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A Coupled Model Between Two Languages Using Fractional Dynamics

Maria Dahmani[1], Lynda Marouf[2] and Zoubir Dahmani[3]

[1]Department of English, UMAB University, Algeria
[2]High School of Ben Guella, Mostaganem, Algeria
[3]Laboratory LPAM, University of Mostaganem, Algeria

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Abstract

We present a new model of fractional dynamics to study the competition between two given languages. The proposed model is studied by considering the Caputo approach which has many applications in real word phenomena. The "U-H" stability is studied and an existence and uniqueness result is discussed. At then end, a conclusion follows and some open questions are posed.

Keywords: *Bilingualism, Caputo derivative, Stability, Competition of two languages.*
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1 Introduction

Language diversity is one of the best examples of the human intelligence. It has been the object of many scientific works from many point of view including physics and mathematics. Among them, the simplest model of two languages competition described by differential equations, see Abrams and Strogatz [1]. The authors assumed that there is no spatial or social structure in the populations, in which all speakers are monolingual and the attractiveness of a language increases in both its number of speakers and its perceived status. The theoretical results successfully fitted to historical data on the competition between Scottish Gaelic and English, Welsh and English, and Quechua and Spanish. Throughout most of the history of languages, many languages appear, evolve and disappear. With partial regional results tallied, the numbers of languages in the world vary between 6000 and 7000. However, with the progress of the globalization, thousands of the world languages are vanishing at an alarming rate, with 90 of them being expected to disappear with the current generation as a result of language competition [2,3,4,5]. In recent years, it has been proved that many phenomena in different fields can be described successfully by the fractional order models; the fractional order derivatives have been introduced almost 300 years ago with a query posed by L'Hospital to Leibnitz. The fractional calculus was reasonably developed by 19th century. It was realized, only in the past few decades that these fractional derivatives are better models to study physical phenomenon [6,7,8,9,10]. These are some of the reasons that motivate us to use fractional order model instead of integer order one in this paper.

So, in this paper, using a fractional calculus approach, we propose a new model of two coupled equations of fractional derivative type, such that the first equation is of non integer order q_1, with $0 < q_1 < 1$ and the second one is of non integer order q_2, with $0 < q_2 < 1$. This model will allow us to study the dynamic behaviour of a competitiveness between two given languages. The proposed $(x(t), y(t))$ fractional model is different from Baggs-Freedman model [2] which involves a coupled system of two autonomous ODEs that are imposed on a time interval of type $[0, 1]$. Our model is also different from that of [1] which is considered as an ordinary differential equation of first order (first derivative with respect to time t).

The approach considered in our model is the Caputo appraoch; three raisons motivate our choice:

1. This fractional theory imposes some regularity conditions on each component $x(t)$ and $y(t)$ of the model (for our case, we need the so called C^1 space of functions).

2. Also, because using the Caputo approach, the initial conditions of integer order on the unknown functions of the model can be well justified.

The proposed model can be applied for: French/English in Canada, Scottish and Irish Gaelic/English, Welsh/English in Wales, Quechua/Spanish in Peru...

3. This approach has proved to be an important tool to model many real word phenomena.

Some results on the Ulam-Hyers stability for our model are presented and a first conclusion related to this stability on the two populations $x(t), y(t)$ is discussed. At the end, some open questions are proposed.

2 The Fractional Model

In this section, we develop a fractional order model of one unilingual component and one bilingual component of a population and we will investigate the dynamics of the interactions of the population in a closed environment. Our system is the following

$$\begin{cases} D^{q_1} x\,(t) = (B_1 - D_1)x(t) - L_1 x^2(t) + P_1 B_2 y(t) - \alpha \frac{x(t)y(t)}{1+x(t)}, \\ \\ D^{q_2} y\,(t) = (P_2 B_2 - D_2)y(t) - L_2 y^2(t) + \alpha \frac{x(t)y(t)}{1+x(t)}, \end{cases} \tag{1}$$

where we take into account the following considerations:
1. $B_1 - D_1 > 0, P_2 B_2 - D_2 > 0, L_1, L_2$ are positive quantities (positive real numbers)
2. The component $x(t)$ represents the concentration, or the rate, of the unilingual component of the population for any positive time t, with $x(0) > 0$ at $t = 0$.
3. The unknown function $y(t)$ represents the concentration of the bilingual component of the population, with $y(0) > 0$.
4. The quantity $(B_1 - D_1)x(t) - L_1 x^2(t)$ represents the growth component of the unilingual population $x(t)$ in isolation (absence, no emigration) from the bilingual population $y(t)$.
5. The quantity $(P_2 B_2 - D_2)y(t) - L_2 y^2(t)$ means that the bilingual population acquires new members at the rate which the complete population gives birth to bilingual children minus the death rate.
6. We take K_1 as the carrying capacity of the environment of population $x(t)$ in the case where $x(t)$ could grow in isolation with the population $y(t)$. The same consideration for K_2.
7. We take the constants $0 < q_i < 1, 0 < P_1 < 1$ and $P_2 + P_1 = 1$, with P_2 represents the rate of the children of the population $y(t)$ which enter the population as unilingual.
8. The function $P_1 B_2 y(t)$ represents children born to bilingual parents who enter the population as unilinguals.
9. The real parameter α in both equations is considered as a conversion parameter.
10. The last term of model (1) indicates that the conversion rate from unilingual to bilingual is proportional to the concentration of both the unilingual and bilingual populations.
It is to note that our model is a fractional modification version of the model developed by I. Baggs (1990). So, the Bagss model can be seen as a "limiting case" of model (1) in the case where both q_1, q_2 are very close to the value $n = 1$; in this case, we obtain the first derivative with respect to time

$x'(t), x^{\cdot}(t), \frac{d}{dt}$
We note again that the derivatives D^{q_1} and D^{q_1} of the left sides of the model are taken in the sense of Caputo, since this approach is very important and has many applications in modeling phenomena in real word...
Let us now consider a general system that includes the above system as a particular case.
So, we consider the coupled problem of fractional order q_1, q_2:

$$\begin{cases} D^{q_1} x(t) = a_1 f_1(t, x(t), y(t)) + b_1 g_1(t, x(t), y(t)), \\ \\ D^{q_2} y(t) = a_2 f_2(t, x(t), y(t)) + b_2 g_2(t, x(t), y(t)), \end{cases} \tag{2}$$

such that $0 \le t \le T, a_i, b_i \in \mathbb{R}, i = 1, 2, < 0q_1, q_2$ <1 and

$$x(0) = x_1, y(0) = x_2 \tag{3}$$

f_1, f_2, g_1, g_2 are nonlinear given functions.
Since the model (1) is a particular case of system (2), so by studying this general system, we will obtain some particular main results regarding the model (1).

3 An Integral Representation

Using some mathematical technics, we can find that the model can be expressed by another equivalent integral system, called also integral representation. This "new" representation allows us to learn more about the unknown functions (the two components of the population) $(x(t), y(t))$.
We can state that the introduced system (2) is equivalent to the following integral representation:

$$\begin{cases} x(t) = x_1 + a_1 \int_0^t \frac{(t-s)^{q_1-1}}{\Gamma(q_1)} f_1(s, x(s), y(s))\, ds + b_1 \int_0^t \frac{(t-s)^{q_1-1}}{\Gamma(q_1)} g_1(s, x(s), y(s))\, ds, \\ \\ y(t) = x_2 + a_2 \int_0^t \frac{(t-s)^{q_2-1}}{\Gamma(q_2)} f_2(s, x(s), y(s))\, ds + b_2 \int_0^t \frac{(t-s)^{q_2-1}}{\Gamma(q_2)} g_2(s, x(s), y(s))\, ds. \end{cases} \tag{4}$$

Instead of the above complicated dynamical system (2), we have proposed an equivalent "easy" system, for which we can use some mathematics technics to study the evolution of its components $(x(t), y(t))$. So, we have transformed the fractional model to an integral model.

4 Solutions of System 2 and Model 1

By imposing some assumptions on the data of the above system, that are the functions f_1, f_2, g_1, g_2..., we can present the following main results:

Theorem 4.1 *Let $f_1, f_2, g_{1,}g_2$ be Lipchitzian given functions. Then, the system (2) has a unique solution $(x(t), y(t))$.*

Proof:
The idea is to show that the following quantities (that are given by the above integral representation):

$$Q_i(x,y)(t) := x_i + a_i \int_0^t \frac{(t-s)^{q_i-1}}{\Gamma(q_i)} f_i\left(s, x(s), y(s)\right) ds + b_i \int_0^t \frac{(t-s)^{q_i-1}}{\Gamma(q_i)} g_i\left(s, x(s), y(s)\right) ds$$

are contractive.
It is sufficient to apply the well known Banach Contraction Principle.

Remark 4.2 *The model(1) admits a unique solution $(x(t), y(t))$ since it can be seen as a particular case of system* (2).

Remark 4.3 *Under some other assumptions on the data, we can also show that there exist at least some other solutions $(x(t), y(t))$ for (2).*
In particular, there are solutions for the fractional model (1).

5 Ulam-Hyres Stability

The Ulam Hyers stability can be applied to nonlinear dynamical models. It has an important significance since it means that if we are studying an Ulam Hyers stable system, then we do not have to reach the exact or the explicit solution, which usually is quite difficult or time consuming. All what is required is to get an approximate solution $(x_a(t), y_a(t))$ which satisfies *an inequality in* (1). It is not necessary that this couple satisfies the equations (1). The Ulam Hyers stability guarantees that there is a close exact solution. This is quite useful in our case where finding the exact and the explicit solution of the above two systems is quite difficult! So the use of Ulam Hyers stability will allow us to progress in this sense to study the above fractional model (1).
We have to present the following result:

Theorem 5.1 *Under the same assumptions as in Theorem 4.1, the system (2) has the Ulam-Hyers stability.*

To prove this result, we just apply the definition of Ulam Hyers stability.
And then, as a remark, we have:

Remark 5.2 *As a particular case of (2), our proposed model is Ulam Hyers stable.*

So we need to find an approximate solution $(x_a(t), y_a(t))$. This can be done by applying a numerical analysis method.

6 Discussion and Conclusion

In this work, we have proposed a new model that is based on Caputo approach. The Caputo derivative is introduced in the model.
We can't use the qualitative theory and the asymptotic behaviour since the fractional derivative is different from the standard derivative of integer order. There is no speed definition in our fractional case. Then, we can't apply the steady states notions! To overcome the problem, we have tried the Ulam Hyers stability because it does not need to obtain the exact or the explicit solution of the model. All what we have needed is a solution of an inequality, instead of an explicit solution of the model. By means of an integral representation, we have proved the existence and uniqueness of solutions of the nonlinear fractional model (1). This study is theoretical. We need real data to be able to show the accuracy of the model.

7 Open Questions

1. The first problem is that we need real data to be able to test our model.
2. In our country, there is a new 2020 politic decision to take English as a second official language, after the announcement of the Minister of Higher Education and Scientific Research early last month: Work will begin on the development of mechanisms to promote the use of English at universities and research centers, at the expense of the French language, and "The French language does not lead anywhere"! This is a real competition between French and English! So, what about our model and the Ministry Decision?
3. Is it possible to apply the Adomian Method to investigate the numerical approach?

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