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Almost contact curves in (κ, μ)-Manifolds

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Abstract

We study almost contact curves in a (κ, μ)-manifold admitting Schouten-van Kampen connection and obtain the curvature and torsion of such curves. The conditions for Legendre curve being of $AW(\omega)$, $\omega = 1, 2, 3$ are obtained. Invariance of curves being biharmonic, C-parallel, C-proper with respect to Schouten-van Kampen connection is proved.

Keywords: *Biharmonic curve, Curve of type AW(ω), (κ, μ)- contact metric manifold, Legendre curve, Schouten-van Kampen Connection.*
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1 Introduction

Almost contact curves are Frenet curves in almost contact metric manifolds which are tangent to the almost contact ditribution [2],[9]. A natural connection adapted to a pair of complementary distributions on a differentiable manifold with an affine connection is the Schouten-van Kampen connection [10]. Olszak [7]studied the Schouten-van Kampen connection in manifolds with almost para contact metric structure. In this paper we study almost contact curves in (κ, μ)-manifolds admitting Schouten-van Kampen connection. The paper is systematized as follows:

After giving introduction and preliminaries in section 1, section 2, in section 3, we find the curvature and torsion of Legendre curve with reference to Schouten-van Kampen connection $\tilde{\nabla}$. In section 4, the conditions for Legendre curve to be of AW(ω) type [1] with respect to $\tilde{\nabla}$ are obtained. In section 5, the invariance of conditions of being biharmonic, proper, recurrent, C-parallel, C-proper curves [6] with respect to ∇(Levi-Civita connection) and $\tilde{\nabla}$ are proved.

2 Preliminaries

A $(2n+1)$-dimensional differential manifold M is called an almost contact metric manifold if there is an almost contact metric structure (ϕ, ξ, η, g) satisfying

$$\phi^2 = -I + \eta \otimes \xi,\ \phi\xi = 0,\ g(X, \xi) = \eta(X), \eta(\xi) = 1, \eta \circ \phi = 0, \tag{1}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2}$$

The equation

$$\nabla_X \xi = -\phi X - \phi h X \tag{3}$$

holds in contact metric manifolds.
Let M be a (κ, μ)-manifold. Then the following relations hold in M[5]:

$$h^2 = (\kappa - 1)\phi^2, \tag{4}$$

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{5}$$

$$(\nabla_X \eta)(Y) = g(X + hX, \phi Y). \tag{6}$$

The relation between Schouten-van Kampen connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ is given by [7].

$$\tilde{\nabla}_Y Z = \nabla_Y Z - \pi(Y, Z), \tag{7}$$

for all vector fields Y and Z on M, where

$$\pi(Y, Z) = g(Y, \phi h Z)\xi - \eta(Z)\phi h Y - g(Y, \phi Z)\xi + \eta(Z)\phi Y. \tag{8}$$

The torsion tensor $\tilde{\mathbf{T}}$ of $\tilde{\nabla}$ is given by,

$$\tilde{\mathbf{T}}(Y, Z) = 2g(Y, \phi Z)\xi - \eta(Y)\phi Z + \eta(Z)\phi Y - \eta(Y)\phi h Z + \eta(Z)\phi h Y. \tag{9}$$

3 Curvature and Torsion of a Legendre curve

We find curvature and torsion of a Legendre curve ,.i.e. an almost contact curve in a (κ, μ) manifold M with respect to $\tilde{\nabla}$.

Suppose σ is a Legendre curve on M. Then $\eta(\alpha) = 0$ [8]. The vector fields α, $\beta = \phi\alpha$, $\gamma = \xi$ form a Frenet frame field on σ. We have

$$g(\nabla_\alpha \alpha, \xi) = 0, \tag{10}$$

$$g(\nabla_\alpha \alpha, \alpha) = 0, \tag{11}$$

which implies that $\nabla_\alpha \alpha$ is perpendicular to α and is in the plane of β [2],[8]. Hence we can write

$$\nabla_\alpha \alpha = -f\beta, \tag{12}$$

where f is a constant function in an interval I. Applying (7) in (12) we get

$$\tilde{\nabla}_\alpha \alpha = -f\phi\alpha, \tag{13}$$

which yields

$$g(\tilde{\nabla}_\alpha \alpha, \tilde{\nabla}_\alpha \alpha) = f^2.$$

$$\tilde{\omega} = f.$$

Hence, we state the subsequent theorem:

Theorem 3.1 *The curvature of a Legendre curve in a (κ, μ) manifold is given by $\tilde{\omega} = f$ with respect to $\tilde{\nabla}$, where f is a constant function in an interval I.*

We consider a Frenet curve $\sigma : I \to M$ and $\{E_1, E_2, E_3\}$ as a Frenet frame of σ. Then we have

$$\tilde{\nabla}_{E_1} E_1 = \tilde{\omega} E_2,$$

$$E_2 = \frac{1}{\tilde{\omega}} \tilde{\nabla}_{E_1} E_1.$$

Using (13) in the above equation, we get

$$E_2 = -\frac{f}{\tilde{\omega}} \phi\alpha.$$

Therefore,

$$\tilde{\nabla}_\alpha E_2 = -\alpha\left(\frac{f}{\tilde{\omega}}\right)\phi\alpha - \left(\frac{f}{\tilde{\omega}}\right)\tilde{\nabla}_\alpha \phi\alpha.$$

Using (5) and (7) in the above equation, we obtain

$$\tilde{\nabla}_{E_1} E_2 = -\alpha\left(\frac{f}{\tilde{\omega}}\right)\phi\alpha - \frac{f(1+\sqrt{\kappa-1})}{\tilde{\omega}}\xi. \tag{14}$$

Since

$$\tilde{\nabla}_{E_1}E_2 = -\tilde{\omega}E_1 + \tilde{\tau}E_3, \tag{15}$$

from (14) and (15), we have

$$\tilde{\tau}E_3 = \tilde{\omega}\alpha - \alpha\left(\frac{f}{\tilde{\omega}}\right)\phi\alpha - \frac{f(1+\sqrt{\kappa-1})}{\tilde{\omega}}\xi.$$

Therefore, the torsion is given by

$$\tilde{\tau} = \sqrt{\tilde{\omega}^2 + \left(\alpha\left(\frac{f}{\tilde{\omega}}\right)\right)^2 + \left(\frac{f(1+\sqrt{\kappa-1}}{\tilde{\omega}}\right)^2}. \tag{16}$$

Therefore,

Theorem 3.2 *If the Legendre curve in a (κ,μ)-manifold is non geodesic, then its torsion is given by*

$$\tilde{\tau} = \sqrt{\tilde{\omega}^2 + \left(\alpha\left(\frac{f}{\tilde{\omega}}\right)\right)^2 + \left(\frac{f(1+\sqrt{\kappa-1})}{\tilde{\omega}}\right)^2}$$

with respect to $\tilde{\nabla}$.

4 AW(ω) type of curve.

Definition 4.1 *An almost contact curve $\sigma(s)$ in a (κ,μ) manifold is said to be*

i) a curve of type AW(1) if $\beta_3 = 0$,

ii) of AW(2) type if

$$\|\beta_2\|^2\beta_3 = \langle\beta_3,\beta_2\rangle\beta_2, \tag{17}$$

iii) a curve of AW(3) type if

$$\|\beta_1\|^2\beta_3 = \langle\beta_3,\beta_1\rangle\beta_1, \tag{18}$$

where $\beta_1 = \sigma''^{\perp}$, $\beta_2 = \sigma'''^{\perp}$, $\beta_3 = \sigma''''^{\perp}$.

Let σ be a almost contact curve in a (κ,μ) contact metric manifold. Then

$$\sigma' = \alpha,$$

$$\sigma'' = \tilde{\nabla}_\alpha\alpha = \omega\beta,$$

$$\sigma''' = \tilde{\nabla}_\alpha \tilde{\nabla}_\alpha \alpha = \omega' \beta - \omega^2 \alpha + \omega(\tau + \sqrt{\kappa - 1} + 1)\xi,$$
$$\sigma'''' = \tilde{\nabla}_\alpha \tilde{\nabla}_\alpha \tilde{\nabla}_\alpha \alpha = -3\omega\omega' \alpha + (\omega'' - \omega^3)\beta + 2\omega'\tau - \omega'\sqrt{\kappa - 1} + 1)\xi.$$

We have by above condition that,

$$\begin{aligned} \beta_1 &= \omega\beta, \\ \beta_2 &= \omega'\beta + \omega(\tau + \sqrt{\kappa - 1} + 1)\xi, \\ \beta_3 &= (\omega'' - \omega^3)\beta + 2\omega'(\tau + \sqrt{\kappa - 1} + 1)\xi. \end{aligned} \tag{19}$$

It follows from the above equations that

$$\begin{aligned} \|\beta_1\|^2 = \omega^2, \|\beta_2\|^2 &= \omega'^2 + \omega^2(\tau + \sqrt{\kappa - 1} + 1)^2, \\ \langle \beta_3, \beta_1 \rangle &= 2\omega'\tau - \omega'(\sqrt{\kappa - 1} + 1)\xi, \\ \langle \beta_3, \beta_2 \rangle &= \omega'(\omega'' - \omega^3) + \omega\omega'(2\tau - \sqrt{\kappa - 1} + 1)(\tau + \sqrt{\kappa - 1} + 1). \end{aligned} \tag{20}$$

From definition (4.1), and equations (19), (20), we state the following:

Theorem 4.2 *An almost contact curve σ in a (κ, μ)- manifold is*

i) AW(1) type if and only if $\omega' = 0$ *or* $\tau = -\sqrt{\kappa - 1} - 1$,

ii) AW(2) type if and only if $2\omega'^2 = \omega(\omega'' - \omega^3)$,

iii) AW(3) type if and only if $\omega' = 0$ *or* $\tau = -\sqrt{\kappa - 1} - 1$.

5 Biharmonic and other type of almost contact curves in (κ, μ)-contact metric manifolds

Definition 5.1 *An almosy=t contact curve σ on a (κ, μ)-manifold M is called Biharmonic equipped with $\tilde{\nabla}$ if it satisfies the equation*

$$\tilde{\nabla}^3_\alpha \alpha + \tilde{\nabla}_\alpha \tilde{\boldsymbol{T}}(\tilde{\nabla}_\alpha \alpha, \alpha) + \tilde{R}(\tilde{\nabla}_\alpha \alpha, \alpha)\alpha = 0, \tag{21}$$

where $\alpha = \dot{\sigma}$, $\tilde{\boldsymbol{T}}$ is the torsion and $\tilde{R}$ is the curvature of $\tilde{\nabla}$.

Consider the Frenet frame field $\{\alpha, \beta, \gamma\}$ of an almost contact curve σ in (M, ϕ, ξ, η, g). Let $\beta = -\phi\alpha$ and $\gamma = \xi$.
Using the definition of R and (7), we get

$$\begin{aligned} \tilde{R}(Y, Z)W = &R(Y, Z)W - \pi(Y, \nabla_Z W) - \nabla_Y \pi(Z, W) + \pi(Y, \pi(Z, W)) + \pi(Z, \nabla_Y W) \\ &+ \nabla_Z \pi(Y, W) - \pi(Z, \pi(Y, W)) + \pi([Y, Z], W). \end{aligned}$$

Take $Y = \beta$, $Z = W = \alpha$, we get

$$\begin{aligned}\tilde{R}(\beta,\alpha)\alpha =& R(\beta,\alpha)\alpha - \pi(\beta,\nabla_\alpha\alpha) - \nabla_\beta\pi(\alpha,\alpha) \\ &+ \pi(\beta,\pi(\alpha,\alpha)) + \pi(\alpha,\nabla_\beta\alpha) \\ &+ \nabla_\alpha\pi(\beta,\alpha) - \pi(\alpha,\pi(\beta,\alpha)) + \pi([\beta,\alpha],\alpha).\end{aligned} \tag{22}$$

In view of (8), we have

$$\begin{aligned}&\pi(\beta,\nabla_\alpha\alpha) = 0, \quad \pi(\alpha,\alpha) = 0, \quad \pi(\alpha,\xi) = -\phi\alpha - \phi h\alpha, \\ &\nabla_\beta\pi(\alpha,\alpha) = 0, \quad \pi(\beta,\pi(\alpha,\alpha)) = 0, \quad \pi(\beta,\alpha) = \xi(1-\sqrt{\kappa-1}), \\ &\nabla_\alpha\pi(\beta,\alpha) = (2-\kappa)\beta, \quad \pi(\alpha,\pi(\beta,\alpha)) = -(1-\sqrt{\kappa-1})^2\beta.\end{aligned} \tag{23}$$

By differentiating, $g(\alpha,\xi) = 0$, covariantly with respect to β, we obtain

$$\eta(\nabla_\beta\alpha) = (1-\sqrt{\kappa-1}). \tag{24}$$

Therefore we obtain

$$\tilde{R}(\beta,\alpha)\alpha = R(\beta,\alpha)\alpha + (1-\sqrt{\kappa-1})^2\beta. \tag{25}$$

Now we have

$$\begin{aligned}\tilde{R}((\tilde{\nabla}_\alpha\alpha,\alpha)\alpha &= \tilde{R}(\nabla_\alpha\alpha,\alpha)\alpha, \\ \tilde{R}((\tilde{\nabla}_\alpha\alpha,\alpha)\alpha &= \omega\tilde{R}(\beta,\alpha)\alpha.\end{aligned} \tag{26}$$

By the use of (7), we get

$$\tilde{\nabla}^3_\alpha\alpha = \nabla^3_\alpha\alpha + 2\omega'\xi(1+\sqrt{\kappa-1}). \tag{27}$$

In view of (9), we obtain

$$\tilde{\nabla}_\alpha\tilde{\mathbf{T}}(\tilde{\nabla}_\alpha\alpha,\alpha) = 2\omega'\xi. \tag{28}$$

By virtue of (26), (27) and (28), we get

$$\begin{aligned}\tilde{\nabla}^3_\alpha\alpha + \tilde{\nabla}_\alpha\tilde{\mathbf{T}}(\tilde{\nabla}_\alpha\alpha,\alpha) + \tilde{R}(\tilde{\nabla}_\alpha\alpha,\alpha)\alpha =& \nabla^3_\alpha\alpha + R(\nabla_\alpha\alpha,\alpha)\alpha + 2\omega'\xi(2+\sqrt{\kappa-1}) \\ &+ \omega((1-\sqrt{\kappa-1})^2)\beta.\end{aligned} \tag{29}$$

Therefore we state the following theorem:

Theorem 5.2 *An almost contact curve σ is biharmonic with reference to $\tilde{\nabla}$ if and only if it is biharmonic with reference to ∇ provided $\kappa = 2$.*

Definition 5.3 *An almost contact curve σ in a (κ,μ)-manifold M*

i) hss proper mean curvature vector $\tilde{H}$ with reference to ∇ if $\tilde{\Delta}\tilde{H} = \lambda\tilde{H}$, for constant λ, where $\tilde{\Delta}\tilde{H} = -\tilde{\nabla}_\alpha\tilde{\nabla}_\alpha\tilde{\nabla}_\alpha\alpha$.

ii) is recurrent if $\tilde{\Delta}_\alpha\tilde{H} = B(\alpha)\tilde{H}$,

iii) is 2-recurrent if $\tilde{\Delta}^2_\alpha\tilde{H} = B(\alpha)\tilde{H}$,

where B is an one form defined on the tangent space of σ.

Definition 5.4 *An almost contact curve σ in a (κ,μ)-manifold M is a C-parallel slant curve with respect to $\tilde{\nabla}$ if there is a differentiable function λ along σ such that*

$$\tilde{\nabla}_\alpha\tilde{H} = \lambda\xi.$$

Definition 5.5 *A curve σ in a (κ,μ)-manifold M is said to be C-proper if*

$$\tilde{\Delta}\tilde{H} = \lambda\xi,$$

for a non-zero differentiable function λ along σ, where $\tilde{\Delta}$ is the Laplacian with reference to $\tilde{\nabla}$ and $\tilde{H} = \tilde{\nabla}_\alpha\alpha$.

By (7) and $H = \nabla_\alpha\alpha$, we have

$$\tilde{\nabla}_\alpha\tilde{H} = \nabla_\alpha H - \omega\pi(\alpha,\beta). \tag{30}$$

Again taking covariant differentiation with respect to γ and by (7), we obtain

$$\tilde{\nabla}_\alpha\tilde{\nabla}_\alpha\tilde{H} = \nabla_\alpha\nabla_\alpha H - \pi(\alpha,\nabla_\alpha H) - \omega'\pi(\alpha,\beta) + \omega\nabla_\alpha\pi(\alpha,\beta) - \omega\pi(\alpha,\pi(\alpha,\beta)). \tag{31}$$

In view of (8), we have

$$\tilde{\nabla}_\alpha\tilde{\nabla}_\alpha\tilde{H} = \nabla_\alpha\nabla_\alpha H + \omega'(\sqrt{\kappa-1}+1)\xi,$$

or

$$\tilde{\nabla}_\alpha\tilde{\nabla}_\alpha\tilde{H} = \nabla_\alpha\nabla_\alpha H + \omega'(\sqrt{\kappa-1}+1)\xi. \tag{32}$$

By the above definition, it follows that, if the mean curvature is proper with reference to ∇ then it is proper with reference to $\tilde{\nabla}$ and the converse holds provided $\kappa = 2$.

By using (8) in (30), we have

$$\tilde{\nabla}_\alpha\tilde{H} = \nabla_\alpha H + \omega(\sqrt{\kappa-1}+1)\xi.$$

By considering the condition for recurrence with reference to ∇, we have

$$\tilde{\nabla}_\alpha\tilde{H} = \gamma(\alpha)H + \omega(\sqrt{\kappa-1}+1)\xi. \tag{33}$$

By (33), we have, if the mean curvature is recurrent with respect to ∇ then it is recurrent with respect to $\tilde{\nabla}$ and conversly provided $\kappa = 2$. By (30), we have

$$\tilde{\nabla}^2_\alpha \tilde{H} = \tilde{\nabla}_\alpha[\nabla_\alpha H - \omega\pi(\alpha, \beta)].$$

In view of (8), the above equation becomes

$$\tilde{\nabla}_\alpha(\nabla_\alpha H) + (\sqrt{\kappa - 1} + 1)(\omega'\xi + \omega\tilde{\nabla}_\alpha\xi).$$

By (3) and (7) we have

$$\tilde{\nabla}^2_\alpha \tilde{H} = \nabla^2_\alpha H + 2\omega'(\sqrt{\kappa - 1} + 1)\xi. \tag{34}$$

Thus we state the following:

Theorem 5.6 *If the almost contact curve in a (κ, μ)-manifold is proper, recurrent, 2-recurrent with respect to ∇ then it is proper, recurrent, 2-recurrent with reference to $\tilde{\nabla}$ and conversly provided $\kappa = 2$.*

By (31), we have

$$\tilde{\nabla}_\alpha \tilde{H} = \nabla_\alpha H - \omega\pi(\alpha, \beta). \tag{35}$$

For a slant curve, in view of (8), we obtain

$$\pi(\alpha, \beta) = (1 + \sqrt{\kappa - 1})(\beta\eta(\beta) - \xi). \tag{36}$$

Using (36) in (35), we obtain

$$\tilde{\nabla}_\alpha \tilde{H} = \nabla_\alpha H - \omega(1 + \sqrt{\kappa - 1})(\beta\eta(\beta) - \xi). \tag{37}$$

From (37) we state the following:

Theorem 5.7 *If an almost contact curve is C-parallel slant curve with respect to ∇ then it is C-parallel slant curve with respect to $\tilde{\nabla}$ and conversely provided $\kappa = 2$.*

We have,

$$-\tilde{\nabla}_\alpha\tilde{\nabla}_\alpha\tilde{H} = -\nabla_\alpha\nabla_\alpha H + \pi(\alpha, \nabla_\alpha H) + \omega'\pi(\alpha, \beta) - \omega\nabla_\alpha\pi(\alpha, \beta) + \omega\pi(\alpha, \pi(\alpha, \beta)). \tag{38}$$

If the almost contact curve is a slant curve, then from (36), we have

$$\nabla_\alpha\pi(\alpha, \beta) = (1 + \sqrt{\kappa - 1})(-\omega\eta(\beta)\alpha + \tau\eta(\beta)\gamma + \omega\eta(\alpha)\beta - \tau\eta(\gamma)\beta). \tag{39}$$

Since $H = \nabla_\alpha \alpha$, from(38) and (39), we have

$$\begin{aligned}\tilde{\Delta}\tilde{H} =& \Delta H + [-\omega^2(1 + \sqrt{\kappa - 1})\eta(\beta)]\alpha \\ &+ [2\omega'(1 + \sqrt{\kappa - 1})\eta(\beta) - \omega(1 + \sqrt{\kappa - 1})^2\eta(\beta)^2 \\ &+ \omega(1 + \sqrt{\kappa - 1})^2]\beta + [\omega\tau(1 + \sqrt{\kappa - 1})\eta(\beta)]\gamma \\ &+ [\omega(1 + \sqrt{\kappa - 1})^2\eta(\beta) - 2\omega'(1 + \sqrt{\kappa - 1})]\xi.\end{aligned} \tag{40}$$

Taking $\xi = a_1\alpha + a_2\beta + a_3\gamma$ in (40), we have

$$\begin{aligned}\tilde{\Delta}\tilde{H} =&\Delta H + (1+\sqrt{\kappa-1})[-\omega^2\eta(\beta) + a_1(\omega\eta(\beta) - 2\omega')]\alpha \\ &+ [2\omega'\eta(\beta) - \omega(1+\sqrt{\kappa-1})\eta(\beta)^2 + \omega(1+\sqrt{\kappa-1}) + a_2(\omega(1+\sqrt{\kappa-1})\eta(\beta) - 2\omega')]\beta \\ &+ [\omega\tau\eta(\beta) + a_3(\omega(1+\sqrt{\kappa-1})\eta(\beta) - 2\omega')]\gamma.\end{aligned} \tag{41}$$

If the almost contact curve is a C-proper slant curve with respect to ∇ and $\tilde{\nabla}$. Then we have

i) $(1+\sqrt{\kappa-1}) = 0$,

ii) $\omega^2\eta(\beta) + a_1(\omega\eta(\beta) - 2\omega') = 0$,

iii) $2\omega'\eta(\beta) - \omega(1+\sqrt{\kappa-1})\eta(\beta)^2 + \omega(1+\sqrt{\kappa-1}) + a_2(\omega(1+\sqrt{\kappa-1})\eta(\beta) - 2\omega') = 0$,

iv) $\omega\tau\eta(\beta) + a_3(\omega(1+\sqrt{\kappa-1})\eta(\beta) - 2\omega') = 0$.

Then by virtue of (ii), (iii) and (iv) and assuming ω to be a constant, we have vector field ξ in the form

$$\xi = \omega\alpha + \frac{\eta(\beta)^2 - 1}{\omega\eta(\beta)}\beta + \frac{\tau}{\omega(1+\sqrt{\kappa-1})}\gamma. \tag{42}$$

Thus we have

Theorem 5.8 *If an almost contact curve σ in a (κ,μ) contact metric manifold is C-proper slant curve with reference to both ∇ and $\tilde{\nabla}$ then the characteristic vector field ξ along σ is given by* (42).

6 Open Problem

In this paper we proved the conditions for Legendre curve to be of AW(ω) and invariance conditions of being different sorts of curves are proved. These sort of work could also be possible for Slant curve, which will be more interesting.

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