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Value Distribution and Uniqueness of Some Difference-Diffrential Polynomials of Meromorphic Functions

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Abstract

In this paper, we discuss the value distribution and uniqueness problems of some differencedifferential polynomials of meromorphic functions sharing a small function. The results of the paper improve and generalize the results due to S.S. Bhoosnurmath and S.R. Kabbur[1].

Keywords: *entire function, uniqueness, small function, difference-differential polynomials, Sharing value.*
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1 Introduction

In this article, we assume that the reader is familiar with the fundamental results and the standard notation of Nevanlinna value distribution theory (see [2], [13]). Let $f(z)$ and $g(z)$ be two non constant meromorphic functions in the complex plane. By $S(r, f)$, we mean any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$, possibly outside a set of finite logarithmic measure.

A meromorphic function $a(z)$ is called a small function with respect to f, provided that $T(r, a) = s(r, f)$. The order and hyper order of meromophic function f are defined respectively by

$$\sigma(f) = \lim_{r\to\infty} sup \frac{logT(r,f)}{logr}$$

and

$$\sigma_2(f) = \lim_{r\to\infty} sup \frac{log\,logT(r,f)}{logr}$$

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions. Let and $a \in \mathbb{C} \cup \{\infty\}$. If the zeros of $f-a$ and $g-a$ coincide in locations and multiplicity, we say that f and g share the value a CM(counting multiplicities). On the other hand, if the zeros of $f-a$ and $g-a$ coincide only in their locations, then we say that f and g share the value a IM(ignoring multiplicities). For a positive integer p we denote by $N_p(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \le p$ and p times if $m > p$.

We denote by $E_{m)}(a,f)$ the set of all a-points of f with multiplicities not exceeding m, where an a-point is counted according to its multiplicity. Also we denote by $\bar{E}_{m)}(a,f)$ the set of distinct a-points of f with multiplicities not greater than m. We denote by $N_{k)}(r,\frac{1}{(f-a)})$ the counting function for zeros of $f-a$ with multiplicity $\le k$, and by $\bar{N}_{k)}(r,\frac{1}{(f-a)})$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r,\frac{1}{(f-a)})$ be the counting function for zeros of $f-a$ with multiplicity atleast k and $\bar{N}_{(k}(r,\frac{1}{(f-a)})$ the corresponding one for which multiplicity is not counted. Set

$$N_k(r,\frac{1}{f-a}) = \bar{N}(r,\frac{1}{f-a}) + \bar{N}_{(2}(r,\frac{1}{f-a}) + ... + \bar{N}_{(k}(r,\frac{1}{f-a}).$$

The following two theorems proved by S.S. Bhoosnurmath and S.R. Kabbur [1].

Theorem A. Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to f and g. Suppose that c is a nonzero complex constant and n,m are positive integers such that $n \ge m+6$. If $f^n(z)(f^m(z)-1)f(z+c)$ and $g^n(z)(g^m(z)-1)g(z+c)$ share $\alpha(z)$ CM, then $f(z) = tg(z)$ where $t^m = 1$.

Theorem B. Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to f and g. Suppose that c is a nonzero complex constant and n,m are positive integers satisfying $n \ge 4m+12$. If $f^n(z)(f^m(z)-1)f(z+c)$ and $g^n(z)(g^m(z)-1)g(z+c)$ share $\alpha(z)$ IM, then $f(z) = tg(z)$ where $t^m = 1$.

I. Lahiri introduced weighted sharing instead of sharing IM or CM, which measure how close a shared value is to being shared CM or to be shared IM. The definition are as follows.

Definition 1 [**7**]. Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \bigcup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer $p, 0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

If α is a small function of f and g, share α with weight k means that $f - \alpha, g - \alpha$ share the value 0 with weight k.

2 Preliminary Lemmas

In this section, we present some lemmas which will be needed in the sequel. We will denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right)$$

Lemma 1 (see[13]). Let f be a meromorphic function of finite order and c is a non-zero complex constant. Then

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r, f).$$

Arguing in a similar manner as in [3], we obtain the following lemma.

Lemma 2. Let f be an meromorphic function of finite order. Then $T(r, f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)) = (n+m+\sigma)T(r, f) + S(r, f)$.

Lemma 3. Let f be an meromorphic function of finite order. Then $T(r, f^n(z)P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j}) = (n+m+\sigma)T(r, f) + S(r, f)$.

Proof. Since f is meromorphic function of finite order, we deduce from Lemma

1 and the standard valiron Mohen'ko theorem

$$\begin{aligned}
(n+m+\sigma)T(r,f) &= T(r,f^{n+m}f^{\sigma})+S(r,f)\\
&= m(r,f^{n+m}f^{\sigma})+N(r,f^{n+m}f^{\sigma})+S(r,f)\\
&= m\left(r,f^{n+m}\prod_{j=1}^{d}f(z+c_j)^{c_j}\frac{f^{\sigma}}{\prod_{j=1}^{d}f(z+c_j)^{c_j}}\right)\\
&+N\left(r,f^{n+m}\prod_{j=1}^{d}f(z+c_j)^{c_j}\frac{f^{\sigma}}{\prod_{j=1}^{d}f(z+c_j)^{c_j}}\right)+S(r,f)\\
&= m\left(r,f^{n+m}\prod_{j=1}^{d}f(z+c_j)^{v_j}\right)+m\left(r,\frac{f^{\sigma}}{\prod_{j=1}^{d}f(z+c_j)^{v_j}}\right)\\
&+N\left(r,f^{n+m}\prod_{j=1}^{d}f(z+c_j)^{v_j}\right)+N\left(r,\frac{f^{\sigma}}{\prod_{j=1}^{d}f(z+c_j)^{v_j}}\right)+S(r,f)\\
&= T(r,f^{n+m}\prod_{j=1}^{d}f(z+c_j)^{v_j})+S(r,f)\\
&= T(r,F)+S(r,f).
\end{aligned}$$

Thus, we get the conclusion.

Lemma 4 (see[13]). Let f be a non-constant meromprphic function and p,k be two positive integers. Then

$$N_p(r,\frac{1}{f^{(k)}})\leq T(r,f^{(k)})-T(r,f)+N_{p+k}(r,\frac{1}{f})+S(r,f)$$
$$N_p(r,\frac{1}{f^{(k)}})\leq k\overline{N}(r,f)+N_{p+k}(r,\frac{1}{f})+S(r,f).$$

Lemma 5 (see[7]). If F and G are two non-constant meromorphic functions and $E_{3)}(1,F)=E_{3)}(1,G)$, then one of the following cases holds.
(1)$T(r,F)+T(r,G)\leq 2N_2(r,\frac{1}{F})+2N_2(r,f)+2N_2(e.\frac{1}{G})+2N_2(r,G)+S(r,F)+S(r,G)$,
(2) $F\equiv G$, (3) $FG\equiv 1$.

Lemma 6 . Let h be a transcendental meromorphic function of finite order. Then we have

$$T\left(r,h^{n+m}\prod_{j=1}^{d}h(z+c_j)^{v_j}\right)\geq(n+m-\sigma)T(r,h)+S(r,f),$$

where $\sigma = v_1 + v_2 + ... + v_d$.

Proof. From Lemma 1, we have

$$\begin{aligned}(n+m+\sigma)T(r,h) &= T(r, h^{n+m}(z)h^{\sigma}) + S(r,h)\\ &= m(r, h^{n+m}(z)h^{\sigma}) + N(r, h^{n+m}(z)h^{\sigma}) + S(r,h)\\ &= m\left(r, h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)^{v_j}\frac{h^{\sigma}}{\prod_{j=1}^{d} h(z+c_j)^{v_j}}\right)\\ &+ N\left(r, h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)^{v_j}\frac{h^{\sigma}}{\prod_{j=1}^{d} h(z+c_j)^{v_j}}\right) + S(r,h)\\ &\leq T\left(r, h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)^{v_j}\right) + 2\sigma T(r,h) + S(r,h).\end{aligned}$$

Thus, we get the conclusion.

3 Main results

Regarding Theorem A-B, a natural question to ask is what can be said if we study the uniqueness of difference polynomials of the form $[f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)_j^v]^{(k)}$ and $[g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+c_j)_j^v]^{(k)}$ where $c_j (j = 1,2,...,d)$ are complex constants, $v_j (j = 1,2,...,d)$ are non-negative integers and $\sigma = v_1 + v_2 + ... + v_d$ without the notion of weighted sharing?. In this section, our main concern is to find the possible answer of the above question. We prove the following result.

Theorem 1. Let f and g be two transcendental meromorphic functions of finite order, and $\alpha(z)(\not\equiv 0)$ be a small function with respect to both f and g. Suppose that $c_j (j = 1,2,...,d)$ are non-zero complex constants, $v_j (j = 1,2,...,d)$ are non-negative integers, $n, m \geq 1$ and $k(\geq 0)$ are integers satisfying $n \geq 4k + m + \sigma + 5$. If $E_{3)}(\alpha(z), [f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}]^{(k)}) = E_{3)}(\alpha(z), [g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+c_j)^{v_j}]^{(k)})$, then $f = hg$, where h is a constant and $h^m = 1$.

Proof.

Let $F_1 = f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}$, $\quad G_1 = g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+c_j)^{v_j}$,

$F = \frac{F_1^{(k)}}{\alpha(z)}$ $G = \frac{G_1^{(k)}}{\alpha(z)}$. Then F and G are transcendental meromorphic functions and $E_{3)}(1,F) = E_{3)}(1,G)$ except the zeros and poles of $\alpha(z)$. By Lemma 2 and Lemma 4 we have

$$N_2(r,\frac{1}{F}) \le N_2(r,\frac{1}{F_1^{(k)}}) + S(r,f)$$
$$\le T(r,F_1^{(k)}) - T(r,F_1) + N_{2+k}(r,\frac{1}{F_1}) + S(r,f) \qquad (1)$$
$$\le T(r,F) - (n+m+\sigma)T(r,f) + N_{2+k}(r,\frac{1}{F_1}) + S(r,f).$$

So we get

$$(n+m+\sigma)T(r,f) \le T(r,F) + N_{2+k}(r,\frac{1}{F_1}) - N_2(r,\frac{1}{F}) + S(r,f). \qquad (2)$$

According to Lemma 4, we can deduce

$$\begin{aligned} N_2(r,\frac{1}{F}) &\le N_2(r,\frac{1}{F_1^{(k)}}) + S(r,f) \\ &\le N_{2+k}(r,\frac{1}{F_1}) + k\overline{N}(r,f) + S(r,f). \end{aligned} \qquad (3)$$

Similarly we have

$$(n+m+\sigma)T(r,g) \le T(r,G) + N_{2+k}(r,\frac{1}{G_1}) - N_2(r,\frac{1}{G}) + S(r,g). \qquad (4)$$

And

$$N_2(r,\frac{1}{G}) \le N_{2+k}(r,\frac{1}{G_1}) + k\overline{N}(r,g) + S(r,g). \qquad (5)$$

Suppose, if possible the (1) of Lemma 5 holds, that is

$$\begin{aligned} T(r,F) + T(r,G) \le\ & 2N_2(r,\frac{1}{F}) + 2N_2(r,F) + 2N_2(r,\frac{1}{G}) \\ & + 2N_2(r,G) + S(r,f) + S(r,g) \end{aligned} \qquad (6)$$

By (2),(3),(4),(5) and (6), we have

$$\begin{aligned} (n+m+\sigma)(T(r,f)+T(r,g)) \le\ & N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + N_{2+k}(r,\frac{1}{F_1}) \\ & + N_{2+k}(r,\frac{1}{G_1}) + S(r,f) + S(r,g) \\ & \le 2N_{2+k}(r,\frac{1}{F_1}) + 2N_{2+k}(r,\frac{1}{G_1}) + 2k(\overline{N}(r,f) + \overline{N}(r,g)) \\ & + S(r,f) + S(r,g) \\ & \le (4k+4+2m+2\sigma)(T(r,f)+T(r,g)) + S(r,f) + S(r,g). \end{aligned} \qquad (7)$$

So

$$(n-4k-m-\sigma-4)(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g), \tag{8}$$

which contradicts the fact that $n \geq 4k+m+\sigma+5$. Therefore, by Lemma 5 we have either $FG = 1$ or $F = G$.
If $FG = 1$, that is

$$[f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}]^{(k)}.[g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+c_j)^{v_j}]^{(k)} = \alpha^2, \tag{9}$$

We can deduce from above that

$$N(r,\frac{1}{f}) = N(r,\frac{1}{f-1}) = S(r,f), \tag{10}$$

which is impossible. So we have $F = G$, that is

$$[f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}]^{(k)} = [g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+c_j)^{v_j}]^{(k)}. \tag{11}$$

Integrating above, we deduce

$$[f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}]^{(k-1)} = [g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+c_j)^{v_j}]^{(k-1)}+c, \tag{12}$$

where c is a constant. If $c \neq 0$, by the second fundamental theorem of Nevanlinna, we have

$$\begin{aligned} T(r,F_1^{(k-1)}) &\leq \overline{N}(r,\frac{1}{f_1^{(k-1)-c}}) + S(r,F) \\ &\leq \overline{N}(r,\frac{1}{F_1^{(k-1)}}) + \overline{N}(r,\frac{1}{G_1^{(k-1)}}) + S(r,F). \end{aligned} \tag{13}$$

By Lemma 4, we obtain

$$\begin{aligned} (n+m+\sigma)T(r,f) &\leq T(r,F_1^{(k-1)}) - \overline{N}(r,\frac{1}{F_1^{(k-1)}}) + N_k(r,\frac{1}{F_1}) + S(r,f) \\ &\leq \overline{N}(r,\frac{1}{G_1^{(k-1)}}) = N_k(r,\frac{1}{F_1}) + S(r,f) \\ &\leq N_k(r,\frac{1}{F_1}) + N_k(r,\frac{1}{G_1}) + S(r,f) + S(r,g) \\ &\leq (k+m+\sigma)(T(r,f)+T(r,g)) + S(r,f) + S(r,g). \end{aligned} \tag{14}$$

Similarly

$$(n+m+\sigma)T(r,g) \leq (k+m+\sigma)(T(r,f)+T(r,g)) + S(r,f) + S(r,g). \quad (15)$$

Combining (14) and (15), we obtain

$$(n-2k-m-\sigma)(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g), \quad (16)$$

which contradicts with $n \geq 2k+m+\sigma+5$. Hence $c=0$. Integrating the (12) $k-1$ times, we can deduce

$$f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j} = g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+c_j)^{v_j}. \quad (17)$$

Set $h=f/g$. If h is not a constant, from (17) we have

$$g^m(z) = \frac{h^{(n)}\prod_{j=1}^{d} h(z+c_j)^{v_j}-1}{h^{(n+m)}\prod_{j=1}^{d} h(z+c_j)^{v_j}-1} \quad (18)$$

If 1 is a picard value of $h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)^{v_j}$, applying the Nevanlinna second fundamental theorem, we get

$$\begin{aligned} T(r,h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)^{v_j}) &\leq \overline{N}r, h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)^{v_j} + \overline{N}\left(r, \frac{1}{h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)^{v_j}}\right) \\ &+ S(r,h) \\ &\leq (2d+2)T(r,h)+S(r,h). \end{aligned} \quad (19)$$

On the other hand, combining the standard Valiron-Mohon'ko theorem, we get

$$\begin{aligned} (n+m+\sigma)T(r,h) &= T(r,h^{n+m}h^{\sigma}) + S(r,h) \\ &\leq T(r,h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)^{v_j}) + T(r,\prod_{j=1}^{d} h(z+c_j)^{v_j}) \\ &\leq (2d+3)T(r,h)+S(r,h) \end{aligned}$$

Therefore, 1 is not a picard exceptional value of $h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)^{v_j}$. Thus $\exists\, z_0$ such that $h^{n+m}(z_0)\prod_{j=1}^{d} h(z_0+c_j)^{v_j}=1$ by (18), we have $h^{n+m}(z_0)\prod_{j=1}^{d} h(z+c_j)^{v_j}=1$. Hence $h_0^m=1$, and

$$\begin{aligned} \overline{N}(r, \frac{1}{h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)^{v_j}-1}) &\leq \overline{N}(r,\frac{1}{h^m-1}) \\ &\leq mT(r,h)+S(r,h). \end{aligned} \quad (20)$$

From the above inequality and by the second fundamental theorem by Nevanlinna, we have

$$\begin{aligned} T(r, h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)^{v_j}) &\leq \overline{N}(r, h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)^{v_j}) \\ &\quad + \overline{N}(r, \frac{1}{h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)^{v_j} - 1}) + S(r,h) \\ &\leq (m+2d+2)T(r,h) + S(r,h) \end{aligned} \tag{21}$$

which is a contradiction with $n \geq 2k+m+\sigma+5$. Therefore h is a constant. Substituting $f = gh$ into (17), we can get

$$\prod_{j=1}^{d} g(z+c_j)^{v_j}(g^{n+m}(z)(h^{n+m+\sigma}-1) + g^n(z)(h^{n+\sigma}-1)) = 0. \tag{22}$$

Since g is an entire function, we have $\prod_{j=1}^{d} g(z+c_j)^{v_j} \neq 0$. Thus

$$g^{n+m}(z)(h^{n+m+\sigma}-1) + g^n(z)(h^{n+\sigma}-1) = 0. \tag{23}$$

If $h^{n+\sigma} \neq 1$, by (23) we can deduce $T(r,g) = S(r,g)$, which contradicts with a transcendental function g. So $h^{n+\sigma} = 1$. We can also deduce that $h^{n+m+\sigma} = 1$. Then $h^m = 1$. This completes the proof of Theorem 1.

4 Open Problem

1. What can be said if we consider the difference-differential polynomials of the form $[f^n P(f)\prod_{j=1}^{d} f(z+c_j)^{v_j}]^{(k)}$, where $P(z) = a_m z^m + a_{m-1}z^{m-1} + ... + a_1 z + a_0, a_0(\neq 0), a_1, ...a_{m-1}, a_m(\neq 0)$ and $c_j(j = 1, 2, ..., d)$
2. Whether it is possible to replace the sharing value small function by polynomial.
3. Is it possible to reduce the condition of the theorem.

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