ and $b_k$ we have

$$|a_k + b_k|^{p_k} \leq C\left(|a_k|^{p_k} + |b_k|^{p_k}\right) \tag{1}$$

where $C = \max(1, 2^{H-1})$.

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Sequence spaces defined by Orlicz functions have been investigated by Et et.al.[21], Tripathy et.al. [22], Tripathy and Dutta [23] and many others.

Let $M$ be an Orlicz function, $s \geq 0$ is a reel number and $p = (p_k)$ be a sequence of positive real numbers such that $0 \leq h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$. Let $\mathrm{P}_s$ denotes the set of all subsets of N, that do not contain more than s elements. With $(\phi_s)$, we will denote a nondecreasing sequence of positive real numbers such that $(s-1)\,\phi_{s-1} \leq (s-1)\,\phi_s$ and $\phi_s \to \infty$ as s$\to \infty$. The class of all the sequences $(\phi_s)$ satisfying this property is denoted by $\Phi$. We introduce the following sequence space of interval number sequences.

$$\bar{\ell}_p(M,p,\Phi) = \{\overline{x} = (\overline{x}_k) \in w^i : \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s}\sum_{k\in\sigma}\left[M\left(\frac{\overline{d}\left(\overline{x}_k, \overline{0}\right)}{r}\right)\right]^{p_k} < \infty, \ for \text{ some } r > 0\}$$

# 3 Main results

**Theorem 3.1.** *The set $\bar{\ell}_p(M,p,\Phi)$ of sequences of interval numbers defined by a Orlicz function are closed under the coordinatewise addition and scalar multiplication.*

*Proof.* Let define the operations "+" and "." as follows:

$$+ : \bar{\ell}_\infty(M,p,s) \times \bar{\ell}_\infty(M,p,s) \to \bar{\ell}_\infty(M,p,s)$$

and

$$\cdot : R \times \bar{\ell}_\infty(M,p,s) \to \bar{\ell}_\infty(M,p,s).$$

Let $\overline{x}, \overline{y} \in \bar{\ell}_p(M,p,\Phi)$, then we may write

$$\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s}\sum_{k\in\sigma}\left[M\left(\frac{\overline{d}\left(\overline{x}_k, \overline{0}\right)}{r}\right)\right]^{p_k} < \infty \text{ and } \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s}\sum_{k\in\sigma}\left[M\left(\frac{\overline{d}\left(\overline{y}_k, \overline{0}\right)}{r}\right)\right]^{p_k} < \infty.$$

Since $\overline{d}(\overline{x}_k + \overline{y}_k, \overline{0}) \leq \overline{d}\left(\overline{x}_k, \overline{0}\right) + \overline{d}(\overline{y}_k, \overline{0})$ and using nondecreasing of $M$ Orlicz function, we obtain

$$\begin{aligned} M(\overline{d}(\overline{x}_k + \overline{y}_k, \overline{0})) &\leq M(\overline{d}\left(\overline{x}_k, \overline{0}\right) + \overline{d}(\overline{y}_k, \overline{0})) \\ &\leq M(\overline{d}\left(\overline{x}_k, \overline{0}\right)) + M(\overline{d}(\overline{y}_k, \overline{0})). \end{aligned}$$

Since the sequence $p=(p_k)$ satisfies $0\le h=\inf_k p_k\le p_k\le \sup_k p_k=H<\infty$ and since $C=\max(1,2^{H-1})$, then

$$\begin{aligned}\sup_{s\ge 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}[M(\overline{d}(\overline{x}_k+\overline{y}_k,\overline{0}))]^{p_k} &\le \sup_{s\ge 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}[M(\overline{d}\left(\overline{x}_k,\overline{0}\right))+M(\overline{d}(\overline{y}_k,\overline{0}))]^{p_k}\\ &\le C\sup_{s\ge 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}[M(\overline{d}\left(\overline{x}_k,\overline{0}\right))]^{p_k}\\ &+C\sup_{s\ge 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}[M(\overline{d}(\overline{y}_k,\overline{0}))]^{p_k}.\end{aligned}$$

Then

$$\begin{aligned}&\sup_{s\ge 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}[M(\overline{d}(\overline{x}_k+\overline{y}_k,\overline{0}))]^{p_k}\\ &\le \sup_{s\ge 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}C[M(\overline{d}\left(\overline{x}_k,\overline{0}\right))^{p_k}]+\sup_{s\ge 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}C[M(\overline{d}(\overline{y}_k,\overline{0}))]^{p_k}<\infty.\end{aligned}$$

Therefore $\overline{x}+\overline{y}\in\overline{\ell}_p(M,p,\Phi)$. Now, let $\overline{x}\in\overline{\ell}_p(M,p,\Phi)$. Then

$$\sup_{s\ge 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\left[M\left(\frac{\overline{d}\left(\overline{x}_k,\overline{0}\right)}{r}\right)\right]^{p_k}<\infty.$$

Let $\alpha\in R$. Since $\overline{d}(\alpha\overline{x}_k,\overline{0})=|\alpha|\overline{d}(\overline{x}_k,\overline{0})$ then $M(\overline{d}(\alpha\overline{x}_k,\overline{0}))=M(|\alpha|\overline{d}(\overline{x}_k,\overline{0}))\le |\alpha|M(\overline{d}(\overline{x}_k,\overline{0}))$ and since $p=(p_k)$ is bounded sequence of positive numbers, then we obtain $[M(\overline{d}(\alpha\overline{x}_k,\overline{0}))]^{p_k}\le|\alpha|^{p_k}[M(\overline{d}(\overline{x}_k,\overline{0}))]^{p_k}$. Then we may write

$$\sup_{s\ge 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}[M(\overline{d}(\alpha x_k,\overline{0}))]^{p_k}\le \sup_{s\ge 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}|\alpha|^{p_k}[M(\overline{d}(\overline{x}_k,\overline{0}))]^{p_k}<\infty.$$

So, $\alpha\overline{x}\in\overline{\ell}_p(M,p,\Phi)$. □

**Theorem 3.2.** *Let $p=(p_k)$ be bounded. The class of sequences of interval numbers $\overline{\ell}_p(M,p,\Phi)$ is complete metric space with respect to the following metric*

$$\overline{d}_p(\overline{x},\overline{y})=\inf\left\{r^{\frac{p_k}{T}}:\sup_{s\ge 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\left[M\left(\frac{\overline{d}\left(\overline{x}_k,\overline{y}_k\right)}{r}\right)\right]^{p_k}\le 1\right\}$$

*where $T=\max(1,\sup_k p_k=H<\infty)$.*

*Proof.* It can be easily verified that the class is metric space. Let $(\overline{x}^i)=\{\overline{x}^i_0,\overline{x}^i_1,\dots\}$ be a Cauchy sequence in $\overline{\ell}_p(M,p,\Phi)$. Then $\overline{d}_\infty(\overline{x}^i,\overline{x}^j)\to 0$ as $i,j\to\infty$. For given $\varepsilon>0$ choose $r>0$ and $x_o>0$ be such that $\frac{\varepsilon}{rx_0}>0$ and

$M\left(\frac{rx_0}{2}\right) \geq 1$. Then there exists $n_o \in N$ such that $\overline{d}_\infty(\overline{x}^i, \overline{x}^j) < \frac{\varepsilon}{rx_0}$ for all $i, j \geq n_o$. This implies

$$\inf\left\{r^{\frac{p_k}{T}} : \sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\left[M\left(\frac{\overline{d}\left(\overline{x}_k^i, \overline{x}_k^j\right)}{r}\right)\right]^{p_k} \leq 1\right\} < \frac{\varepsilon}{rx_0}.$$

Now, $M\left(\frac{\overline{d}\left(\overline{x}_k^i, \overline{x}_k^j\right)}{r}\right) \leq 1 \leq M\left(\frac{rx_0}{2}\right)$ implies that $\frac{\overline{d}\left(\overline{x}_k^i, \overline{x}_k^j\right)}{\overline{d}_\infty(\overline{x}^i, \overline{x}^j)} \leq \frac{rx_0}{2}$. So we obtain $\overline{d}\left(\overline{x}_k^i, \overline{x}_k^j\right) < \frac{rx_0}{2}\frac{\varepsilon}{rx_0} = \frac{\varepsilon}{2}$. This implies $\left(\overline{x}_k^{(i)}\right)$ is a Cauchy sequence of interval numbers in I$R$ for all fixed $k \in N$. Since the set of interval numbers set I$R$ is complete, so there exists an interval number $\overline{x} = (\overline{x}_k)$, such that $\overline{x}_k^i \to \overline{x}_k$ as i$\to \infty$. Now

$$\lim_{j\to\infty}\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\left[M\left(\frac{\overline{d}\left(\overline{x}_k^i, \overline{x}_k^j\right)}{r}\right)\right] \leq 1 \Longrightarrow \sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\left[M\left(\frac{\overline{d}\left(\overline{x}_k^i, \overline{x}_k\right)}{r}\right)\right] \leq 1.$$

Let $j \geq n_o$ then taking infimum of such $r$, we have $\overline{d}\left(\overline{x}_k^i, \overline{x}_k\right) < \varepsilon$. Now using $\overline{d}\left(\overline{x}_k, \overline{0}\right) \leq \overline{d}\left(\overline{x}_k, \overline{x}_k^i\right) + \overline{d}\left(\overline{x}_k^i, \overline{0}\right)$ we get $\overline{x} = (\overline{x}_k) \in \bar{\ell}_p(M, p, \Phi)$. Since $\{\overline{x}_k^{(i)}\}$ is arbitrary Cauchy sequence then the space $\bar{\ell}_p(M, p, \Phi)$ is complete. Now we give relation between $\bar{\ell}_p(M, p, \Phi^1)$ and $\bar{\ell}_p(M, p, \Phi^2)$ with respect to an Orlicz function. □

*Theorem.* $\bar{\ell}_p(M, p, \Phi^1) \subset \bar{\ell}_p(M, p, \Phi^2)$ if and only if A$=\sup_{s\geq 1}\frac{\phi_s^1}{\phi_s^2} < \infty$. □

*Proof.* Let $\overline{x} \in \bar{\ell}_p(M, p, \Phi^1)$ and A$=\sup_{s\geq 1}\frac{\phi_s^1}{\phi_s^2} < \infty$. Then we can write

$$\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s^2}\sum_{k\in\sigma}\left[M\left(\frac{\overline{d}\left(\overline{x}_k, \overline{0}\right)}{r}\right)\right]^{p_k} \leq A\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s^1}\sum_{k\in\sigma}\left[M\left(\frac{\overline{d}\left(\overline{x}_k, \overline{0}\right)}{r}\right)\right]^{p_k}.$$

Therefore $\overline{x} \in \bar{\ell}_p(M, p, \Phi^2)$. Conversely, let $\bar{\ell}_p(M, p, \Phi^1) \subset \bar{\ell}_p(M, p, \Phi^2)$ and $\overline{x} \in \bar{\ell}_p(M, p, \Phi^1)$. Then there exists $r > 0$ such that

$$\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s^1}\sum_{k\in\sigma}\left[M\left(\frac{\overline{d}\left(\overline{x}_k, \overline{0}\right)}{r}\right)\right]^{p_k} < \infty.$$

Now suppose that $A = \sup_{s\geq 1}\frac{\phi_s^1}{\phi_s^2} = \infty$. Then there exists a sequence of natural numbers $(s_j)$ such that $\lim_{j\to\infty}\frac{\phi_{s_j}^1}{\phi_{s_j}^2} = \infty$. Hence we can write

$$\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s^2}\sum_{k\in\sigma}\left[M\left(\frac{\overline{d}\left(\overline{x}_k, \overline{0}\right)}{r}\right)\right]^{p_k} \geq A\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s^1}\sum_{k\in\sigma}\left[M\left(\frac{\overline{d}\left(\overline{x}_k, \overline{0}\right)}{r}\right)\right]^{p_k} = \infty.$$

Therefore $\overline{x} \notin \bar{\ell}_p(M,p,\Phi^2)$ which is contradiction to the fact that $\bar{\ell}_p(M,p,\Phi^1) \subset \bar{\ell}_p(M,p,\Phi^2)$. Hence A=$\sup_{s\geq 1}\frac{\phi_s^1}{\phi_s^2} < \infty$. The following result is a consequence of this theorem. □

*Proposition.* Let M be an Orlicz function. Then $\bar{\ell}_p(M,p,\Phi^1) = \bar{\ell}_p(M,p,\Phi^2)$ if and only if $\sup_{s\geq 1}\frac{\phi_s^1}{\phi_s^2} < \infty$ and $\sup_{s\geq 1}\frac{\phi_s^2}{\phi_s^1} < \infty$. □

*Theorem.* Let $M_1$ and $M_2$ be two Orlicz functions which satisfying the $\Delta_2-$ *condition*. Then $\bar{\ell}_p(M_1,p,\Phi) \subset \bar{\ell}_p(M_1 o M_2,p,\Phi)$. □

*Proof.* Let $\bar{\ell}_p(M_1,p,\Phi)$ and $\varepsilon > 0$ be given and choose $\delta$ with $0 < \delta < 1$ such that $M_1(t) < \varepsilon$ for $0 \leq t \leq \delta$. We may write

$$\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\left[M_1 o M_2\left(\frac{\overline{d}\left(\overline{x}_k,\overline{0}\right)}{r}\right)\right]^{p_k} = \sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{1}\left[M_1 o M_2\left(\frac{\overline{d}\left(\overline{x}_k,\overline{0}\right)}{r}\right)\right]^{p_k} + \sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{2}\left[M_1 o M_2\left(\frac{\overline{d}\left(\overline{x}_k,\overline{0}\right)}{r}\right)\right]^{p_k}$$

where the summation $\sum_1$ is over $M_2\left(\frac{\overline{d}(\overline{x}_k,\overline{0})}{r}\right) \leq \delta$ and the summation $\sum_2$ is over $M_2\left(\frac{\overline{d}(\overline{x}_k,\overline{0})}{r}\right) > \delta$. Since $M_1$ is continuous, we have

$$\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{1}\left[M_1 o M_2\left(\frac{\overline{d}\left(\overline{x}_k,\overline{0}\right)}{r}\right)\right]^{p_k} \leq \max(1,M_1(1))^H \sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{1}\left[M_2\left(\frac{\overline{d}\left(\overline{x}_k,\overline{0}\right)}{r}\right)\right]^{p}$$

Now for $M_2\left(\frac{\overline{d}(\overline{x}_k,\overline{0})}{r}\right) > \delta$, we use the fact that

$$M_2\left(\frac{\overline{d}\left(\overline{x}_k,\overline{0}\right)}{r}\right) < M_2\left(\frac{\overline{d}\left(\overline{x}_k,\overline{0}\right)}{r}\right)\delta^{-1} \leq 1 + M_2\left(\frac{\overline{d}\left(\overline{x}_k,\overline{0}\right)}{r}\right)\delta^{-1}.$$

Since $M_1$ satisfies $\Delta_2 -$ *condition*,then there exists $B > 1$ such that

$$M_1\left[M_2\left(\frac{\overline{d}\left(\overline{x}_k,\overline{0}\right)}{r}\right)\right] \leq M_1\left[1 + M_2\left(\frac{\overline{d}\left(\overline{x}_k,\overline{0}\right)}{r}\right)\delta^{-1}\right]$$

$$\leq \frac{1}{2}M_1\left(2\right) + \frac{1}{2}M_1(2M_2\left(\frac{\overline{d}\left(\overline{x}_k,\overline{0}\right)}{r}\right)\delta^{-1}) < \frac{1}{2}BM_1(2)M_2\left(\frac{\overline{d}\left(\overline{x}_k,\overline{0}\right)}{r}\right)\delta^{-1} + \frac{1}{2}BM_1(2)M_2\left(\frac{\overline{d}\left(\overline{x}_k,\overline{0}\right)}{r}\right.$$

$$= BM_1(2)M_2\left(\frac{\overline{d}\left(\overline{x}_k,\overline{0}\right)}{r}\right)\delta^{-1}.$$

Then we have

$$\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{2}\left[M_1 o M_2\left(\frac{\overline{d}\left(\overline{x}_k,\overline{0}\right)}{r}\right)\right]^{p_k}$$

$$\leq \max(1, BM_1(2)\delta^{-1})^H \sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{2}\left[M_2\left(\frac{\overline{d}\left(\overline{x}_k,\overline{0}\right)}{r}\right)\right]^{p_k}$$

$$\leq \max(1, BM_1(2)\delta^{-1})^H \sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\left[M_2\left(\frac{\overline{d}\left(\overline{x}_k,\overline{0}\right)}{r}\right)\right]^{p_k}.$$

Hence

$$\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\left[M_1 o M_2\left(\frac{\overline{d}\left(\overline{x}_k,\overline{0}\right)}{r}\right)\right]^{p_k}$$

$$\leq \max(1, M_1(1))^H \sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\left[M_2\left(\frac{\overline{d}\left(\overline{x}_k,\overline{0}\right)}{r}\right)\right]^{p_k}$$

$$+ \max(1, BM_1(2)\delta^{-1})^H \sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\left[M_2\left(\frac{\overline{d}\left(\overline{x}_k,\overline{0}\right)}{r}\right)\right]^{p_k}.$$

It follows that $\overline{x}\in \overline{\ell}_p(M_1 o M_2, p, \Phi)$. □

**Theorem 3.3.** *Let $M$ and $N$ be two Orlicz functions. Then $\overline{\ell}_p(M,p,\Phi)\cap \overline{\ell}_p(N,p,\Phi)\subset \overline{\ell}_p(M+N,p,\Phi)$.*

*Proof.* Let $\overline{x}=(\overline{x}_k)\in \overline{\ell}_p(M,p,\Phi)\cap\overline{\ell}_p(N,p,\Phi)$. From (1), we have

$$\begin{aligned}[(M+N)(\overline{d}\left(\overline{x}_k,\overline{0}\right))]^{p_k} &= [M\left(\overline{d}\left(\overline{x}_k,\overline{0}\right)\right)+N\left(\overline{d}\left(\overline{x}_k,\overline{0}\right)\right)]^{p_k}\\ &\leq C[M(\overline{d}\left(\overline{x}_k,\overline{0}\right))]^{p_k}+C[N(\overline{d}\left(\overline{x}_k,\overline{0}\right))]^{p_k}.\end{aligned}$$

Then we can write

$$\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}[(M+N)(\overline{d}\left(\overline{x}_k,\overline{0}\right))]^{p_k}$$

$$\leq C\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}[M(\overline{d}\left(\overline{x}_k,\overline{0}\right))]^{p_k}+C\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}[N(\overline{d}\left(\overline{x}_k,\overline{0}\right))]^{p_k}.$$

Hence we obtain $\overline{x}=(\overline{x}_k)\in\overline{\ell}_p(M+N,p,\Phi)$. □

**Theorem 3.4.** *Let $M$ be an Orlicz function, then **a)** $\overline{\ell}_\infty\subset\overline{\ell}_p(M,p,\Phi)$ , **b)** If $M$ is bounded then $\overline{\ell}_p(M,p,\Phi)=w^i$.*

*Proof.* **a)** *Let* $\overline{x} = (\overline{x}_k) \in \bar{\ell}_\infty$. Then there exists a positive integer $G \geq 0$ such that $\overline{d}\left(\overline{x}_k, \overline{0}\right) \leq G$. Then the sequence $(M(\overline{d}\left(\overline{x}_k, \overline{0}\right)))$ is also bounded. Hence

$$[M(\overline{d}(\overline{x}_k, \overline{0}))]^{p_k} \leq [GM(1)]^{p_k} \leq [GM(1)]^H < \infty.$$

Therefore $\overline{x} = (\overline{x}_k) \in \bar{\ell}_p(M, p, \Phi)$. **b)** If $M$ Orlicz function is bounded then for any $\overline{x} = (\overline{x}_k) \in w^i$, $[M(\overline{d}(\overline{x}_k, \overline{0}))]^{p_k} \leq L^{p_k} \leq L^H < \infty$. Hence we obtain $\bar{\ell}_p(M, p, \Phi) = w^i$. Let X be a sequence space. Then X is called solid (or normal) if $(\alpha_k x_k) \in X$ whenever $(x_k) \in X$ for all sequences $(\alpha_k)$ of scalars with $|\alpha_k| \leq 1$ for all k$\in N$. □

**Theorem 3.5.** *The space* $\bar{\ell}_p(M, p, \Phi)$ *is solid space.*

*Proof.* Let $\alpha = (\alpha_k)$ be a sequence of scalars such that $|\alpha_k| \leq 1$ for all k$\in N$. We get

$$\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} \left[ M\left( \frac{\overline{d}\left(\alpha_k \overline{x}_k, \overline{0}\right)}{r} \right) \right]^{p_k}$$

$$\leq \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} \left[ M\left( \frac{\sup|\alpha_k|\,\overline{d}\left(\overline{x}_k, \overline{0}\right)}{r} \right) \right]^{p_k} \leq \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} \left[ M\left( \frac{\overline{d}\left(\overline{x}_k, \overline{0}\right)}{r} \right) \right]^{p_k}.$$

Then the result follows from the above inequality. □

**Theorem 3.6.** *Let* $M$ *be an Orlicz function.* ***a)*** *If* $0 < \inf p_k = h \leq p_k \leq 1$, *then* $\bar{\ell}_p(M, p, \Phi) \subset \bar{\ell}_p(M, \Phi)$, ***b)*** *If* $1 \leq p_k \leq \sup p_k < \infty$, *then* $\bar{\ell}_p(M, \Phi) \subset \bar{\ell}_p(M, p, \Phi)$, ***c)*** *Let* $0 < p_k \leq q_k$ *for each* $k \in \mathbb{N}$. *Then we have* $\bar{\ell}_p(M, p, \Phi) \subset \bar{\ell}_p(M, q, \Phi)$.

*Proof.* **a)** The proof is obtained by using the following inequality:

$$\left[ M\left( \frac{\overline{d}\left(\overline{x}_k, \overline{0}\right)}{r} \right) \right] \leq \left[ M\left( \frac{\overline{d}\left(\overline{x}_k, \overline{0}\right)}{r} \right) \right]^{p_k}.$$

**b)** The proof is obtained from the following inequality:

$$\left[ M\left( \frac{\overline{d}\left(\overline{x}_k, \overline{0}\right)}{r} \right) \right]^{p_k} \leq \left[ M\left( \frac{\overline{d}\left(\overline{x}_k, \overline{0}\right)}{r} \right) \right].$$

**c)** Let $\overline{x} = (\overline{x}_k) \in \bar{\ell}_p(M, p, \Phi)$ , that is $\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} \left[ M\left( \frac{\overline{d}(\overline{x}_k, \overline{0})}{r} \right) \right]^{p_k} < \infty$.

This implies that $\left[ M\left( \frac{\overline{d}(\overline{x}_k, \overline{x}_0)}{r} \right) \right]^{p_k} \leq 1$ for sufficiently large $k$. Since $M$ Orlicz function is non-decreasing we have

$$\left[ M\left( \frac{\overline{d}\left(\overline{x}_k, \overline{0}\right)}{r} \right) \right]^{q_k} \leq \left[ M\left( \frac{\overline{d}\left(\overline{x}_k, \overline{0}\right)}{r} \right) \right]^{p_k},$$

i.e., $\overline{x} = (\overline{x}_k) \in \bar{\ell}_p(M, q, \Phi)$. This completes the proof. □

# 4 Open Problem

We introduced a new generalized sequence space of interval numbers using by Orlicz function and examine some properties of resulting sequence classes of interval numbers. It is open problem that this class of interval number sequences has similar properties in two or three-dimensional space?

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