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Further generalizations involving open problems of F. Qi

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Abstract

Some integral inequalities related to the open problem of F. Qi are recently obtained. In the present work we generalize these inequalities. First, is given the answer to an open problem. Secondly, these inequalities are generalized for many functions and different parameters. In addition , we consider two other theorems by introducing a certain parameter λ.

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1 Introduction

In [3], F. Qi proved the following Theorem.

Theorem 1.1 *For $(x_1, x_2, ..., x_n) \in [0, \infty)^n$ and $n \geq 2$ inequality*

$$\frac{e^2}{4} \sum_{i=1}^{n} x_i^2 \leq \exp\left(\sum_{i=1}^{n} x_i \right) \tag{1}$$

is valid. Equality in (1) holds if $x_i = 2$ for some given $1 \leq i \leq n$ and $x_j = 0$ for all $1 \leq j \leq n$ with $j \neq i$. Thus, the constant $\frac{e^2}{4}$ in (1) is the best possible.

In [1] the inequality (1) was generalized as follows.

Theorem 1.2 *Let $p \geq 1$ be a real number. For $(x_1, x_2, ..., x_n) \in [0, \infty)^n$ and $n \geq 2$, the inequality*

$$\frac{e^p}{p^p} \sum_{i=1}^{n} x_i^p \leq \exp\left(\sum_{i=1}^{n} x_i\right) \tag{2}$$

is valid. Equality in (2) holds if $x_i = p$ for some given $1 \leq i \leq n$ and $x_j = 0$ for all $1 \leq j \leq n$ with $j \neq i$. Thus, the constant $\frac{e^p}{p^p}$ in (2) is the best possible.

A similar result for sequences of positive operators in Hilbert space was obtained (for details see [2]).

Theorem 1.3 *Let $0 < p \leq 1$ be a real number. For $(x_1, x_2, ..., x_n) \in [0, \infty)^n$ and $n \geq 2$, the inequality*

$$n^{p-1} \frac{e^p}{p^p} \sum_{i=1}^{n} x_i^p \leq \exp\left(\sum_{i=1}^{n} x_i\right) \tag{3}$$

is valid. Equality in (3) holds if $x_i = \frac{p}{n}$ for all $1 \leq i \leq n$. Thus, the constant $n^{p-1}\frac{e^p}{p^p}$ in (3) is the best possible.

In [1] the following Lemmas and Theorem were proved.

Lemma 1.4 *For $x \in [0, \infty)$ and $p > 0$, the inequality*

$$\frac{e^p}{p^p} x^p \leq e^x \tag{4}$$

is valid. Equality in (4) holds if $x = p$. Thus, the constant $\frac{e^p}{p^p}$ in (4) is the best possible.

Lemma 1.5 *Let $p > 0$ be a real number. For $(x_1, x_2, ..., x_n) \in [0, \infty)^n$ and $n \geq 2$, we have:*

(i) If $p \geq 1$, then the inequality

$$\sum_{i=1}^{n} x_i^p \leq \left(\sum_{i=1}^{n} x_i\right)^p \tag{5}$$

is valid.

(ii) If $o<p\le 1$, then inequality

$$n^{p-1}\sum_{i=1}^{n} x_i^p \le \left(\sum_{i=1}^{n} x_i\right)^p \tag{6}$$

is valid.

Theorem 1.6 *Let $0<p\le 1$, be a real number and let f be a non negative continuous function on $[a,b]$, then the inequality*

$$\int_a^b f^p\,dx \le (b-a)^{1-p}\,\frac{p^p}{e^p}\,\exp\left(\int_a^b f\,dx\right) \tag{7}$$

is valid. Equality in (7) holds if $f(x)=p(b-a)^{-1}$. Thus, the constant $\frac{p^p}{e^p}\,(b-a)^{1-p}$ in (7) is the best possible.

In [4] the following Theorem was proved.

Theorem 1.7 *Let $p>0$ be a real number and f be a non negative continuous function on $[a,b]$ such that $0<f(x)\le p(b-a)^{-1}$.*

1) If $p\ge 1$, then

$$\left(\int_a^b f(x)\,dx\right)^p \le \frac{p^p}{e^p}\,\exp\left(\int_a^b f(x)\,dx\right) \le \frac{p^{2p}}{(b-a)^{1+p}}\int_a^b f^{-p}(x)\,dx. \tag{8}$$

2) If $0<p\le 1$, then

$$\int_a^b f^p(x)\,dx \le (b-a)^{1-p}\,\frac{p^p}{e^p}\,\exp\left(\int_a^b f(x)\,dx\right) \le \frac{p^{2p}}{(b-a)^{2p}}\int_a^b f^{-p}(x)\,dx. \tag{9}$$

Equalities in (2.4) and (9) hold if $f=p(b-a)^{-1}$. Thus, the constants $C_1=\frac{p^{2p}}{(b-a)^{p+1}}$, $C_2=\frac{p^{2p}}{(b-a)^{2p}}$, are the best possible.

Remark 1.8 *The two sided inequality (8) holds also for $0<p\le 1$, but inequality $\int_a^b f^p(x)\,dx \le (b-a)^{1-p}\,\frac{p^p}{e^p}\,\exp\left(\int_a^b f(x)\,dx\right)$ is valid only for $o<p\le 1$, therefore in the rest of this work we consider this case (inequality (9)) separately.*

In [5] the following Lemma was established and proved.

Lemma 1.9 *(Power mean inequality).*

If $x_i \geq 0$, $\underset{i}{\lambda} > 0$, $i = 1, 2, ..., n$, $0 < p \leq 1$. *Tnen*

$$\sum_{i=1}^{n} \underset{i}{\lambda} x_i^p \leq \left(\sum_{i=1}^{n} \underset{i}{\lambda} \right)^{1-p} \left(\sum_{i=1}^{n} \underset{i}{\lambda} x_i \right)^p . \tag{10}$$

Inequality (10) is reversed for $p \geq 1$ *or* $p < 0$.

The aim of this paper is to give an answer to an open Problem posed in [4]: in Theorem 1.7 replace condition $0 < f(x) \leq p(b-a)^{-1}$ by weaker one. Moreover we generalize the Theprem 1.7 for many functions f_i with different parameters p_i. Also we generalize Theorem 1.2 and Theorem 1.3 by using Lemma 1.9.

2 Main results

We give an answer to the open problem posed in [4]. We replace condition $0 < f(x) \leq p(b-a)^{-1}$ by $0 < f(x) \leq M < \infty$ for any $M > 0$.

Theorem 2.1 *Let* $p > 0$, *be a real number and* f *be a non negative Lebesgue measurable function on [a,b] such that* $0 < f(x) \leq M < \infty$, *then*

$$e^{\int_a^b f dx} \leq M^p (b-a)^{-1} e^{(b-a)M} \left(\int_a^b f^{-p} dx \right) . \tag{11}$$

Equality in (11) holds if $f(x) = M$.

Proof 2.2 *If* $0 < f(x) \leq M < \infty$, *therefore* $\int_a^b f(x)dx \leq M(b-a)$, *and*

$$e^{\int_a^b f dx} \leq e^{M(b-a)}. \tag{$*$}$$

From condition $0 < f(x) \leq M < \infty$, *it follows that*

$$M^{-p}(b-a) \leq \int_a^b f^{-p} dx. \tag{$**$}$$

By $(*)$ *and* $(**)$ *we obtain*

$$e^{\int_a^b f dx} \leq M^{-p}(b-a) M^p (b-a)^{-1} e^{(b-a)M} \leq$$
$$\leq e^{(b-a)M} M^p (b-a)^{-1} \left(\int_a^b f^{-p} dx \right) .$$

Thus

$$e^{\int_a^b f dx} \leq M^p (b-a)^{-1} e^{(b-a)M} \left(\int_a^b f^{-p} dx \right) .$$

Theorem 2.3 *Let $p > 0$ be a real number and f be a non negative Lebesgue measurable function on [a,b] such that $0 < f(x) \leq M < \infty$.*

1) If $p \geq 1$, then

$$\left(\int_a^b f\right)^p \leq \frac{p^p}{e^p} e^{\int_a^b f dx} \leq \frac{p^p}{e^p} \; \frac{M^p}{(b-a)} e^{(b-a)M} \int_a^b f^{-p} dx. \tag{12}$$

2) If $0 < p \leq 1$, then

$$\int_a^b f^p dx \leq (b-a)^{1-p} \frac{p^p}{e^p} e^{\int_a^b f dx} \leq \frac{M^p}{(b-a)^p} \; \frac{p^p}{e^p} e^{(b-a)M} \left(\int_a^b f^{-p} dx\right). \tag{13}$$

Proof 2.4 *First we put $x = \int_a^b f dx$.*

1) If $p \geq 1$, by lemma 1.4,we have

$$\left(\int_a^b f(x)\, dx\right)^p \leq \frac{p^p}{e^p}\, e^{\int_a^b f(x)\, dx}$$

and by using (11), we get (12).

2) If $0 < p \leq 1$, by (7) the left-hand side inequality of (13) holds for $0 < p \leq 1$. Now combining this inequality and (11), we deduce the desired inequalities (13).

Remark 2.5 *If in (12) and (13), we put $M = p(b-a)^{-1}$, we get two sided inequalities (8) and (9) respectively.*

The following Lemma is the generalization of Lemma 1.4.

Lemma 2.6 *Let $p_i > 0$ be real numbers, $i = 1, 2, ..., m$, $m \in \mathbb{N}$ $x_i > 0$, then*

$$\frac{e^{\sum_{i=1}^m p_i}}{\prod_{i=1}^m p_i^{p_i}} \prod_{i=1}^m x_i^{p_i} \leq e^{\sum_{i=1}^m x_i}. \tag{14}$$

Equality in [14] holds if $x_i = p_i$. Thus, the constant $\dfrac{e^{\sum_{i=1}^m p_i}}{\prod_{i=1}^m p_i^{p_i}}$ is the best possible.

Proof 2.7 *By Lemma 1.4, we have*

$$\frac{e^{p_1}}{p_1^{p_1}} x_1^{p_1} \le e^{x_1}$$
$$\vdots$$
$$\frac{e^{p_2}}{p_2^{p_2}} x_2^{p_2} \le e^{x_2}$$
$$\vdots$$
$$\frac{e^{p_m}}{p_m^{p_m}} x_m^{p_m} \le e^{x_m}$$

By multiplying these inequalities, we get (14).

Lemma 2.8 *Let $0 < p_i \le 1$, be real numbers, $i = 1, 2, ..., m$, $m \in \mathbb{N}$ and f_i be non negative continuous functions on [a,b], then*

$$\prod_{i=1}^{m} \int_a^b f_i^{p_i} dx \le \frac{\prod\limits_{i=1}^{m} (b-a)^{1-p_i}}{e^{\sum\limits_{i=1}^{m} p_i}} \prod_{i=1}^{m} p_i^{p_i} e^{\sum\limits_{i=1}^{m} \int_a^b f_i dx}. \tag{15}$$

Equality in (15) holds if $f_i = p_i(b-a)^{-1}$. Thus, the constant $\frac{\prod\limits_{i=1}^{m} p_i^{p_i}}{e^{\sum\limits_{i=1}^{m} p_i}} (b-a)^{1-p_i}$ is the best possible.

Proof 2.9 *In (14) we put $x_i = \int_a^b f_i dx$ and we apply inequality (7). The rest is similar to the proof of Lemma 2.6.*

Remark 2.10 *The previous Lemma is the generalization of Theorem 1.6.*

The following Theorem is the generalization of Theorem 1.7.

Theorem 2.11 *Let $p_i > 0$ be real numbers and f_i be non negative continuous functions on (a, b), $x_i \in (a, b)$, $i = 1, 2, ..., m$, $m \in \mathbb{N}$, such that $f_i(x) \le p_i(b-a)^{-1}$.*

1) If $p_i \ge 1$, then

$$\prod_{i=1}^{m} \left(\int_a^b f_i dx \right)^{p_i} \le \frac{\prod\limits_{i=1}^{m} p_i^{p_i}}{e^{\sum\limits_{i=1}^{m} p_i}} e^{\sum\limits_{i=1}^{m} \int_a^b f_i dx} \le \frac{\prod\limits_{i=1}^{m} p_i^{2p_i}}{\prod\limits_{i=1}^{m} (b-a)^{p_i+1}} \prod_{i=1}^{m} \left(\int_a^b f_i^{-p_i} dx \right). \tag{16}$$

2) If $0 < p_i \le 1$, then

$$\prod_{i=1}^{m}\int_a^b f^{p_i}dx \le \frac{\prod\limits_{i=1}^{m}(b-a)^{1-p_i}}{e^{\sum\limits_{i=1}^{m}p_i}}\prod_{i=1}^{m}p_i^{p_i}e^{\sum\limits_{i=1}^{m}\int_a^b f_i dx} \le \frac{\prod\limits_{i=1}^{m}p_i^{2p_i}}{\prod\limits_{i=1}^{m}(b-a)^{2p_i}}\prod_{i=1}^{m}\left(\int_a^b f_i^{-p_i}dx\right). \tag{17}$$

Equalities in (16) and (17) hold if $f_i = p_i(b-a)^{-1}$. *Thus, the constants* $C_3 = \frac{\prod\limits_{i=1}^{m}p_i^{2p_i}}{\prod\limits_{i=1}^{m}(b-a)^{2p_i+1}}$, $C_4 = \frac{\prod\limits_{i=1}^{m}p_i^{2p_i}}{\prod\limits_{i=1}^{m}(b-a)^{2p_i+1}}$ *are the best possible.*

Proof 2.12 *By setting* $x_i = \int_a^b f_i dx$, *(14) takes the form*

$$\frac{e^{\sum\limits_{i=1}^{m}p_i}}{\prod\limits_{i=1}^{m}p_i^{p_i}}\prod_{i=1}^{m}\left(\int_a^b f_i dx\right)^{p_i} \le e^{\sum\limits_{i=1}^{m}\int_a^b f_i dx}. \tag{18}$$

If

$$0 < f_i(x) \le p_i(b-a)^{-1}, \tag{19}$$

therefore

$$\int_a^b f_i dx \le p_i,$$

then

$$e^{\sum\limits_{i=1}^{m}\int_a^b f_i dx} \le e^{\sum\limits_{i=1}^{m}p_i} \tag{20}$$

From condition (19) it follows that

$$p_i^{-p_i}(b-a)^{p_i+1} \le \int_a^b f_i^{-p_i}dx,$$

thus

$$\prod_{i=1}^{m}p_i^{-p_i}(b-a)^{p_i+1} \le \prod_{i=1}^{m}\left(\int_a^b f_i^{-p_i}dx\right). \tag{21}$$

Inequalities (20) and (21) imply to

$$e^{\sum\limits_{i=1}^{m}\int_a^b f_i(x)dx} \le e^{\sum\limits_{i=1}^{m}p_i}\prod_{i=1}^{m}p_i^{-p_i}(b-a)^{p_i+1}\prod_{i=1}^{m}p_i^{p_i}(b-a)^{-p_i-1} \le$$

$$\le \frac{e^{\sum\limits_{i=1}^{m}p_i}\prod\limits_{i=1}^{m}p_i^{p_i}}{\prod\limits_{i=1}^{m}(b-a)^{p_i+1}}\prod_{i=1}^{m}\int_a^b f_i^{-p_i}dx,$$

then

$$e^{\sum_{i=1}^{m}\int_a^b f_i dx} \leq \frac{e^{\sum_{i=1}^{m} p_i}\prod_{i=1}^{m} p_i^{p_i}}{\prod_{i=1}^{m}(b-a)^{p_i+1}}\prod_{i=1}^{m}\left(\int_a^b f_i^{-p_i}dx\right).$$

Finally by (18) we get (16).

Taking in account Remark 1.8, one can use (16) for $0 < p_i \leq 1$.

Now by applying (15) and right -hand side inequality (16), we get

$$\prod_{i=1}^{m}\int_a^b f^{p_i}dx \leq \frac{\prod_{i=1}^{m}(b-a)^{1-p_i}}{e^{\sum_{i=1}^{m} p_i}}\prod_{i=1}^{m} p_i^{p_i} e^{\sum_{i=1}^{m}\int_a^b f_i dx} \leq \frac{\prod_{i=1}^{m} p_i^{2p_i}}{\prod_{i=1}^{m}(b-a)^{2p_i}}\prod_{i=1}^{m}\left(\int_a^b f_i^{-p_i}dx\right).$$

The Proof is complete.

The following Theorem is the generalization of Theorem 1.2 and Theorem 1.3.

Theorem 2.13 *Let* $p > 0$ *be a real number for* $x_i \geq 0$, $i = 1, 2, ..., m$, $m \in \mathbb{N}$, $m \geq 2, \lambda_i > 0$.

1) If $p \geq 1$, *then*

$$\frac{e^p}{p^p}\sum_{i=1}^{m}\lambda_i^p x_i^p \leq e^{\sum_{i=1}^{m}\lambda_i x_i}. \tag{22}$$

2) If $0 < p \leq 1$, *then*

$$\frac{e^p}{p^p}\sum_{i=1}^{m}\lambda_i x_i^p \leq \left(\sum_{i=1}^{m}\lambda_i\right)^{1-p} e^{\sum_{i=1}^{m}\lambda_i x_i}. \tag{23}$$

Equality in (22) holds if $x_i = \frac{p}{\lambda_i}$. *Thus, the constant* $C_5 = \frac{e^p}{p^p}$, *is the best possible.*

Proof 2.14 *Let* $x = \sum_{i=1}^{m}\lambda_i x_i$.

1) If $p \geq 1$,

Let $y_i = x_i\lambda_i$. *By (2) (Theorem 1.2), we obtain*

$$\frac{e^p}{p^p}\sum_{i=1}^{m} y_i^p \leq e^{\sum_{i=1}^{m} y_i}.$$

Thus

$$\frac{e^p}{p^p}\sum_{i=1}^{m}\lambda_i^p x_i^p \leq e^{\sum_{i=1}^{m}\lambda_i x_i}.$$

2) If $0 < p \leq 1$,

by Lemma 1.4 we have

$$\left(\sum_{i=1}^{m} \lambda_i x_i\right)^p \leq \frac{p^p}{e^p} e^{\sum\limits_{i=1}^{m} \lambda_i x_i},$$

then

$$\left(\sum_{i=1}^{m} \lambda_i\right)^{1-p} \left(\sum_{i=1}^{m} \lambda_i x_i\right)^p \leq \frac{p^p}{e^p} \left(\sum_{i=1}^{m} \lambda_i\right)^{1-p} e^{\sum\limits_{i=1}^{m} \lambda_i x_i}.$$

According to inequality (10) (Lemma 1.9), we get

$$\sum_{i=1}^{m} \lambda_i x_i^p \leq \left(\sum_{i=1}^{m} \lambda_i\right)^{1-p} \left(\sum_{i=1}^{m} \lambda_i x_i\right)^p \leq \frac{p^p}{e^p} \left(\sum_{i=1}^{m} \lambda_i\right)^{1-p} e^{\sum\limits_{i=1}^{m} \lambda_i x_i}.$$

3 Open Problems

Problem 3.1 *Let* p, q *be real parameters such that* $0 < p < q \leq \infty$ *and* $x_i \geq 0$, $\lambda_i > 0$, $i = 1, 2, ..., m$, $m \geq 2$. *Give the integral analogue of inequalities (22) and (23) of Theorem 2.13 for two parameters p and q with the best possible constant.*

Problem 3.2 *In [4] tne following result was established: let* $0 < p < q < \infty$, *and* f, w *be non negative Lebesgue measurable functions on* $[a, b]$ *such that* $\int_a^b f^q \, w \, dx < \infty$, *then*

$$\left(\int_a^b f^p \, w \, dx\right)^{\frac{1}{p}} \leq \left(\int_a^b w \, dx\right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_a^b f^q w \, dx\right)^{\frac{1}{q}}. \tag{24}$$

Question: does the inequality (24) remain valid if $q < p$*?*

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