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On an Open Problems of F. Qi

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Abstract

In this paper, we propose the answer to an open problem recently posed. In addition, we derive some inequalities by introducing certain parameters λ, p and q.

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1 Introduction

Some results involving open problems of F. Qi were established and proved earlier by K. Tibor Pogany (cf. [6]). Other results were obtained for positive operators in Hilbert space (for details, see [2]). Then, B. Belaidi et al. (see [1]) were particularly interested in the following two problems posed in [3]:

Problem 1.1 *For $x_i \geq 0$, $i = 1, 2, .., n$, $n \in \mathbb{N}$, $n \geq 2$, determine the best possible constants α_n $\lambda_n \in \mathbb{R}$ and $\beta_n > 0$, $\mu_n < \infty$ such that*

$$\beta_n \sum_{i=1}^{n} x_i^{\alpha_n} \leq \exp\left(\sum_{i=1}^{n} x_i\right) \leq \mu_n \sum_{i=1}^{n} x_i^{\lambda_n}. \tag{1}$$

Problem 1.2 *What is the integral analogue of the two-sided inequality (1)?*

In the following theorems and lemmas were recently given the answers to open Problems 1.1 and 1.2 (for details See[1]):

Theorem 1.3 *Let* $p \geq 1$ $x_i \in (0, \infty)$, $i = 1, 2, ..., n,$ $n \in \mathbb{N}$ *and* $n \geq 2$, *then*

$$\sum_{i=1}^{n} x_i^p \leq \frac{p^p}{e^p} \exp\left(\sum_{i=1}^{n} x_i\right). \tag{2}$$

The factor $\frac{p^p}{e^p}$ *is the best possible in this inequality (the smallest constant independent of* x_i*).*

Theorem 1.4 *Let* $0 < p \leq 1$ $x_i \in (0, \infty)$, $i = 1, 2, ..., n,$ $n \in \mathbb{N}$ *and* $n \geq 2$, *then*

$$\sum_{i=1}^{n} x_i^p \leq n^{1-p}\frac{p^p}{e^p} \exp\left(\sum_{i=1}^{n} x_i\right). \tag{3}$$

The constant $n^{1-p}\frac{p^p}{e^p}$ *is the best possible in (3).*

Lemma 1.5 *Let* $p > o$ $x \in (0, \infty)$, *then*

$$x^p \leq \frac{p^p}{e^p} e^x. \tag{4}$$

The constant $\frac{p^p}{e^p}$ *is the best possible in (4).*

Lemma 1.6 *Let* $p > 0$, $x_i \in (0, \infty)$, $i = 1, 2, ..., n,$ $n \in \mathbb{N}$ *and* $n \geq 2$, *then*

(i) For $p \geq 1$

$$\sum_{i=1}^{n} x_i^p \leq \left(\sum_{i=1}^{n} x_i\right)^p. \tag{5}$$

(ii) For $o < p \leq 1$

$$n^{p-1}\sum_{i=1}^{n} x_i^p \leq \left(\sum_{i=1}^{n} x_i\right)^p. \tag{6}$$

Theorem 1.7 *Let* $p > 0$ $n \geq 2$, *then*

$$\exp\left(\sum_{i=1}^{n} x_i\right) \leq \frac{p^p}{n} e^{np} \sum_{i=1}^{n} x_i^{-p} \tag{7}$$

holds for $0 < x_i \leq p$, $1 \leq i \leq n$. *The constant* $\frac{p^p}{n} e^{np}$ *is the best possible.*

In [4] the following lemmas and theorem were proved.

Lemma 1.8 *Let (x_i) be a sequence of non negative real numbers and let $0 < p \leq q \leq \infty$, then*

$$\left(\sum_{i=1}^{n} x_i^q\right)^{\frac{1}{q}} \leq \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \leq n^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{i=1}^{n} x_i^q\right)^{\frac{1}{q}}. \tag{8}$$

Theorem 1.9 *Let $0 < p \leq q$, and $x_i > 0$, $i = 1, 2, .., n$, $n \in N$, then*

$$\sum_{i=1}^{n} x_i^p \leq n^{1-\frac{p}{q}} \frac{p^p}{e^p} \exp\left(\sum_{i=1}^{n} x_i^q\right)^{\frac{1}{q}}. \tag{9}$$

The constant $n^{1-\frac{p}{q}} \dfrac{p^p}{e^p}$ is the best possible.

Lemma 1.10 *Let $0 < p < q \leq \infty$, and f, w be non negative Lebesgue measurable function on $[a, b]$ such that $\int_a^b f^q\, w\, dx < \infty$, then*

$$\left(\int_a^b f^p\, w\, dx\right)^{\frac{1}{p}} \leq \left(\int_a^b w\, dx\right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_a^b f^q w\, dx\right)^{\frac{1}{q}}. \tag{10}$$

In [5] was proved the following theorem.

Theorem 1.11 *Let $p > 0$, $x_i \geq 0$, $i = 1, 2, ..., n$, $n \in \mathbb{N}$, $n \geq 2, \lambda_i > 0$.*

1) If $p \geq 1$, then

$$\sum_{i=1}^{n} \lambda_i^p x_i^p \leq \frac{p^p}{e^p} \exp \sum_{i=1}^{n} \lambda_i x_i. \tag{11}$$

2) If $0 < p \leq 1$, then

$$\sum_{i=1}^{n} \lambda_i x_i^p \leq \frac{p^p}{e^p} \left(\sum_{i=1}^{n} \lambda_i\right)^{1-p} \exp \sum_{i=1}^{n} \lambda_i x_i. \tag{12}$$

The constant $\frac{p^p}{e^p}$ is the best possible.

In [5], the following problem is posed: let p, q be real parameters such that $0 < p \leq q \leq \infty$ and $x_i \geq 0$, $\lambda_i > 0$, $i = 1, 2, ..., n$, $n \geq 2$. Give the integral analogue of inequalities (11) and (12) for two parameters p and q with the best possible constant.
The aim of the present article, in one hand, is to give an answer to this open problem and on the other hand, to generalize some inequalities obtained in [1] and [4].

2 Main results

In the following theorem, we give an answer to the proposed open problem 3.1 in [5]. The sharp constant (best possible) is determined only for the case $p = q$.

Theorem 2.1 *Let* $0 < p \leq q \leq \infty$, $0 < \lambda < \infty$, f *be a non negative Lebesgue measurable function on [a,b] such that* $\int_a^b f^q dx < \infty$.

1) If $p \geq 1$, *then*

$$\int_a^b \lambda^p f^p dx \leq (b-a)^{\frac{1}{p}-\frac{1}{q}} \frac{p^p}{e^p} \exp\left(\int_a^b \lambda^q f^q dx\right)^{\frac{1}{q}}. \tag{13}$$

2) If $0 < p \leq 1$, *then*

$$\int_a^b \lambda f^p dx \leq (b-a)^{\frac{1}{p}-\frac{1}{q}} \frac{p^p}{e^p} \exp\left(\int_a^b \lambda^{\frac{q}{p}} f^q dx\right)^{\frac{1}{q}}. \tag{14}$$

Proof 2.2 *1)* $p \geq 1$.

Let $g = \lambda f$. *By applying Lemma 1.5 with* $x = \left(\int_a^b g^q dx\right)^{\frac{1}{q}}$, *we get*

$$\left(\int_a^b g^q dx\right)^{\frac{p}{q}} \leq \frac{p^p}{e^p} \exp\left(\int_a^b g^q dx\right)^{\frac{1}{q}}. \tag{15}$$

Now, taking into account (10) with $w = 1$ *and inequality (15), we conclude that*

$$\left(\int_a^b g^p dx\right) \leq (b-a)^{1-\frac{p}{q}} \left(\int_a^b g^q dx\right)^{\frac{p}{q}}$$
$$\leq (b-a)^{1-\frac{p}{q}} \frac{p^p}{e^p} \exp\left(\int_a^b g^q dx\right)^{\frac{p}{q}}.$$

Thus

$$\left(\int_a^b \lambda^q f^q dx\right) \leq (b-a)^{1-\frac{p}{q}} \frac{p^p}{e^p} \exp\left(\int_a^b \lambda^q f^q dx\right)^{\frac{1}{q}}.$$

2) $0 < p \leq 1$.

By setting $g = \lambda^{\frac{1}{q}} f$, *similar arguments lead to the inequality (14).*

If in (13) and (14) we put $q = p$, we obtain the following corollary.

Corollary 2.3 *Let* $0 < p \leq \infty$, $0 < \lambda < \infty$, f *be a non negative Lebesgue measurable function on [a,b] such that* $\int_a^b \lambda^p f^p dx < \infty$.

1) If $p \geq 1$, then

$$\int_a^b \lambda^p f^p dx \leq \frac{p^p}{e^p} \exp\left(\int_a^b \lambda^p f^p dx\right)^{\frac{1}{p}}. \tag{16}$$

Equality in (16) holds if $f(x) = \frac{p(b-a)^{-\frac{1}{p}}}{\lambda}$. *Thus the constant* $\frac{p^p}{e^p}$ *is the best possible (the smallest constant independent of function f).*

2) If $0 < p \leq 1$, then

$$\int_a^b \lambda f^p dx \leq \frac{p^p}{e^p} \exp\left(\int_a^b \lambda f^p dx\right)^{\frac{1}{p}}. \tag{17}$$

Equality in (17) holds if $f(x) = \frac{p(b-a)^{-\frac{1}{p}}}{\lambda^{\frac{1}{p}}}$. *Thus the constant* $\frac{p^p}{e^p}$ *is the best possible.*

By putting $y_i = \lambda_i x_i$ in inequality (8), we have the following corollary.

Corollary 2.4 *Let (x_i) be a sequence of non negative real numbers and $0 < p \leq< q \leq \infty$, $\lambda_i > 0$, $i = 1, 2, ..., n$, then*

$$\left(\sum_{i=1}^n \lambda_i^q x_i^q\right)^{\frac{1}{q}} \leq \left(\sum_{i=1}^n \lambda_i^p x_i^p\right)^{\frac{1}{p}} \leq n^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{i=1}^n \lambda_i^q x_i^q\right)^{\frac{1}{q}}. \tag{18}$$

Theorem 2.5 *Let $0 < p \leq q \leq \infty$, $x_i > 0$, $\lambda_i > 0$ $i = 1, 2, ..., n$, then*

$$\sum_{i=1}^n \lambda_i^p x_i^p \leq n^{1-\frac{p}{q}} \frac{p^p}{e^p} \exp\left(\sum_{i=1}^n (\lambda_i^q x_i^q)\right)^{\frac{1}{q}}. \tag{19}$$

The constant $n^{1-\frac{p}{q}} \frac{p^p}{e^p}$ *is the best possible.*

Proof 2.6 *By using the right hand side of double inequality (18) and Lemma 1.5, we obtain*

$$\left(\sum_{i=1}^n \lambda_i^p x_i^p\right) \leq n^{1-\frac{p}{q}} \left(\sum_{i=1}^n (\lambda_i^q x_i^q)\right)^{\frac{p}{q}}$$
$$\leq n^{1-\frac{p}{q}} \frac{p^p}{e^p} \exp\left(\sum_{i=1}^n (\lambda_i^q x_i^q)\right)^{\frac{1}{q}}.$$

By taking in (19) $x_i = \dfrac{pn^{-\frac{1}{q}}}{\lambda_i}$, *we have equality and we conclude that* $n^{1-\frac{p}{q}} \frac{p^p}{e^p}$ *is the best possible constant.*

Remark 2.7 *1) If in (19) $\lambda_i = 1$, $i = 1, ..., n$, then we have inequality (9) of Theorem 1.9 with the best constant $n^{1-\frac{p}{q}} \frac{p^p}{e^p}$.*
2) If in (19) $q = 1$, we get the following Corollary.

Corollary 2.8 *Let $0 < p \leq 1 \leq \infty$, $x_i > 0$, $\lambda_i > 0$, $i = 1, 2, ..., n$, $n \in \mathbb{N}$, then*

$$\sum_{i=1}^{n} \lambda_i^p x_i^p \leq n^{1-p} \frac{p^p}{e^p} \exp^{\sum_{i=1}^{n} \lambda_i x_i}. \tag{20}$$

Remark 2.9 *If in (20) $\lambda_i = 1$, $i = 1, ..., n$ we obtain inequality (3).*

Theorem 2.10 *Let $p > 0$ $x_i \geq 0$, $0 < x_i \leq p$, $\lambda_i > 0$, $i = 1, 2, ..., n$, $n \geq 2$.*

1) If $p \geq 1$, then

$$\sum_{i=1}^{n} \lambda_i^p x_i^p \leq \frac{e^{p(x-1)}}{n} p^{2p} \sum_{i=1}^{n} (\lambda_i x_i)^{-p}. \tag{21}$$

2) If $0 < p \leq 1$, then

$$\sum_{i=1}^{n} \lambda_i x_i^p \leq \sum_{i=1}^{n} (\lambda_i)^{1-p} \frac{e^{p(n-1)}}{n} p^{2p} \sum_{i=1}^{n} (\lambda_i x_i)^{-p}. \tag{22}$$

Proof 2.11 *1) $p \geq 1$. If in (7) we put $y_i = \lambda_i x_i$, we have*

$$\exp \sum_{i=1}^{n} y_i \leq \frac{p^p}{n} e^{np} \sum_{i=1}^{n} y_i^{-p},$$

thus

$$\exp \sum_{i=1}^{n} \lambda_i x_i \leq \frac{p^p}{n} e^{np} \sum_{i=1}^{n} (\lambda_i x_i)^{-p}. \tag{$*$}$$

By (11) and (*), we get

$$\frac{e^p}{p^p} \sum_{i=1}^{n} \lambda_i^p x_i^p \leq \exp \sum_{i=1}^{n} \lambda_i x_i \leq \frac{p^p}{n} e^{np} \sum_{i=1}^{n} (\lambda_i x_i)^{-p}.$$

Consequently

$$\sum_{i=1}^{n} \lambda_i^p x_i^p \leq \frac{e^{p(n-1)p2^p}}{n} \sum_{i=1}^{n} (\lambda_i x_i)^{-p}.$$

2) $0 < p \leq 1$.
By using (12) and (*) we obtain inequality (22).

The proof is complete.

Theorem 2.12 *Let* $p > 0$ $x_i \geq 0$, $\lambda_i > 0$, $i = 1, 2, ..., n$, *such that* $\sum_{i=1}^{+\infty} \lambda_i x_i < \infty$, $\left(\sum_{i=1}^{+\infty} \lambda_i\right) < \infty$.

1) If $p \geq 1$, *then*

$$\sum_{i=1}^{+\infty} \lambda_i^p x_i^p \leq \frac{p^p}{e^p} \exp \sum_{i=1}^{+\infty} \lambda_i x_i. \tag{23}$$

2) If $0 < p \leq 1$, *then*

$$\sum_{i=1}^{+\infty} \lambda_i x_i^p \leq \frac{p^p}{e^p} \left(\sum_{i=1}^{+\infty} \lambda_i\right)^{1-p} \exp \sum_{i=1}^{+\infty} \lambda_i x_i. \tag{24}$$

The constant $\frac{p^p}{e^p}$ *is the best possible.*

Proof 2.13 *Letting* $n \to +\infty$ *in Theorem 1.11, we obtain (23) and (24). If in (23)* $x_i = \frac{p}{\lambda_i}$, *we get equality, then the constant* $\frac{p^p}{e^p}$ *is the best possible.*

3 Open Problems

Problem 3.1 *Establish a relation between the inequalities (13) and (14) of the Theorem 2.1 and the well-known Tchebychev inequality.*

Problem 3.2 *Let* $E \subset \mathbb{R}^n$, $0 < \text{mes} E < \infty$.
If $0 < p_1 < p < p_2 \leq \infty$, *then*

$$\|f\|_{L_p(E)} \leq \|f\|^{\alpha}_{L_{p_1}(E)} \|f\|^{1-\alpha}_{L_{p_2}(E)}, \tag{25}$$

where $\alpha \in (0, 1)$. *Under what conditions on the parameters* p_1, p, p_2 *and* α *does the inequality (25) hold?*

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