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A new criterion for commutative groups

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Abstract

Tarnauceanu [1] proved that the group G is commutative if and only if every two of its k-subsets commute, $k \in \{1, 2, 3\}$***. In the same study, some open problems related to the commutativity of groups were given. The main purpose of this paper is to demonstrate that a finite group G of order at least*** 8 ***is commutative if and only if every two of its 4-subsets commute.***

Keywords: *commutative group, k-subset.*

1 Introduction

A commutative group is a group in which every two elements commute. Many studies have investigated the criteria of commutativity of a group [2, 3, 4]. Tarnauceanu [1] obtained that the group G is commutative if and only if every two of its k-subsets commute, $k \in \{1, 2, 3\}$. In this paper, we show that this theorem also is true for $k = 4$.

Let G be a group and k be a positive integer. A subset A of G is called a k-subset if $|A| = k$. Given two subsets A and B of G, we will denote by AB their product, i.e. $AB = \{ab | a \in A, b \in B\}$. Also, we will say that A and B commute if AB = BA.

Theorem 1.1. [**1**] For $k \in \{1, 2, 3\}$, the group G is commutative if and only if every two of its k-subsets commute.

Lemma 1.2. [**1**] Let G be a finite group and A, B be two subsets of G satisfying $|A| + |B| > |G|$. Then $AB = G$.

Let G be a group and $a \in G$. The order of a is the order of the cyclic subgroup $< a >$ and is denoted $|a|$. A group in which every element has order a

power(≥ 0) of some fixed prime p is called p-group. As a consequence, a finite group G is a p-group if and only if $|G|$ is a power of p.
Theorem 1.3 [**2**] The center $Z(G)$ of a nontrivial finite p-group G contains more than one element.

2 Main Results

Open Problem [**1**] Let $k \geq 4$ be an integer and let G be a finite group of order at least $2k$. If every two k-subsets of G commute, then is G a commutative group?
As seen in Lemma 1.2, if $|A| + |B| > |G|$, then $AB = G = BA$. Therefore, we must investigate whether a group of at least 8 elements is commutative if and only if every two of its 4-subsets commute.

Theorem 2.1 Let G be a finite group of order at least 8. G is commutative if and only if every two of its 4-subsets commute.
Proof. Clearly, if the group G is commutative, then every two of its k-subsets commute for any k.
Conversely, if any two 4-subsets of G commute, we need to show that G is commutative. Assume the contrary, and consider the following two cases:

- **Case 1.** $|Z(G)| \neq 1$.
  Then there exists $a \in Z(G)$ such that $a \neq 1$. As G is not commutative, there exists $x \in G \setminus Z(G)$ such that $x \neq x^{-1}$. Moreover, there exists $y \in G$ satisfying $xy \neq yx$.

  Now, let $A = \{1, x, a, y\}$ and $B = \{1, x^{-1}, a, y\}$.
  $BA = \{1, x, a, y, x^{-1}, x^{-1}a, x^{-1}y, ax, a^2, ay, yx, y^2\}$
  Notice that $xy \in AB$, but $xy \notin BA$.
  It is clear that $xy \notin \{1, x, y, x^{-1}y, a^2, ay, yx, y^2\}$. Indeed, we will check the following cases:

  - If $xy = a$, then $xy \in Z(G)$. So $(xy)x^{-1} = x^{-1}(xy) = y$. Hence $xy = yx$ and it is a conflict with our assumption.

  - If $xy = x^{-1}$, then we also get that $xy = yx$.

  - If $xy = x^{-1}a$, then $xyx = a \in Z(G)$. Since $x^{-1}(xyx) = (xyx)x^{-1}$, we obtain that $xy = yx$.
  - If $xy = ax$, then $y = a \in Z(G)$ and it is a confliction.

  So it is a conflict with the assumption of that A and B being commute. Until now, by Theorem 1.3, we obtain that the theorem true if G is a p-group.

- **Case 2.** $|Z(G)| = 1$.
  Since G is not commutative, there exists $a \in G$ such that $a \neq 1$ and $a^2 \neq 1$. Moreover, as $a \notin Z(G)$, there exists $b \in G$ such that $ab \neq ba$. We will make evaluation according to the order of a and b.

  1. If $|a| \neq 3$ and $|b| \neq 2$:
     Let $A = \{1, a, a^{-1}, b\}$ and $B = \{1, a, b^2, a^{-1}b\}$.
     $AB = \{1, a, b^2, a^{-1}b, a^2, ab^2, b, a^{-1}, a^{-1}b^2, (a^{-1})^2b, ba, b^3, ba^{-1}b\}$
     Here, $ab \in BA$ but $ab \notin AB$.
     So it is a conflict.

  2. If $|a| = 3$ and $|b| = 3$:
     Let $A = \{1, a, a^2, b\}$ and $B = \{1, a, a^2, b^2\}$.
     $AB = \{1, a, a^2, b^2, ab^2, b, a^2b^2, ba, ba^2\}$
     Here, $ab \in BA$ but $ab \notin AB$. It is clear that $ab \notin \{1, a, a^2, b^2, ab^2, b, a^2b^2, ba\}$.
     Now we have to show that $ab \neq ba^2$.
     Let $ab = ba^2$. Then

     $$\begin{aligned} a^2(ab)a &= a^2(ba^2)a \\ ba &= a^2b \\ bab &= a^2b^2 \\ b^2a^2 &= a^2b^2 \\ b(b^2a^2)b &= b(a^2b^2)b \\ a^2b &= ba^2 \\ a(a^2b)a &= a(ba^2)a \\ ba &= ab \end{aligned}$$

     So it is a conflict.

  3. If $|a| = 3$ and $|b| \neq 2$, $|b| \neq 3$:
     Let $A = \{1, a, a^2, b\}$ and $B = \{1, a, b^2, b^3\}$.
     $AB = \{1, a, b^2, b^3, a^2, ab^2, ab^3, a^2b^2, a^2b^3, b, ba, b^3, b^4\}$

Here, $ab \in BA$ but $ab \notin AB$. It is clear that $ab \notin \{1, a, b^2, b^3, a^2, ab^2, ab^3, a^2b^2, b, ba, b^3, b^4\}$. Let $ab = a^2b^3$. Then we have that,

$$\begin{aligned} ab &= a^2b^3 \\ a^2(ab) &= a^2(a^2b^3) \\ b &= ab^3 \\ 1 &= ab^2 \\ (b^{-1})^2 &= a \\ ba &= ab \end{aligned}$$

So it is a conflict.

4. If $|a| \neq 3$ and $|b| = 2$:
   Let $A = \{b, ba, a, ba^{-1}\}$ and $B = \{a^{-1}b, b, a^{-1}, ab\}$.
   $AB = \{ba^{-1}b, 1, ba^{-1}, bab, b, ba^2b, ab, a^2b, b(a^{-1})^2b, b(a^{-1})^2\}$.
   Here, $ba \in BA$ but $ba \notin AB$. It easily seen that $ba \notin \{ba^{-1}b, 1, ba^{-1}, bab, b, ba^2b, ab, b(a^{-1})^2b, b(a^{-1})^2\}$. We must show that $ba \neq a^2b$. Suppose that $ba = a^2b$. Then

$$\begin{aligned} ba &= a^2b \\ bab &= a^2b^2 \\ babbab &= a^4 \\ ba^2b &= a^4 \\ b(bab)b &= a^4 \\ a &= a^4 \end{aligned}$$

   It is a conflict with $|a| \neq 3$.

5. If $|a| = 3$ and $|b| = 2$:
   Let $A = \{1, a^2, ab, ba\}$ and $B = \{a, a^2, ba^2, b\}$.
   $AB = \{a, a^2, ba^2, b, 1, a^2ba^2, a^2b, aba, aba^2, baba^2, bab\}$.

   - If $ba \neq a^2b$, then we have that $ba \in BA$ but $ba \notin AB$. So it is a conflict.

   - If $ba = a^2b$, then $< a, b | a^3 = b^2 = 1 \ \ ba = a^2b >$ is a subgroup of G and $< a, b >= \{1, a, a^2, b, ab, a^2b\}$.
     Since $|G| \geq 8$, there exist two distinct elements c and d in $G \backslash < a, b >$.
     Let $A = \{1, a, b, ab\}$ and $B = \{1, b, c, d\}$. Then we obtain that $AB = \{1, b, c, d, a, ab, ac, ad, bc, bd, abc, abd\}$.

Hence we get that, $ba \in BA$ but $ba \notin AB$. So it is a conflict.

As a result, we have obtained a contradiction in every situation. In this case, our assumption is wrong. G is a commutative group.

3 Open Problem

We obtain that the group G is commutative if and only if every two of its 4-subsets commute. We think that a more elegant solution can be made. In addition, we partly give an answer for the open problem given by Tarnauceanu [1]. We prove a theorem for only 4-subsets. We believe that a more general solution can be found in future studies.

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