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Quantitative uncertainty principles for the Dunkl wavelet transform

Hatem Mejjaoli and Nadia Sraieb

Department of Mathematics, College of Sciences, Taibah University
PO BOX 30002 Al Madinah AL Munawarah, Saudi Arabia
e-mail: hmejjaoli@gmail.com
Department of Mathematics, Faculty Sciences of Gabes, Gabes University
Omar Ibn Khattab Street 6029 Gabes, Tunisia
e-mail: nadia.sraieb@fsg.rnu.tn

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Abstract

In this paper, we present the basic Dunkl wavelet theory. Next, we prove several uncertainty principles for this transform such as Heisenberg type inequalities, Shannon's uncertainty principle, Faris-Price type uncertainty principle and local uncertainty principles.

Keywords: *Dunkl wavelet transform, Heisenberg's type inequality, local uncertainty principles, Faris-Price's uncertainty principle, Shannon's uncertainty principle.*
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1 Introduction

We consider the differential-difference operators T_j, $j = 1, 2, ..., d$, associated with a root system R and a multiplicity function k, introduced by Dunkl in [15], and called the Dunkl operators in the literature.

During the last years, these operators have gained considerable interest in various fields of mathematics and also in physical applications, especially in conformal field theory. In fact these operators they are naturally connected with certain Schrödinger operators for Calogero-Sutherland-type quantum many body systems. A good bibliography is contained in [14, 27].

The Dunkl theory is based on the Dunkl kernel $K(\lambda, .)$, $\lambda \in \mathbb{C}^d$, which is the unique analytic solution of the system

$$T_j u(x) = \lambda_j u(x), \quad j = 1, 2, ..., d,$$

satisfying the normalizing condition $u(0) = 1$.

With the kernel $K(\lambda,.)$, Dunkl have defined in [16] the Dunkl transform $\mathscr{F}_D$. For a family of weighted functions, ω_k, invariant under a finite reflection group W, Dunkl transform is an extension of the Fourier transform that defines an isometry of $L^2(\mathbb{R}^d, \omega_k(x)dx)$ onto itself. The basic properties of the Dunkl transforms have been studied by several authors, see [13, 15, 16, 53] and the references therein.

Very recently, many authors have been investigating the behavior of the Dunkl transform to several problems already studied for the Fourier transform; for instance, Babenko inequality [7], uncertainty principles [8, 25], real Paley-Wiener theorems [29], heat equation [43], Dunkl Gabor transform [30, 31, 33], Dunkl wavelet transform [30, 32, 34, 54], and so on.

In quantum mechanics, the Heisenberg uncertainty principle states that the position and momentum of a particle described by a wave function in $L^2(\mathbb{R})$ cannot be simultaneously and arbitrary small. Motivated by this principle in 1946, D. Gabor, who won the 1971 Nobel Prize in physics, first recognized the great importance of localized time and frequency concentrations in signal processing [20]. In order to incorporate both time and frequency localization properties in one single transform function, Gabor introduced the windowed Fourier transform (or Gabor transform) by using a Gaussian distribution function as window function. Subsequently, various other functions have been used as window functions instead of the Gaussian function that was originally introduced by Gabor. The Gabor transformation has been found to be very useful in many physical and engineering applications, including wave propagation, signal processing and quantum optics [10, 11]. The major drawback of the Gabor transform is the fixed width of the analysing window. Indeed, in many applications, the high frequency content of a signal is more time/space-localized than the low-frequency one. Removing of the rigidity of the window function is one of the motivations for continuous wavelet transform.

In the classical setting, the notion of wavelets was first introduced by Morlet, a French petroleum engineer at ELF-Aquitaine, in connection with his study of seismic traces. The mathematical foundations were given by Grossmann and Morlet in [22]. The harmonic analyst Meyer and many other mathematicians became aware of this theory and they recognized many classical results inside it (see [9, 26, 38, 52]). Classical wavelets have wide applications, ranging from signal analysis in geophysics and acoustics to quantum theory and pure mathematics (see [12, 23] and the references therein).

Next, the theory of wavelets and continuous wavelet transform has been extended in the context of the Dunkl seeting (see [54]).

Very recently, many authors have been investigating the behavior of the Dunkl wavelet transform to several problems already studied for the classical wavelet transform; for instance, Uncertainty principles [21, 30, 32, 34], Localization theory [32], Reproducing kernel theory [50], and so on.

This paper is a continuation of the papers [30, 32, 34] in the study of the quantitative uncertainty principles for the Dunkl wavelet transform on $\mathbb{R}^d$. In the classical setting, the notion of the quantitative uncertainty principles for the wavelet transform was first introduced by Wilczok [56]. Next, this subject has been extended for the generalized wavelet transforms (see [1, 4, 5, 30, 32, 34, 42, 45, 46] and others).

We recall that the classical quantitative uncertainty principles is just another name for some special inequalities. These inequalities give us information about how a function and its Fourier transform relate. They are called uncertainty principles since they are similar to the classical Heisenberg uncertainty principle, which has had a big part to play in the development and understanding of quantum physics.

The quantitative uncertainty principles have been studied by many authors for various Fourier transforms, for examples (cf. [3, 25, 28, 55]) and others.

To date, several generalizations, modifications and variations of the harmonic based uncertainty principles have appeared in the open literature, for instance, the logarithmic uncertainty principles, Benedick's uncertainty principle, Amrein's and Berthier's uncertainty principles, local uncertainty principles and much more [2, 6, 17, 18, 19, 24, 39, 40, 41, 48, 49]. Thus, it is therefore interesting and worthwhile to investigate these kinds of uncertainty principles for the Dunkl wavelet transforms in arbitrary space dimensions.

The aim of this article is to formulate some novel uncertainty principles for the Dunkl wavelet transform. In fact, the main contributions of this article are as follows:

- To obtain the Faris-Price uncertainty for the Dunkl wavelet transforms.
- To derive the Heisenberg uncertainty principle for the Dunkl wavelet transforms by using the Dunkl entropy.
- To obtain some L^p-Heisenberg's uncertainty inequalities for the Dunkl wavelet transforms.
- To study some local-type uncertainty principles for the Dunkl wavelet transform.
- To derive some generalized Heisenberg-type inequalities for the proposed transform.

We note that recently in [34] we have established an analogue of the well-known Pitt's inequality, Beckners uncertainty principle, Benedick-Amrein-Berthier's uncertainty principle and Dunkl logarithmic Sobolev uncertainty inequality for the Dunkl wavelet transforms.

The remaining part of the paper is organized as follows. In §2 we recall the main results of the harmonic analysis associated with the Dunkl operators and we present the basic results for the Dunkl wavelet transform. §3 is devoted to give many versions of Heisenberg's inequalities for this transform. In §4 we present the L^p local uncertainty principles for the Dunkl wavelet transform.

2 Preliminaries

This section gives an introduction to the Dunkl theory. Main references are [13, 15, 16, 44, 51, 53, 54].

2.1 The Dunkl operators

We consider $\mathbb{R}^d$ with the Euclidean scalar product $\langle , \rangle$ for which the basis $\{e_i,\ i = 1, ..., d\}$ is orthogonal and $\|x\| = \sqrt{\langle x, x\rangle}$. For α in $\mathbb{R}^d\backslash\{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α, i.e.

$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{\|\alpha\|^2}\alpha. \tag{2.1}$$

A finite set $R \subset \mathbb{R}^d\backslash\{0\}$ is called a root system if $R \bigcap \mathbb{R}\alpha = \{\pm\alpha\}$ and $\sigma_\alpha(R) = R$ for all $\alpha \in R$. For a given root system R the reflections $\sigma_\alpha, \alpha \in R$, generate a finite group $W \subset O(d)$, called the reflection group associated with R.

We fix a positive root system $R_+ = \Big\{\alpha \in R\ : \langle \alpha, \beta\rangle > 0\Big\}$ for some $\beta \in \mathbb{R}^d\backslash \bigcup_{\alpha\in R} H_\alpha$. We will assume that $\langle \alpha, \alpha\rangle = 2$ for all $\alpha \in R_+$. A function $k : \mathscr{R} \longrightarrow [0, \infty)$ is called a multiplicity function if it is invariant under the action of the associated reflection group W. For abbreviation, we introduce the index

$$\gamma = \gamma(k) = \sum_{\alpha\in R_+} k(\alpha). \tag{2.2}$$

Moreover, let ω_k denotes the weight function

$$\omega_k(x) = \prod_{\alpha\in R_+} |\langle\alpha, x\rangle|^{2k(\alpha)}, \tag{2.3}$$

which is W-invariant and homogeneous of degree 2γ. We introduce the Mehta-type constant

$$c_k = \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}} \omega_k(x)\, dx. \tag{2.4}$$

In the following we denote by

$C^p(\mathbb{R}^d)$ the space of functions of class C^p on $\mathbb{R}^d$.

$\mathscr{E}(\mathbb{R}^d)$ the space of C^∞-functions on $\mathbb{R}^d$.

$\mathscr{S}(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$.

$D(\mathbb{R}^d)$ the space of C^∞-functions on $\mathbb{R}^d$ which are of compact support.

The Dunkl operators T_j, $j = 1, ..., d$, on $\mathbb{R}^d$ associated with the finite reflection group W and multiplicity function k are given by

$$T_j f(x) := \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha\in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle\alpha, x\rangle}, \quad f \in C^1(\mathbb{R}^d), \tag{2.5}$$

where $\alpha_j = \langle\alpha, e_j\rangle$.

We define the Dunkl-Laplacian operator $\triangle_k$ on $\mathbb{R}^d$ by

$$\triangle_k f(x) := \sum_{j=1}^{d} T_j^2 f(x) = \triangle f(x) + 2\sum_{\alpha\in R^+} k(\alpha)\Big(\frac{\langle\nabla f(x), \alpha\rangle}{\langle\alpha, x\rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle\alpha, x\rangle^2}\Big),$$

where $\triangle$ and ∇ are the usual Euclidean Laplacian and the gardient operators on $\mathbb{R}^d$ respectively.

For $y \in \mathbb{R}^d$, the system

$$\begin{cases} T_j u(x, y) = y_j u(x, y), & j = 1, ..., d, \\ u(0, y) = 1, \end{cases} \tag{2.6}$$

admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $K(x, y)$ and called Dunkl kernel. This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$.

The function $K(x, z)$ admits for all $x \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$ the following Laplace type integral representation

$$K(x, z) = \int_{\mathbb{R}^d} e^{\langle y,z\rangle} d\mu_x(y), \tag{2.7}$$

where μ_x is the positive probability measure on $\mathbb{R}^d$, with support in the closed ball $B_d(0, \|x\|)$ of center 0 and radius $\|x\|$. (cf. [44]).

2.2 The Dunkl transform

Notation. We denote by $L^p_k(\mathbb{R}^d)$ the space of measurable functions on $\mathbb{R}^d$ such that

$$\|f\|_{L^p_k(\mathbb{R}^d)} := \Big(\int_{\mathbb{R}^d} |f(x)|^p \, d\gamma_k(x)\Big)^{\frac{1}{p}} < \infty, \quad \text{if} \quad 1 \le p < \infty,$$
$$\|f\|_{L^\infty_k(\mathbb{R}^d)} := \operatorname{ess}\sup_{x\in\mathbb{R}^d} |f(x)| < \infty,$$

where

$$d\gamma_k(x) := \omega_k(x)dx.$$

For $p = 2$, we provide this space with the scalar product

$$\langle f, g\rangle_{L^2_k(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f(x)\overline{g(x)}d\gamma_k(x).$$

If $\mathscr{F}$ is a space of a $\mathbb{C}$-valued functions on $\mathbb{R}^d$, denote by

$$\mathscr{F}_{rad} := \left\{f \in \mathscr{F} : f \circ A = f \text{ for all } A \in O(d, \mathbb{R})\right\}$$

the subspace of those $f \in \mathscr{F}$ which are radial. For $f \in \mathscr{F}_{rad}$ there exists a unique function $F : \mathbb{R}_+ \to \mathbb{C}$ such that $f(x) = F(\|x\|)$ for all $x \in \mathbb{R}^d$.

Remark 2.1. *By using the homogeneity of ω_k it is shown in [44] that for a radial function $f \in L^1_k(\mathbb{R}^d)$ the function F defined on $[0, \infty)$ by $f(x) = F(\|x\|)$, for all $x \in \mathbb{R}^d$ is integrable with respect to the measure $r^{2\gamma+d-1}dr$. More precisely,*

$$\int_{\mathbb{R}^d} f(x)d\gamma_k(x) = d_k \int_0^{\infty} F(r)r^{2\gamma+d-1}dr, \tag{2.8}$$

where

$$d_k := \frac{c_k}{2^{\gamma+\frac{d}{2}-1}\Gamma(\gamma + \frac{d}{2})}. \tag{2.9}$$

The Dunkl transform of a function f in $L^1_k(\mathbb{R}^d)$ is given by

$$\mathscr{F}_D(f)(y) = \frac{1}{c_k}\int_{\mathbb{R}^d} f(x)K(-ix, y)d\gamma_k(x), \quad \text{for all } y \in \mathbb{R}^d. \tag{2.10}$$

In the following we give some properties of this transform (cf. [13, 16]).

i) For f in $L^1_k(\mathbb{R}^d)$ we have

$$\|\mathscr{F}_D(f)\|_{L^\infty_k(\mathbb{R}^d)} \leq \frac{1}{c_k}\|f\|_{L^1_k(\mathbb{R}^d)}. \tag{2.11}$$

ii) Inversion formula: Let f be a function in $L^1_k(\mathbb{R}^d)$, such that $\mathscr{F}_D(f) \in L^1_k(\mathbb{R}^d)$. Then

$$\mathscr{F}_D^{-1}(f)(x) = \mathscr{F}_D(f)(-x), \quad a.e.\ x \in \mathbb{R}^d. \tag{2.12}$$

Proposition 2.1. *The Dunkl transform $\mathscr{F}_D$ is a topological isomorphism from $\mathscr{S}(\mathbb{R}^d)$ onto itself. If we put for f in $\mathscr{S}(\mathbb{R}^d)$*

$$\overline{\mathscr{F}_D}(f)(y) = \mathscr{F}_D(f)(-y), \quad y \in \mathbb{R}^d, \tag{2.13}$$

we have

$$\mathscr{F}_D\overline{\mathscr{F}_D} = \overline{\mathscr{F}_D}\mathscr{F}_D = Id.$$

Proposition 2.2. i*) Plancherel's formula for $\mathscr{F}_D$.*
For all f in $\mathscr{S}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} |f(x)|^2 d\gamma_k(x) = \int_{\mathbb{R}^d} |\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi). \tag{2.14}$$

ii*) Plancherel's theorem for $\mathscr{F}_D$.*
The Dunkl transform $f \mapsto \mathscr{F}_D(f)$ can be uniquely extended to an isometric isomorphism on $L^2_k(\mathbb{R}^d)$.
iii*) Parseval's formula for $\mathscr{F}_D$.*
For all f, g in $\mathscr{S}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} f(x)\overline{g(x)}d\gamma_k(x) = \int_{\mathbb{R}^d} \mathscr{F}_D(f)(\xi)\overline{\mathscr{F}_D(g)(\xi)}d\gamma_k(\xi). \tag{2.15}$$

Definition 2.1. *([44]) Let $x \in \mathbb{R}^d$. The Dunkl translation operator $f \mapsto \tau_x f$ is defined on $L^2_k(\mathbb{R}^d)$ by*

$$\mathscr{F}_D(\tau_x f) = K(ix, .)\mathscr{F}_D(f). \tag{2.16}$$

Using the Dunkl translation operator, we define the Dunkl convolution product of functions as follows (see [51, 53]).

Definition 2.2. *For f, g in $D(\mathbb{R}^d)$, we define the Dunkl convolution product by*

$$\forall\, x \in \mathbb{R}^d, \quad f *_D g(x) = \frac{1}{c_k}\int_{\mathbb{R}^d} \tau_x f(-y)g(y)d\gamma_k(y). \tag{2.17}$$

2.3 Basic Dunkl wavelet theory

In this subsection we recall some results introduced and proved by Trimèche in [54].

Let $b > 0$. The dilation operator Δ_b of a measurable function h, is defined by

$$\forall\, y \in \mathbb{R}^d,\ \Delta_b h(y) := \frac{1}{b^{\gamma+\frac{d}{2}}} h(\frac{y}{b}). \tag{2.18}$$

This operator satisfies.

Proposition 2.3. (i*) For all a, b in $(0, \infty)$, we have*

$$\Delta_a\Delta_b = \Delta_{ab}. \tag{2.19}$$

*(*ii*) Let $b > 0$. For all h in $L^2_k(\mathbb{R}^d)$, the function $\Delta_b h$ belongs to $L^2_k(\mathbb{R}^d)$ and we have*

$$||\Delta_b h||_{L^2_k(\mathbb{R}^d)} = ||h||_{L^2_k(\mathbb{R}^d)}, \tag{2.20}$$

and

$$\mathscr{F}_D(\Delta_b h)(\xi) = b^{\gamma+\frac{d}{2}}\mathscr{F}_D(h)(b\xi), \quad \xi \in \mathbb{R}^d. \tag{2.21}$$

*(*iii*) Let $b > 0$. For all h, g in $L^2_k(\mathbb{R}^d)$, we have*

$$\langle \Delta_b h, g\rangle_{L^2_k(\mathbb{R}^d)} = \langle h, \Delta_{\frac{1}{b}} g\rangle_{L^2_k(\mathbb{R}^d)}. \tag{2.22}$$

*(*iv*) Let $b > 0$ and $y \in \mathbb{R}^d$. We have*

$$\Delta_b\tau_y = \tau_{by}\,\Delta_b. \tag{2.23}$$

Definition 2.3. *A Dunkl wavelet on $\mathbb{R}^d$ is a measurable function h on $\mathbb{R}^d$ satisfying for almost all y in $\mathbb{R}^d\backslash\{0\}$, the condition*

$$0 < C_h := \int_0^\infty |\mathscr{F}_D(h)(\lambda y)|^2 \frac{d\lambda}{\lambda} < \infty. \tag{2.24}$$

Example 2.1. *The function α_t, $t > 0$, defined on $\mathbb{R}^d$ by*

$$\alpha_t(y) = \frac{1}{(2t)^{\gamma+\frac{d}{2}}} e^{-\frac{\|y\|^2}{4t}}, \tag{2.25}$$

satisfies

$$\forall\, \xi \in \mathbb{R}^d,\ \mathscr{F}_D(\alpha_t)(\xi) = e^{-t\|\xi\|^2}. \tag{2.26}$$

The function $h(y) = -\dfrac{d}{dt}\alpha_t(y)$ is a Dunkl wavelet on $\mathbb{R}^d$ in $\mathscr{S}(\mathbb{R}^d)$, and we have $C_h = \dfrac{1}{8t^2}$.

Let $b > 0$ and h be a Dunkl wavelet in $L^2_k(\mathbb{R}^d)$. We consider the family $h_{b,y}$, $y \in \mathbb{R}^d$, of functions on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$ defined by

$$h_{b,y}(x) := \tau_y(\Delta_b h)(x),\ x \in \mathbb{R}^d, \tag{2.27}$$

where τ_y, $y \in \mathbb{R}^d$, are the Dunkl translation operators given by (2.16).

We note that we have

$$\forall\, b > 0,\ \forall\, y \in \mathbb{R}^d, \quad \|h_{b,y}\|_{L^2_k(\mathbb{R}^d)} \leq \|h\|_{L^2_k(\mathbb{R}^d)}. \tag{2.28}$$

Notation. We denote by

- $\mathbb{R}^{d+1}_+ = \Big\{(b,y) = (b, y_1, ..., y_d) \in \mathbb{R}^{d+1} :\ b > 0\Big\}$.
- $L^p_{\mu_k}(\mathbb{R}^{d+1}_+)$, $p \in [1,\infty]$, the space of measurable functions f on $\mathbb{R}^{d+1}_+$ such that

$$\begin{aligned} \|f\|_{L^p_{\mu_k}(\mathbb{R}^{d+1}_+)} &:= \left(\int_{\mathbb{R}^{d+1}_+} |f(b,y)|^p d\mu_k(b,y)\right)^{\frac{1}{p}} < \infty,\quad 1 \leq p < \infty, \\ \|f\|_{L^\infty_{\mu_k}(\mathbb{R}^{d+1}_+)} &:= \underset{(b,y)\in\mathbb{R}^{d+1}_+}{ess\sup} |f(b,y)| < \infty, \end{aligned}$$

where the measure μ_k is defined by

$$\forall\, (b,y) \in \mathbb{R}^{d+1}_+, \quad d\mu_k(b,y) = \frac{d\gamma_k(y)db}{b^{2\gamma+d+1}}.$$

Definition 2.4. *Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$. The Dunkl continuous wavelet transform Φ^D_h on $\mathbb{R}^d$ is defined for regular functions f on $\mathbb{R}^d$ by*

$$\Phi^D_h(f)(b,y) = \frac{1}{c_k}\int_{\mathbb{R}^d} f(x)\overline{h_{b,y}(x)}d\gamma_k(x) = \frac{1}{c_k}\langle f, \tau_y(\Delta_b h)\rangle_{L^2_k(\mathbb{R}^d)},\ b > 0,\ y \in \mathbb{R}^d. \tag{2.29}$$

This transform can also be written in the form

$$\Phi^D_h(f)(b,y) = \check{f} *_D \overline{\Delta_b h}(y), \tag{2.30}$$

where $\check{f}(x) := f(-x)$, and $*_D$ is the Dunkl convolution product given by (2.17).

Remark 2.2. *Let h be a Dunkl wavelet in $L^2_k(\mathbb{R}^d)$. Then from Cauchy-Schwarz's inequality, the relations (2.29) and* (2.28)*, for all f in $L^2_k(\mathbb{R}^d)$ we have*

$$\|\Phi_h^D(f)\|_{L^\infty_{\mu_k}(\mathbb{R}^{d+1}_+)} \leq \frac{1}{c_k}\|f\|_{L^2_k(\mathbb{R}^d)}\|h\|_{L^2_k(\mathbb{R}^d)}. \tag{2.31}$$

Theorem 2.1. *(Plancherel's formula for Φ_h^D). Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$. For all f in $L^2_k(\mathbb{R}^d)$ we have*

$$\int_{\mathbb{R}^d} |f(x)|^2 d\gamma_k(x) = \frac{1}{C_h}\int_{\mathbb{R}^{d+1}_+} |\Phi_h^D(f)(b,y)|^2 d\mu_k(b,y). \tag{2.32}$$

Corollary 2.1. *(Parseval's formula for Φ_h^D). Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$ and f_1, f_2 in $L^2_k(\mathbb{R}^d)$. Then, we have*

$$\int_{\mathbb{R}^d} f_1(x)\overline{f_2(x)}\omega_k(x)dx = \frac{1}{C_h}\int_{\mathbb{R}^{d+1}_+} \Phi_h^D(f_1)(b,y)\overline{\Phi_h^D(f_2)(b,y)}d\mu_k(b,y). \tag{2.33}$$

By Riesz-Thorin's interpolation theorem we obtain the following.

Proposition 2.4. *Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$, $f \in L^2_k(\mathbb{R}^d)$ and p belongs in* $[2,\infty]$*. We have*

$$\|\Phi_h^D(f)\|_{L^p_{\mu_k}(\mathbb{R}^{d+1}_+)} \leq (C_h)^{\frac{1}{p}}\left(\frac{\|h\|_{L^2_k(\mathbb{R}^d)}}{c_k}\right)^{\frac{p-2}{p}}\|f\|_{L^2_k(\mathbb{R}^d)}. \tag{2.34}$$

Theorem 2.2. *(Inversion formula for Φ_h^D). Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$. For all f in $L^1_k(\mathbb{R}^d)$ (resp. $L^2_k(\mathbb{R}^d)$) such that $\mathscr{F}_D(f)$ belongs to $L^1_k(\mathbb{R}^d)$ (resp. $L^1_k(\mathbb{R}^d)\bigcap L^\infty_k(\mathbb{R}^d)$) we have*

$$f(x) = \frac{1}{c_k\,C_h}\int_0^\infty\int_{\mathbb{R}^d} \Phi_h^D(f)(b,y)h_{b,y}(x)d\mu_k(b,y), \ a.e., \tag{2.35}$$

where for each $x \in \mathbb{R}^d$, both the inner integral and the outer integral are absolutely convergent, but possible not the double integral.

Proposition 2.5. *Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$. For any f in $L^2_k(\mathbb{R}^d)$, and for any* $t > 0$*, we have*

$$\forall\,(b,y) \in \mathbb{R}^{d+1}_+, \quad \Phi_h^D(\Delta_t f)(b,y) = \Phi_h^D(f)(\frac{b}{t},\frac{y}{t}). \tag{2.36}$$

Henceforth, the function h will denote an arbitrary Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$.

3 Heisenberg types uncertainty inequalities for the Dunkl wavelet transform

In this section, we establish many versions of Heisenberg-type inequality for the Dunkl wavelet transform.

3.1 Heisenberg types uncertainty inequalities via the Dunkl Entropy

A probability density function ρ on $\mathbb{R}_+^{d+1}$ is a nonnegative measurable function on $\mathbb{R}_+^{d+1}$ satisfying

$$\int_{\mathbb{R}_+^{d+1}} \rho(b,y)d\mu_k(b,y) = 1.$$

Following Shannon [47], the Dunkl entropy of a probability density function ρ on $\mathbb{R}_+^{d+1}$ is defined by

$$E_k(\rho) := -\int_{\mathbb{R}_+^{d+1}} \ln(\rho(b,y))\rho(b,y)d\mu_k(b,y).$$

Henceforth, we extend the definition of the Dunkl entropy of a nonnegative measurable function ρ on $\mathbb{R}_+^{d+1}$ whenever the previous integral on the right hand side is well defined.

The aim of this part is to study the localization of the Dunkl entropy of the Dunkl wavelet transform over the space $\mathbb{R}_+^{d+1}$, indeed we have the following result.

Proposition 3.1. *Let $f \in L_k^2(\mathbb{R}^d)$ be nonzero function, then*

$$E_k(|\Phi_h^D(f)|^2) \geqslant -2C_h\|f\|^2_{L_k^2(\mathbb{R}^d)} \ln\left(\frac{\|f\|_{L_k^2(\mathbb{R}^d)}\|h\|_{L_k^2(\mathbb{R}^d)}}{c_k}\right). \tag{3.1}$$

Proof. Assume that $\|f\|_{L_k^2(\mathbb{R}^d)}\|h\|_{L_k^2(\mathbb{R}^d)} = c_k$, then by Relation (2.31) we deduce that

$$\forall (b,y) \in \mathbb{R}_+^{d+1},\ |\Phi_h^D(f)(b,y)| \leqslant \frac{1}{c_k}\|f\|_{L_k^2(\mathbb{R}^d)}\|h\|_{L_k^2(\mathbb{R}^d)} = 1. \tag{3.2}$$

In particular $E_k(|\Phi_h^D(f)|^2) \geqslant 0$ and therefore if the entropy $E_k(|\Phi_h^D(f)|^2)$ is infinite, the inequality (3.1) holds obviously. Suppose now that the entropy $E_k(|\Phi_h^D(f)|^2)$ is finite. We return now to the general case, so let $f \in L_k^2(\mathbb{R}^d)$ and $h \in L_k^2(\mathbb{R}^d)$ be nonzero functions and let

$$\varphi = c_k\frac{f}{\|f\|_{L_k^2(\mathbb{R}^d)}} \quad \text{and} \quad \psi = \frac{h}{\|h\|_{L_k^2(\mathbb{R}^d)}}.$$

Then, $\varphi \in L_k^2(\mathbb{R}^d)$, $\psi \in L_k^2(\mathbb{R}^d)$ and $\|\varphi\|_{L_k^2(\mathbb{R}^d)}\|\psi\|_{L_k^2(\mathbb{R}^d)} = c_k$, hence

$$E_k(|\Phi_\psi^D(\varphi)|^2) \geqslant 0.$$

However, $\Phi_\psi^D(\varphi) = \frac{c_k}{\|f\|_{L_k^2(\mathbb{R}^d)}\|h\|_{L_k^2(\mathbb{R}^d)}}\Phi_h^D(f)$ and

$$E_k(|\Phi_\psi^D(\varphi)|^2) = \frac{c_k^2}{\|h\|^2_{L_k^2(\mathbb{R}^d)}\|f\|^2_{L_k^2(\mathbb{R}^d)}}E_k(|\Phi_h^D(f)|^2) + \ln\left(\frac{\|f\|_{L_k^2(\mathbb{R}^d)}\|h\|_{L_k^2(\mathbb{R}^d)}}{c_k}\right)\frac{2C_hc_k^2}{\|h\|^2_{L_k^2(\mathbb{R}^d)}}.$$

So,

$$E_k(|\Phi_h^D(f)|^2) \geqslant -2C_h\|f\|^2_{L_k^2(\mathbb{R}^d)} \ln\left(\frac{\|f\|_{L_k^2(\mathbb{R}^d)}\|h\|_{L_k^2(\mathbb{R}^d)}}{c_k}\right).$$

□

Using the Dunkl entropy of the Dunkl wavelet transform, we can obtain the following Heisenberg uncertainty principle for Φ_h^D.

Theorem 3.1. *Let p and q be two positive real numbers. Then, there exists a positive constant $M_{p,q}(k)$ such that for every function $f \in L^2_k(\mathbb{R}^d)$ we have*

$$\left(\int_{\mathbb{R}^{d+1}_+} \|y\|^p |\Phi^D_h(f)(b,y)|^2 d\mu_k(b,y)\right)^{\frac{q}{p+q}} \left(\int_{\mathbb{R}^{d+1}_+} b^{-q} |\Phi^D_h(f)(b,y)|^2 d\mu_k(b,y)\right)^{\frac{p}{p+q}} \geqslant M_{p,q}(k) C_h \|f\|^2_{L^2_k(\mathbb{R}^d)}, \tag{3.3}$$

where

$$M_{p,q}(k) = \frac{d+2\gamma}{p^{\frac{q}{p+q}} q^{\frac{p}{p+q}}} e^{A_{p,q}(k)},$$

here

$$A_{p,q}(k) := pq \frac{\ln\left(\frac{pq}{d_k \Gamma(\frac{d+2\gamma}{p}) \Gamma(\frac{d+2\gamma}{q})}\right)}{(d+2\gamma)(p+q)} - 1.$$

Proof. Assume that $\|f\|_{L^2_k(\mathbb{R}^d)} \|h\|_{L^2_k(\mathbb{R}^d)} = c_k$. For every positive real numbers t, p, q, let $\eta_{t,p,q}$ be the function defined on $\mathbb{R}^{d+1}_+$ by

$$\eta_{t,p,q}(b,y) = \frac{pq}{d_k \Gamma(\frac{d+2\gamma}{p}) \Gamma(\frac{d+2\gamma}{q})} \frac{e^{-\frac{\|y\|^p + b^{-q}}{t}}}{t^{\frac{(d+2\gamma)(p+q)}{pq}}}.$$

So by simple calculus we see that

$$\int_{\mathbb{R}^{d+1}_+} \eta_{t,p,q}(b,y) d\mu_k(b,y) = 1,$$

in particular the measure $d\sigma_{t,p,q}(b,y) = \eta_{t,p,q}(b,y) d\mu_k(b,y)$ is a probability measure on $\mathbb{R}^{d+1}_+$. Since the function $\varphi(t) = t\ln(t)$ is convex over $(0,\infty)$, hence by using Jensen's inequality for convex functions, we get

$$\int_{\mathbb{R}^{d+1}_+} \frac{|\Phi^D_h(f)(b,y)|^2}{\eta_{t,p,q}(b,y)} \ln\left(\frac{|\Phi^D_h(f)(b,y)|^2}{\eta_{t,p,q}(b,y)}\right) d\sigma_{t,p,q}(b,y) \geqslant 0,$$

which implies in terms of Dunkl entropy that for every positive real numbers t, p, q, we have

$$E_k(|\Phi^D_h(f)|^2) + \ln\left(\frac{pq}{d_k \Gamma(\frac{d+2\gamma}{p}) \Gamma(\frac{d+2\gamma}{q})}\right) C_h \|f\|^2_{L^2_k(\mathbb{R}^d)} \leqslant \ln(t^{\frac{(d+2\gamma)(p+q)}{pq}}) C_h \|f\|^2_{L^2_k(\mathbb{R}^d)} + \frac{1}{t} \int_{\mathbb{R}^{d+1}_+} (\|y\|^p + b^{-q}) |\Phi^D_h(f)(b,y)|^2 d\mu_k(b,y).$$

Therefore, by Proposition 3.1 we get

$$\int_{\mathbb{R}^{d+1}_+} (\|y\|^p + b^{-q}) |\Phi^D_h(f)(b,y)|^2 d\mu_k(b,y) \geqslant t\left(\ln\left(\frac{pq}{d_k \Gamma(\frac{d+2\gamma}{p}) \Gamma(\frac{d+2\gamma}{q})}\right) - \ln(t^{\frac{(d+2\gamma)(p+q)}{pq}})\right) \|\Phi^D_h(f)\|^2_{L^2_{\mu_k}(\mathbb{R}^{d+1}_+)}.$$

However, the expression

$$t\left(\ln\left(\frac{pq}{d_k \Gamma(\frac{d+2\gamma}{p}) \Gamma(\frac{d+2\gamma}{q})}\right) - \ln(t^{\frac{(d+2\gamma)(p+q)}{pq}})\right) \|\Phi^D_h(f)\|^2_{L^2_{\mu_k}(\mathbb{R}^{d+1}_+)}$$

attains its upper bound at $t_0 = e^{A_{p,q}(k)}$, and consequently

$$\int_{\mathbb{R}^{d+1}_+} (\|y\|^p + b^{-q})|\Phi_h^D(f)(b,y)|^2 d\mu_k(b,y) \geqslant C_{p,q}(k) C_h \|f\|^2_{L^2_k(\mathbb{R}^d)},$$

where

$$C_{p,q}(k) = \frac{(d+2\gamma)(p+q)}{pq} e^{A_{p,q}(k)}.$$

Now, the general formula follows from above by substituting f by $c_k f/\{\|f\|_{L^2_k(\mathbb{R}^d)}\}$ and h by $h/\|h\|_{L^2_k(\mathbb{R}^d)}$. Therefore, for every $f \in L^2_k(\mathbb{R}^d)$ and $h \in L^2_k(\mathbb{R}^d)$, we get

$$\int_{\mathbb{R}^{d+1}_+} \|y\|^p |\Phi_h^D(f)(b,y)|^2 d\mu_k(b,y) + \int_{\mathbb{R}^{d+1}_+} b^{-q} |\Phi_h^D(f)(b,y)|^2 d\mu_k(b,y) \geqslant C_{p,q}(k) C_h \|f\|^2_{L^2_k(\mathbb{R}^d)}. \tag{3.4}$$

Now, for every $\lambda > 0$ the dilates $\Delta_{\frac{1}{\lambda}} f \in L^2_k(\mathbb{R}^d)$. Then by Relation (3.4), we have

$$\int_{\mathbb{R}^{d+1}_+} \|y\|^p |\Phi_h^D(\Delta_{\frac{1}{\lambda}} f)(b,y)|^2 d\mu_k(b,y) + \int_{\mathbb{R}^{d+1}_+} b^{-q} |\Phi_h^D(\Delta_{\frac{1}{\lambda}} f)(b,y)|^2 d\mu_k(b,y) \geqslant C_{p,q}(k) C_h \|\Delta_{\frac{1}{\lambda}} f\|^2_{L^2(\mathbb{R}^d)}$$

Thus using the fact that $\|\Delta_{\frac{1}{\lambda}} f\|^2_{L^2_k(\mathbb{R}^d)} = \|f\|^2_{L^2_k(\mathbb{R}^d)}$ and (2.36), we get for every positive real number λ

$$\lambda^{-p} \int_{\mathbb{R}^{d+1}_+} \|y\|^p |\Phi_h^D(f)(b,y)|^2 d\mu_k(b,y) + \lambda^q \int_{\mathbb{R}^{d+1}_+} b^{-q} |\Phi_h^D(f)(b,y)|^2 d\mu_k(b,y) \geqslant C_{p,q}(k) C_h \|f\|^2_{L^2_k(\mathbb{R}^d)}.$$

In particular, the inequality holds at the critical point

$$\lambda = \left(\frac{p \int_{\mathbb{R}^{d+1}_+} \|y\|^p |\Phi_h^D(f)(b,y)|^2 d\mu_k(b,y)}{q \int_{\mathbb{R}^{d+1}_+} b^{-q} |\Phi_h^D(f)(b,y)|^2 d\mu_k(b,y)} \right)^{\frac{1}{p+q}},$$

which implies that

$$\left(\int_{\mathbb{R}^{d+1}_+} \|y\|^p |\Phi_h^D(f)(b,y)|^2 d\mu_k(b,y) \right)^{\frac{q}{p+q}} \left(\int_{\mathbb{R}^{d+1}_+} b^{-q} |\Phi_h^D(f)(b,y)|^2 d\mu_k(b,y) \right)^{\frac{p}{p+q}} \geqslant M_{p,q}(k) C_h \|f\|^2_{L^2_k(\mathbb{R}^d)},$$

where

$$M_{p,q}(k) = C_{p,q}(k) \frac{p^{\frac{p}{p+q}} q^{\frac{q}{p+q}}}{p+q} = \frac{d+2\gamma}{p^{\frac{q}{p+q}} q^{\frac{p}{p+q}}} e^{A_{p,q}(k)}.$$

□

Remark 3.1. *When $p = q = 2$, we get*

$$\big\| \|y\| \Phi_h^D(f) \big\|_{L^2_{\mu_k}(\mathbb{R}^{d+1}_+)} \big\| b^{-1} \Phi_h^D(f) \big\|_{L^2_{\mu_k}(\mathbb{R}^{d+1}_+)} \geqslant \Big(\frac{4}{d_k (\Gamma(\frac{d+2\gamma}{2}))^2} \Big)^{\frac{1}{d+2\gamma}} \frac{d+2\gamma}{2e} C_h \|f\|^2_{L^2_k(\mathbb{R}^d)}.$$

3.2 L^p-Heisenberg's uncertainty principles for the Dunkl wavelet transform

Let $t > 0$. We put

$$E_t(b, y) := e^{-t\|(\frac{1}{b}, y)\|^2}, \quad \text{for all } (b, y) \in \mathbb{R}_+^{d+1}.$$

By simple calculations it is easy to prove the following.

Lemma 3.1. *Let $1 \le q < \infty$ and $t > 0$. There exists a positive constant C, such that we have*

$$\|E_t\|_{L^q_{\mu_k}(\mathbb{R}_+^{d+1})} = Ct^{-\frac{d+2\gamma}{q}}.$$

Lemma 3.2. *Let $1 < p \le 2$ and $0 < s < \frac{d+2\gamma}{2p'}$. Then, there exists a positive constant C such that, for all $f \in L^2_k(\mathbb{R}^d)$ and $t > 0$,*

$$\|e^{-t\|(\frac{1}{b}, y)\|^2}\Phi_h^D(f)\|_{L^{p'}_{\mu_k}(\mathbb{R}_+^{d+1})} \le C(C_h)^{\frac{1}{p'}}\left(\frac{\|h\|_{L^2_k(\mathbb{R}^d)}}{c_k}\right)^{\frac{p'-2}{p'}} t^{-2s}\Big[\| \|y\|^s f\|_{L^2_k(\mathbb{R}^d)} + \| \|y\|^s f\|_{L^{2p}_k(\mathbb{R}^d)}\Big]. \tag{3.5}$$

Proof. Inequality (3.5) holds if $\| \|y\|^s f\|_{L^2_k(\mathbb{R}^d)} + \| \|y\|^s f\|_{L^{2p}_k(\mathbb{R}^d)} = \infty$. Assume that

$$\| \|y\|^s f\|_{L^2_k(\mathbb{R}^d)} + \| \|y\|^s f\|_{L^{2p}_k(\mathbb{R}^d)} < \infty.$$

For $r > 0$, let $f_r = f\mathbb{1}_{B_d(0,r)}$ and $f^r = f - f_r$.

Using (2.34) and the fact that

$$|f^r(y)| \le r^{-s}| \|y\|^s f(y)|$$

we get

$$\begin{aligned}
\| e^{-t\|(\frac{1}{b}, y)\|^2}\Phi_h^D(f\mathbb{1}_{B^c_d(0,r)})\|_{L^{p'}_{\mu_k}(\mathbb{R}_+^{d+1})} &\le \| e^{-t\|(\frac{1}{b}, y)\|^2}\|_{L^{\infty}_{\mu_k}(\mathbb{R}_+^{d+1})} \|\Phi_h^D(f\mathbb{1}_{B^c_d(0,r)})\|_{L^{p'}_{\mu_k}(\mathbb{R}_+^{d+1})}\\
&\le (C_h)^{\frac{1}{p'}}\left(\frac{\|h\|_{L^2_k(\mathbb{R}^d)}}{c_k}\right)^{\frac{p'-2}{p'}}\|f\mathbb{1}_{B^c_d(0,r)}\|_{L^2_k(\mathbb{R}^d)}\\
&\le (C_h)^{\frac{1}{p'}}\left(\frac{\|h\|_{L^2_k(\mathbb{R}^d)}}{c_k}\right)^{\frac{p'-2}{p'}} r^{-s}\| \|y\|^s f\|_{L^2_k(\mathbb{R}^d)}.
\end{aligned}$$

On the other hand, by (2.31) and Hölder's inequality

$$\begin{aligned}
\| e^{-t\|(\frac{1}{b}, y)\|^2}\Phi_h^D(f\mathbb{1}_{B_d(0,r)})\|_{L^{p'}_{\mu_k}(\mathbb{R}_+^{d+1})} &\le \|e^{-t\|(\frac{1}{b}, y)\|^2}\|_{L^{p'}_{\mu_k}(\mathbb{R}_+^{d+1})} \|\Phi_h^D(f\mathbb{1}_{B_d(0,r)})\|_{L^{\infty}_k(\mathbb{R}^d)}\\
&\le \frac{\|h\|_{L^2_k(\mathbb{R}^d)}}{c_k}\|e^{-t\|(\frac{1}{b}, y)\|^2}\|_{L^{p'}_{\mu_k}(\mathbb{R}_+^{d+1})} \|f\mathbb{1}_{B_d(0,r)}\|_{L^2(\mathbb{R}^d)}\\
&\le \frac{\|h\|_{L^2_k(\mathbb{R}^d)}}{c_k}\|e^{-t\|(\frac{1}{b}, y)\|^2}\|_{L^{p'}_{\mu_k}(\mathbb{R}_+^{d+1})}\| \|y\|^{-s}\mathbb{1}_{B_d(0,r)}\|_{L^{2p'}_{k,a}(\mathbb{R}^d)}\| \|y\|^s f\|_{L^{2p}_k(\mathbb{R}^d)}.
\end{aligned}$$

A simple calculation give that there exists a positive constant C, such that

$$\| \|y\|^{-s}\mathbb{1}_{B_d(0,r)}\|_{L^{2p'}_{k,a}(\mathbb{R}^d)} = Cr^{-s+\frac{d+2\gamma}{2p'}}.$$

So

$$\begin{aligned}
\|e^{-t\|(\frac{1}{b}, y)\|^2}\Phi_h^D(f)\|_{L^{p'}_{\mu_k}(\mathbb{R}_+^{d+1})} &\le \|e^{-t\|(\frac{1}{b}, y)\|^2}\Phi_h^D(f_r)\|_{L^{p'}_{\mu_k}(\mathbb{R}_+^{d+1})} + \|e^{-t\|(\frac{1}{b}, y)\|^2}\Phi_h^D(f^r)\|_{L^{p'}_{\mu_k}(\mathbb{R}_+^{d+1})}\\
&\le Cr^{-s}\frac{\|h\|_{L^2_k(\mathbb{R}^d)}}{c_k}\Big[\Big(\frac{C_h c_k^2}{\|h\|^2_{L^2_k(\mathbb{R}^d)}}\Big)^{\frac{1}{p'}}\| \|y\|^s f\|_{L^2_k(\mathbb{R}^d)} + r^{\frac{d+2\gamma}{2p'}}\|E_t\|_{L^{p'}_{\mu_k}(\mathbb{R}_+^{d+1})}\| \|y\|^s f\|_{L^{2p}_k(\mathbb{R}^d)}\Big].
\end{aligned}$$

Choosing $r = \Big(\frac{C_h c_k^2}{\|h\|^2_{L^2_k(\mathbb{R}^d)}}\Big)^{\frac{2}{d+2\gamma}} t^2$, we obtain (3.5). □

Theorem 3.2. *Let $1 < p \leq 2$ and $0 < s < \frac{d+2\gamma}{2p'}$ and $r > 0$. Then there exists a positive constant C such that, for all $f \in L^2_k(\mathbb{R}^d)$*

$$\|\Phi^D_h(f)\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)} \leq C[(C_h)^{\frac{1}{p'}}(\frac{\|h\|_{L^2_k(\mathbb{R}^d)}}{c_k})^{\frac{p'-2}{p'}}]^{\frac{r}{s+r}}\Big[\| \, \|y\|^s f\|_{L^2_k(\mathbb{R}^d)}+\| \, \|y\|^s f\|_{L^{2p}_k(\mathbb{R}^d)}\Big]^{\frac{r}{s+r}}\| \, \|(\frac{1}{b},y)\|^{4r}\Phi^D_h(f)\|^{\frac{s}{s+r}}_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}. \tag{3.6}$$

Proof. Let $1 < p \leq 2$ and $0 < s < \frac{d+2\gamma}{2p'}$. Assume that $r \leq \frac{1}{2}$. From the previous lemma, for all $t > 0$

$$\begin{aligned}\|\Phi^D_h(f)\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)} &\leq \|e^{-t\|(\frac{1}{b},y)\|^2}\Phi^D_h(f)\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)} + \|(1-e^{-t\|(\frac{1}{b},y)\|^2})\Phi^D_h(f)\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}\\ &\leq C(C_h)^{\frac{1}{p'}}(\frac{\|h\|_{L^2_k(\mathbb{R}^d)}}{c_k})^{\frac{p'-2}{p'}}t^{-2s}\Big[\| \, \|y\|^s f\|_{L^2_k(\mathbb{R}^d)} + \| \, \|y\|^s f\|_{L^{2p}_k(\mathbb{R}^d)}\Big]\\ &+ \|(1-e^{-t\|(\frac{1}{b},y)\|^2})\Phi^D_h(f)\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}.\end{aligned}$$

On the other hand,

$$\|(1-e^{-t\|(\frac{1}{b},y)\|^2})\Phi^D_h(f)\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)} = t^{2r}\|(t\|(\frac{1}{b},y)\|^2)^{-2r}(1-e^{-t\|(\frac{1}{b},y)\|^2})\|(\frac{1}{b},y)\|^{4r}\Phi^D_h(f)\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}.$$

Since $(1-e^{-u})u^{-2r}$ is bounded for $u \geq 0$ if $r \leq \frac{1}{2}$. Then, we obtain

$$\|\Phi^D_h(f)\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)} \leq C(C_h)^{\frac{1}{p'}}(\frac{\|h\|_{L^2_k(\mathbb{R}^d)}}{c_k})^{\frac{p'-2}{p'}}t^{-2s}\Big[\| \, \|y\|^s f\|_{L^2_k(\mathbb{R}^d)}+\| \, \|y\|^s f\|_{L^{2p}_k(\mathbb{R}^d)}\Big]+Ct^{2r}\| \, \|(\frac{1}{b},y)\|^{4r}\Phi^D_h(f)\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)},$$

from which, optimizing in t, we obtain (3.6) for $0 < s < \frac{d+2\gamma}{2p'}$ and $r \leq \frac{1}{2}$.

If $r > \frac{1}{2}$, let $r' \leq \frac{1}{2}$. For $u \geq 0$ we have $u^{4r'} \leq 1 + u^{4r}$, which for $u = \frac{\|(\frac{1}{b},y)\|}{\varepsilon}$ gives the inequality

$$(\frac{\|(\frac{1}{b},y)\|}{\varepsilon})^{4r'} < 1 + (\frac{\|(\frac{1}{b},y)\|}{\varepsilon})^{4r}, \quad \text{for all } \varepsilon > 0.$$

It follows that

$$\| \, \|(\frac{1}{b},y)\|^{4r'}\Phi^D_h(f)\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)} \leq \varepsilon^{4r'}\|\Phi^D_h(f)\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)} + \varepsilon^{4(r'-r)}\| \, \|(\frac{1}{b},y)\|^{4r}\Phi^D_h(f)\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}.$$

Optimizing in ε, we get

$$\| \, \|(\frac{1}{b},y)\|^{4r'}\Phi^D_h(f)\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)} \leq \|\Phi^D_h(f)\|^{\frac{r-r'}{r}}_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}\| \, \|(\frac{1}{b},y)\|^{4r}\Phi^D_h(f)\|^{\frac{r'}{r}}_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}.$$

Together with (3.6) for r', we get the result for $r > \frac{1}{2}$. □

Corollary 3.1. *Let $0 < s < \frac{d+2\gamma}{4}$ and $r > 0$. Then there exists a positive constant C such that, for all $f \in L^2_k(\mathbb{R}^d)$, we have*

$$\|f\|_{L^2_k(\mathbb{R}^d)} \leq C(C_h)^{\frac{-s}{2(s+r)}}\Big[\| \, \|y\|^s f\|_{L^2_k(\mathbb{R}^d)} + \| \, \|y\|^s f\|_{L^4_k(\mathbb{R}^d)}\Big]^{\frac{r}{s+r}}\| \, \|(\frac{1}{b},y)\|^{4r}\Phi^D_h(f)\|^{\frac{s}{s+r}}_{L^2_{\mu_k}(\mathbb{R}^{d+1}_+)}. \tag{3.7}$$

Proof. Using the previous theorem for $p = 2$, and applying Plancherel's formula (2.31) we obtain the result. □

4 L^p-Local uncertainty principle of Φ_h^D

Theorem 4.1. *(Faris-Price's uncertainty principle for Φ_h^D)*

Let η, p be two real numbers such that $0 < \eta < 2\gamma + d$ and $p \geqslant 1$. Then, there is a nonnegative constant $C_k(\eta, p)$ such that for every function f in $L_k^2(\mathbb{R}^d)$ and for every measurable subset $T \subset \mathbb{R}_+^{d+1}$ such that $0 < \mu_k(T) := \int_T d\mu_k(b,y) < \infty$, we have

$$\left(\int_T |\Phi_h^D(f)(b,y)|^p d\mu_k(b,y)\right)^{\frac{1}{p}} \leqslant C_k(\eta,p)\big(\mu_k(T)\big)^{\frac{1}{p(p+1)}} \Big\| \|(\frac{1}{b},y)\|^\eta \Phi_h^D(f)\Big\|_{L^2_{\mu_k}(\mathbb{R}_+^{d+1})}^{\frac{4\gamma+2d}{(2\gamma+d+\eta)(p+1)}} \Big(\|f\|_{L_k^2(\mathbb{R}^d)}\|h\|_{L_k^2(\mathbb{R}^d)}\Big)^{\frac{(2\gamma+d+\eta)(p+1)-(4\gamma+2d)}{(2\gamma+d+\eta)(p+1)}}.$$

Proof. One can assume that $\|f\|_{L_k^2(\mathbb{R}^d)}\|h\|_{L_k^2(\mathbb{R}^d)} = c_k$, then for every positive real number $s > 1$, we have

$$\|\Phi_h^D(f)\|_{L^p_{\mu_k}(T)} \leqslant \|\Phi_h^D(f)\mathbb{1}_{V_s}\|_{L^p_{\mu_k}(T)} + \|\Phi_h^D(f)\mathbb{1}_{V_s^c}\|_{L^p_{\mu_k}(T)},$$

where V_s denotes the subset of $\mathbb{R}_+^{d+1}$ given by

$$V_s := \Big\{(b,y) \in \mathbb{R}_+^{d+1} : \quad \|(\frac{1}{b},y)\| \leq s\Big\}.$$

However, by Hölder's inequality and Relation (2.31) we get for every $\eta \in (0, 2\gamma + d)$

$$\begin{aligned}
\|\Phi_h^D(f)\mathbb{1}_{V_s}\|_{L^p_{\mu_k}(T)} &= \left(\int_{\mathbb{R}_+^{d+1}} |\Phi_h^D(f)(b,y)|^p \mathbb{1}_{V_s}(b,y)\mathbb{1}_T(b,y)d\mu_k(b,y)\right)^{\frac{1}{p}}\\
&\leqslant \|\Phi_h^D(f)\|_{L^\infty_{\mu_k}(\mathbb{R}_+^{d+1})}^{\frac{p}{p+1}} \left(\int_{\mathbb{R}_+^{d+1}} |\Phi_h^D(f)(b,y)|^{\frac{p}{p+1}} \mathbb{1}_{V_s}(b,y)\mathbb{1}_T(b,y)d\mu_k(b,y)\right)^{\frac{1}{p}}\\
&\leqslant \big(\mu_k(T)\big)^{\frac{1}{p(p+1)}} \|\Phi_h^D(f)\mathbb{1}_{V_s}\|_{L^1_{\mu_k}(\mathbb{R}_+^{d+1})}^{\frac{1}{p+1}}\\
&\leqslant \big(\mu_k(T)\big)^{\frac{1}{p(p+1)}} \Big\| \|(\frac{1}{b},y)\|^\eta \Phi_h^D(f)\Big\|_{L^2_{\mu_k}(\mathbb{R}_+^{d+1})}^{\frac{1}{p+1}} \Big\| \|(\frac{1}{b},y)\|^{-\eta} \mathbb{1}_{V_s}\Big\|_{L^2_{\mu_k}(\mathbb{R}_+^{d+1})}^{\frac{1}{p+1}}.
\end{aligned}$$

On the other hand by simple calculations we see that

$$\Big\| \|(\frac{1}{b},y)\|^{-\eta}\mathbb{1}_{V_s}\Big\|_{L^2_{\mu_k}(\mathbb{R}_+^{d+1})} \leq \frac{\sqrt{d_k}\Gamma(\gamma+\frac{d}{2})}{2\sqrt{(2\gamma+d-\eta)\Gamma(2\gamma+d)}} s^{2\gamma+d-\eta}.$$

Thus we get

$$\begin{aligned}
\|\Phi_h^D(f)\mathbb{1}_{V_s}\|_{L^p_{\mu_k}(T)} \quad \leqslant \quad & \big(\mu_k(T)\big)^{\frac{1}{p(p+1)}} \left(\frac{\sqrt{d_k}\Gamma(\gamma+\frac{d}{2})}{2\sqrt{(2\gamma+d-\eta)\Gamma(2\gamma+d)}}\right)^{\frac{1}{p+1}}\\
& s^{\frac{2\gamma+d-\eta}{p+1}} \Big\| \|(\tfrac{1}{b},y)\|^\eta \Phi_h^D(f)\Big\|_{L^2_{\mu_k}(\mathbb{R}_+^{d+1})}^{\frac{1}{p+1}}.
\end{aligned}$$

On the other hand, and again by Hölder's inequality and Relation (2.31), we deduce that

$$\begin{aligned}
\|\Phi_h^D(f)\mathbb{1}_{V_s^c}\|_{L^p_{\mu_k}(T)} &\leqslant \|\Phi_h^D(f)\|_{L^\infty_{\mu_k}(\mathbb{R}_+^{d+1})}^{\frac{p-1}{p+1}} \left(\int_{\mathbb{R}_+^{d+1}} |\Phi_h^D(f)(b,y)|^{\frac{2p}{p+1}} \mathbb{1}_{V_s^c}(b,y)\mathbb{1}_T(b,y)d\mu_k(b,y)\right)^{\frac{1}{p}}\\
&\leqslant \big(\mu_k(T)\big)^{\frac{1}{p(p+1)}} \left(\int_{\mathbb{R}_+^{d+1}} |\Phi_h^D(f)(b,y)|^2 \mathbb{1}_{V_s^c}(b,y)d\mu_k(b,y)\right)^{\frac{1}{p+1}}\\
&\leqslant \big(\mu_k(T)\big)^{\frac{1}{p(p+1)}} \Big\| \|(\frac{1}{b},y)\|^\eta \Phi_h^D(f)\Big\|_{L^2_{\mu_k}(\mathbb{R}_+^{d+1})}^{\frac{2}{p+1}} s^{-\frac{2\eta}{p+1}}.
\end{aligned}$$

Hence, for every $\eta \in (0, 2\gamma + d)$,

$$\left(\int_T |\Phi_h^D(f)(b,y)|^p d\mu_k(b,y)\right)^{\frac{1}{p}} \leqslant (\mu_k(T))^{\frac{1}{p(p+1)}} \left\| \|(\frac{1}{b},y)\|^\eta \Phi_h^D(f)\right\|_{L^2_{\mu_k}(\mathbb{R}^{d+1}_+)}^{\frac{1}{p+1}} \left(\left(\frac{\sqrt{d_k}\Gamma(\gamma+\frac{d}{2})}{2\sqrt{(2\gamma+d-\eta)\Gamma(2\gamma+d)}}\right)^{\frac{1}{p+1}} s^{\frac{2\gamma+d-\eta}{p+1}} + \left\| \|(\frac{1}{b},y)\|^\eta \Phi_h^D(f)\right\|_{L^2_{\mu_k}(\mathbb{R}^{d+1}_+)}^{\frac{1}{p+1}} s^{-\frac{2\eta}{p+1}}\right).$$

In particular the inequality holds for

$$s_0 = \frac{\left(\frac{2\eta}{2\gamma+d-\eta}\right)^{\frac{p+1}{2\gamma+d+\eta}}}{\left(\frac{\sqrt{d_k}\Gamma(\gamma+\frac{d}{2})}{2\sqrt{(2\gamma+d-\eta)\Gamma(2\gamma+d)}}\right)^{\frac{1}{2\gamma+d+\eta}}} \left\| \|(\frac{1}{b},y)\|^\eta \Phi_h^D(f)\right\|_{L^2_{\mu_k}(\mathbb{R}^{d+1}_+)}^{\frac{1}{2\gamma+d+\eta}}$$

and therefore

$$\left(\int_T |\Phi_h^D(f)(b,y)|^p d\mu_k(b,y)\right)^{\frac{1}{p}} \leqslant (\mu_k(T))^{\frac{1}{p(p+1)}} \left(\frac{\sqrt{d_k}\Gamma(\gamma+\frac{d}{2})}{2\sqrt{(2\gamma+d-\eta)\Gamma(2\gamma+d)}}\right)^{\frac{2\eta}{(2\gamma+d+\eta)(p+1)}}$$
$$\left\| \|(\frac{1}{b},y)\|^\eta \Phi_h^D(f)\right\|^2_{L^2_{\mu_k}(\mathbb{R}^{d+1}_+)})^{\frac{4\gamma+2d}{(2\gamma+d+\eta)(p+1)}} \left(\frac{2\eta}{2\gamma+d-\eta}\right)^{\frac{-2\eta}{2\gamma+d+\eta}} \left(\frac{2\gamma+d+\eta}{2\gamma+d+-\eta}\right).$$

Now, the general formula follows from above by substituting f by $c_k f/\{\|f\|_{L^2_k(\mathbb{R}^d)}\}$ and h by $h/\|h\|_{L^2_k(\mathbb{R}^d)}$. □

Theorem 4.2. *Let $1 < p \leqslant 2$, $t > 0$ and T be a measurable subset in $\mathbb{R}^{d+1}_+$ satisfying $0 < \mu_k(T) < \infty$. Then, for all $f \in L^2_k(\mathbb{R}^d)$, we have*

$$\left\|\mathbb{1}_T \Phi_h^D(f)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)} \leqslant \begin{cases} C_1(t,k,h)(\mu_k(T))^{\frac{2t}{2\gamma+d}}\Big[\left\| \|y\|^t f\right\|_{L^2_k(\mathbb{R}^d)} + \left\| \|y\|^t f\right\|_{L^{2p}_k(\mathbb{R}^d)}\Big], & \text{for } t < \frac{d+2\gamma}{2p'} \\ C_2(t,k,h)(\mu_k(T))^{\frac{1}{p'}} \|f\|_{L^{2p}_k(\mathbb{R}^d)}^{1-\frac{d+2\gamma}{2tp'}} \| \|y\|^t f\|_{L^{2p}_k(\mathbb{R}^d)}^{\frac{d+2\gamma}{2tp'}}, & \text{for } t > \frac{d+2\gamma}{2p'} \\ C_3(t,k,h)(\mu_k(T))^{\frac{1}{2p'}} \\ \quad \Big[\|f\|^{\frac12}_{L^2_k(\mathbb{R}^d)} \| \|y\|^t f\|^{\frac12}_{L^2_k(\mathbb{R}^d)} + \|f\|^{\frac12}_{L^{2p}_k(\mathbb{R}^d)} \| \|y\|^t f\|^{\frac12}_{L^{2p}_k(\mathbb{R}^d)}\Big], & \text{for } t = \frac{d+2\gamma}{2p'} \end{cases}$$

where

$$\begin{aligned} C_1(t,k,h) &= \left(\frac{d_k}{2\gamma+d-2tp'}\right)^{\frac{t}{2\gamma+d}} (C_h)^{\frac{2\gamma+d-2tp'}{p'(2\gamma+d)}} \left(\frac{c_k}{\|h\|_{L^2_k(\mathbb{R}^d)}}\right)^{\frac{(2\gamma+d)(2-p')-4tp'}{p'(2\gamma+d)}}, \\ C_2(t,k,h) &= \frac{1}{c_k}\left(\frac{2tp'}{2tp'-2\gamma-d}\right)^{\frac{1}{2p}} \left(\frac{2tp'}{2\gamma+d}-1\right)^{\frac{d+2\gamma}{4tpp'}} \left(\frac{d_k}{2pt}\frac{\Gamma(\frac{d+2\gamma}{pt})\Gamma(\frac{2p't-2\gamma-d}{2pt})}{\Gamma(\frac{p'}{p})}\right)^{\frac{1}{2p'}} \|h\|_{L^2_k(\mathbb{R}^d)}, \\ C_3(t,k,h) &= 2^{1+\frac{1}{4p'}} \left(\frac{(C_h)^2 d_k}{2\gamma+d}\right)^{\frac{1}{4p'}} \left(\frac{c_k}{\|h\|_{L^2_k(\mathbb{R}^d)}}\right)^{\frac{-1}{p}}. \end{aligned}$$

Proof. We proceed as the proof of Lemma 3.2 and we assume that

$$\|f\|_{L^{2p}_k(\mathbb{R}^d)} + \| \|y\|^t f\|_{L^{2p}_k(\mathbb{R}^d)} < \infty.$$

(i) For $s > 0$, let $f_s = \mathbb{1}_{B_d(0,s)} f$ and $f^s = f - f_s$.

Using (2.34) and the fact that $|f^s(y)| \leqslant s^{-t}|\,\|y\|^t f(y)|$, we get

$$\begin{aligned}\left\|\mathbb{1}_T\Phi_h^D(\mathbb{1}_{B^c_d(0,s)}f)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)} &\leqslant \left\|\Phi_h^D(\mathbb{1}_{B^c_d(0,s)}f)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}\\ &\leqslant (C_h)^{\frac{1}{p'}}(\frac{\|h\|_{L^2_k(\mathbb{R}^d)}}{c_k})^{\frac{p'-2}{p'}}\left\|\mathbb{1}_{B^c_d(0,s)}f\right\|_{L^2_k(\mathbb{R}^d)}\\ &\leqslant (C_h)^{\frac{1}{p'}}(\frac{\|h\|_{L^2_k(\mathbb{R}^d)}}{c_k})^{\frac{p'-2}{p'}}s^{-t}\left\|\,\|y\|^t f\right\|_{L^2_k(\mathbb{R}^d)}.\end{aligned}$$

On the other hand, by (2.31) and Hölder's inequality, we have

$$\begin{aligned}\left\|\mathbb{1}_T\Phi_h^D(\mathbb{1}_{B_d(0,s)}f)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)} &\leqslant \|\mathbb{1}_T\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}\left\|\Phi_h^D(\mathbb{1}_{B_d(0,s)}f)\right\|_{L^{\infty}_{\mu_k}(\mathbb{R}^{d+1}_+)}\\ &\leqslant \frac{1}{c_k}\|h\|_{L^2_k(\mathbb{R}^d)}(\mu_k(T))^{\frac{1}{p'}}\left\|\mathbb{1}_{B_d(0,s)}f\right\|_{L^2_k(\mathbb{R}^d)}\\ &\leqslant \frac{1}{c_k}\|h\|_{L^2_k(\mathbb{R}^d)}(\mu_k(T))^{\frac{1}{p'}}\left\|\,\|y\|^{-t}\mathbb{1}_{B_d(0,s)}\right\|_{L^{2p'}_k(\mathbb{R}^d)}\left\|\,\|y\|^t f\right\|_{L^{2p}_k(\mathbb{R}^d)}.\end{aligned}$$

A simple calculation gives

$$\left\|\,\|y\|^{-t}\mathbb{1}_{B_d(0,s)}\right\|_{L^{2p'}_k(\mathbb{R}^d)} = (\frac{d_k}{2\gamma+d-2tp'})^{\frac{1}{2p'}}s^{-t+\frac{d+2\gamma}{2p'}}.$$

Therefore,

$$\begin{aligned}\left\|\mathbb{1}_T\Phi_h^D(f)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)} &\leqslant \left\|\mathbb{1}_T\Phi_h^D(f_s)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)} + \left\|\mathbb{1}_T\Phi_h^D(f^s)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}\\ &\leqslant s^{-t}\frac{\|h\|_{L^2_k(\mathbb{R}^d)}}{c_k}\Big[(C_h)^{\frac{1}{p'}}(\frac{\|h\|_{L^2_k(\mathbb{R}^d)}}{c_k})^{\frac{-2}{p'}}\left\|\,\|y\|^t f\right\|_{L^2_k(\mathbb{R}^d)}\\ &+ (\frac{d_k}{2\gamma+d-2tp'})^{\frac{1}{2p'}}(\mu_k(T))^{\frac{1}{p'}}s^{\frac{d+2\gamma}{2p'}}\left\|\,\|y\|^t f\right\|_{L^{2p}_k(\mathbb{R}^d)}\Big].\end{aligned}$$

By choosing

$$s = \Big(\frac{C_h(c_k)^2}{\|h\|^2_{L^2_k(\mathbb{R}^d)}}\Big)^{\frac{2}{2\gamma+d}}(\frac{d_k}{2\gamma+d-2tp'})^{\frac{-1}{2\gamma+d}}(\mu_k(T))^{\frac{-2}{2\gamma+d}}$$

we obtain the first inequality.

(ii) As above, using (2.31) and Hölder's inequality, we get

$$\begin{aligned}\left\|\mathbb{1}_T\Phi_h^D(f)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)} &\leqslant \|\mathbb{1}_T\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}\left\|\Phi_h^D(f)\right\|_{L^{\infty}_{\mu_k}(\mathbb{R}^{d+1}_+)}\\ &\leqslant \frac{1}{c_k}\|h\|_{L^2_k(\mathbb{R}^d)}(\mu_k(T))^{\frac{1}{p'}}\|f\|_{L^2_k(\mathbb{R}^d)}.\end{aligned} \tag{4.1}$$

On the other hand, using Hölder's inequality and the fact that $t > \frac{d+2\gamma}{2p'}$, a simple calculation gives

$$\begin{aligned}\|f\|^{2p}_{L^2_k(\mathbb{R}^d)} &= \Big(\int_{\mathbb{R}^d}(1+\|y\|^{2pt})^{\frac{1}{p}}|f(y)|^2\frac{d\gamma_k(y)}{(1+\|y\|^{2pt})^{\frac{1}{p}}}\Big)^p\\ &\leqslant \Big(\int_{\mathbb{R}^d}\frac{d\gamma_k(y)}{(1+\|y\|^{2pt})^{\frac{p'}{p}}}\Big)^{\frac{p}{p'}}\Big(\|f\|^{2p}_{L^{2p}_k(\mathbb{R}^d)} + \|\,\|y\|^t f\|^{2p}_{L^{2p}_k(\mathbb{R}^d)}\Big)\\ &\leqslant \Big(\frac{d_k}{2pt}\frac{\Gamma(\frac{d+2\gamma}{2pt})\Gamma(\frac{2p't-2\gamma-d}{2pt})}{\Gamma(\frac{p'}{p})}\Big)^{\frac{p}{p'}}\Big(\|f\|^{2p}_{L^{2p}_k(\mathbb{R}^d)} + \|\,\|y\|^t f\|^{2p}_{L^{2p}_k(\mathbb{R}^d)}\Big) < \infty.\end{aligned}$$

Thus, f belongs to $L^2_k(\mathbb{R}^d)$. Replacing $f(y)$ by $\delta_\lambda f(y) := f(\lambda y)$, with $\lambda > 0$, in the above last inequality we get

$$\|f\|^{2p}_{L^2_k(\mathbb{R}^d)} \leqslant \Big(\frac{d_k}{2pt}\frac{\Gamma(\frac{d+2\gamma}{2pt})\Gamma(\frac{2p't-2\gamma-d}{2pt})}{\Gamma(\frac{p'}{p})}\Big)^{\frac{p}{p'}}\Big(\lambda^{(2\gamma+d)(p-1)}\|f\|^{2p}_{L^{2p}_k(\mathbb{R}^d)} + \lambda^{(2\gamma+d)(p-1)-2pt}\|\,\|y\|^t f\|^{2p}_{L^{2p}_k(\mathbb{R}^d)}\Big).$$

In particular, the inequality holds at

$$\lambda = \Big(\frac{2p't}{2\gamma + d} - 1\Big)^{\frac{1}{2pt}}\Big(\frac{\| \|y\|^t f\|_{L^{2p}_k(\mathbb{R}^d)}}{\|f\|_{L^{2p}_k(\mathbb{R}^d)}}\Big)^{\frac{1}{t}},$$

which implies

$$\|f\|_{L^2_k(\mathbb{R}^d)} \leqslant C(d,p,t,k)\|f\|^{1-\frac{d+2\gamma}{2tp'}}_{L^{2p}_k(\mathbb{R}^d)} \| \|y\|^t f\|^{\frac{d+2\gamma}{2tp'}}_{L^{2p}_k(\mathbb{R}^d)}, \tag{4.2}$$

with

$$C(d,p,t,k) = \Big(\frac{2p't}{2tp' - (2\gamma + d)}\Big)^{\frac{1}{2p}}\Big(\frac{2tp'}{2\gamma + d} - 1\Big)^{\frac{d+2\gamma}{4tpp'}}\Big(\frac{d_k}{2pt}\frac{\Gamma(\frac{d+2\gamma}{2pt})\Gamma(\frac{2p't-(2\gamma+d)}{2pt})}{\Gamma(\frac{p'}{p})}\Big)^{\frac{1}{2p'}}. \tag{4.3}$$

The desired result follows immediately from (4.1), (4.2) and (4.3).

(iii) Let $r > 0$. Using the inequality

$$\Big(\frac{\|y\|}{r}\Big)^{\frac{d+2\gamma}{4p'}} \leqslant 1 + \Big(\frac{\|y\|}{r}\Big)^{\frac{d+2\gamma}{2p'}},$$

we get

$$\| \|y\|^{\frac{d+2\gamma}{4p'}} f\|_{L^{2p}_k(\mathbb{R}^d)} \leqslant r^{\frac{d+2\gamma}{4p'}}\|f\|_{L^{2p}_k(\mathbb{R}^d)} + r^{-\frac{d+2\gamma}{4p'}}\| \|y\|^{\frac{d+2\gamma}{2p'}} f\|_{L^{2p}_k(\mathbb{R}^d)}.$$

Optimizing in r, we obtain

$$\| \|y\|^{\frac{d+2\gamma}{4p'}} f\|_{L^{2p}_k(\mathbb{R}^d)} \leqslant 2\|f\|^{\frac12}_{L^{2p}_k(\mathbb{R}^d)}\| \|y\|^{\frac{d+2\gamma}{2p'}} f\|^{\frac12}_{L^{2p}_k(\mathbb{R}^d)}.$$

Similarly, we prove that

$$\| \|y\|^{\frac{d+2\gamma}{4p'}} f\|_{L^{2}_k(\mathbb{R}^d)} \leqslant 2\|f\|^{\frac12}_{L^{2}_k(\mathbb{R}^d)}\| \|y\|^{\frac{d+2\gamma}{2p'}} f\|^{\frac12}_{L^{2}_k(\mathbb{R}^d)}.$$

Thus, we deduce that

$$\begin{aligned}\|\mathbb{1}_T\Phi^D_h(f)\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)} &\leqslant C_1\big(\tfrac{d+2\gamma}{4p'},k,h\big)(\mu_k(T))^{\frac{1}{2p'}}\Big[\| \|y\|^{\frac{d+2\gamma}{4p'}} f\|_{L^{2}_k(\mathbb{R}^d)} + \| \|y\|^{\frac{d+2\gamma}{4p'}} f\|_{L^{2p}_k(\mathbb{R}^d)}\Big]\\ &\leqslant 2C_1\big(\tfrac{d+2\gamma}{4p'},k,h\big)(\mu_k(T))^{\frac{1}{2p'}}\\ &\quad\Big[\|f\|^{\frac12}_{L^{2}_k(\mathbb{R}^d)}\| \|y\|^{\frac{d+2\gamma}{2p'}} f\|^{\frac12}_{L^{2}_k(\mathbb{R}^d)} + \|f\|^{\frac12}_{L^{2p}_k(\mathbb{R}^d)}\| \|y\|^{\frac{d+2\gamma}{2p'}} f\|^{\frac12}_{L^{2p}_k(\mathbb{R}^d)}\Big].\end{aligned}$$

□

Remark 4.1. *We note that when $t = \frac{d+2\gamma}{2p'}$, we can obtain a family of inequalities and improve the inequality given in Theorem 4.2. Indeed, if we apply the first inequality with*

$$s = (1-\varepsilon)\frac{d+2\gamma}{2p'},\ \varepsilon \in (0,1),$$

and then apply the classical inequality

$$\| \|y\|^{t-t\varepsilon} f\|_{L^{r}_k(\mathbb{R}^d)} \leqslant C\|f\|^{\varepsilon}_{L^{r}_k(\mathbb{R}^d)}\| \|y\|^t f\|^{1-\varepsilon}_{L^{r}_k(\mathbb{R}^d)}, \tag{4.4}$$

we obtain for all $\varepsilon \in (0,1)$,

$$\|\mathbb{1}_T\Phi^D_h(f)\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)} \leqslant C_4(\varepsilon,t,k,h)(\mu_k(T))^{\frac{1-\varepsilon}{p'}}\Big[\|f\|^{\varepsilon}_{L^{2}_k(\mathbb{R}^d)}\| \|y\|^t f\|^{1-\varepsilon}_{L^{2}_k(\mathbb{R}^d)} + \|f\|^{\varepsilon}_{L^{2p}_k(\mathbb{R}^d)}\| \|y\|^t f\|^{1-\varepsilon}_{L^{2p}_k(\mathbb{R}^d)}\Big]. \tag{4.5}$$

Definition 4.1. *Let* $0 \le \varepsilon < 1$, $S \subset \mathbb{R}^d$ *and* $T \subset \mathbb{R}_+^{d+1}$.

(1) *A nonzero* $f \in L_k^2(\mathbb{R}^d)$*, is called an* ε*-concentrated function on* S *in* $L_k^2(\mathbb{R}^d)$*-norm, if*

$$\|\mathbb{1}_{S^c} f\|_{L_k^2(\mathbb{R}^d)} \le \varepsilon \|f\|_{L_k^2(\mathbb{R}^d)}.$$

(2) *A nonzero* $f \in L_k^2(\mathbb{R}^d)$*, is called an* ε*-bandlimited function on* T *in* $L^2_{\mu_k}(\mathbb{R}_+^{d+1})$*-norm, if*

$$\|\mathbb{1}_{T^c} \Phi_h^D(f)\|_{L^2_{\mu_k}(\mathbb{R}_+^{d+1})} \le \varepsilon \|f\|_{L_k^2(\mathbb{R}^d)}.$$

Here A^c *is the complement of* A.

Corollary 4.1. *Let* h *be a Dunkl wavelet on* $\mathbb{R}^d$ *in* $L_k^2(\mathbb{R}^d)$ *such that* $C_h = 1$.

(1) *If* $0 < t < \frac{d+2\gamma}{4}$*, then there exists a positive constant* $\mathfrak{C}_1(t,k,h)$ *such that for every function* f *which is* ε*-bandlimited on* T,

$$(\mu_k(T))^{\frac{4t}{2\gamma+2d}} \Big[\| \, \|y\|^t f\|_{L_k^2(\mathbb{R}^d)} + \| \, \|y\|^t f\|_{L_k^4(\mathbb{R}^d)}\Big]^2 \ge \mathfrak{C}_1(t,k,h) \left(1 - \varepsilon^2\right) \|f\|^2_{L_k^2(\mathbb{R}^d)}. \tag{4.6}$$

(2) *If* $t > \frac{d+2\gamma}{4}$*, then there exists a positive constant* $\mathfrak{C}_2(t,k,h)$ *such that for every function* f *which is* ε*-bandlimited on* T,

$$\mu_k(T) \|f\|^{2-\frac{d+2\gamma}{2t}}_{L_k^4(\mathbb{R}^d)} \| \, \|y\|^t f\|^{\frac{d+2\gamma}{2t}}_{L_k^4(\mathbb{R}^d)} \ge \mathfrak{C}_2(t,k,h) \left(1 - \varepsilon^2\right) \|f\|^2_{L_k^2(\mathbb{R}^d)}. \tag{4.7}$$

(3) *For all* $s \in (0,1)$*, there exists a positive constant* $\mathfrak{C}_3(t,s,k,h)$ *such that for every function* f *which is* ε*-bandlimited on* T,

$$(\mu_k(T))^{1-s} \Big[\|f\|^s_{L_k^2(\mathbb{R}^d)} \| \, \|y\|^t f\|^{1-s}_{L_k^2(\mathbb{R}^d)} + \|f\|^s_{L_k^4(\mathbb{R}^d)} \| \, \|y\|^t f\|^{1-s}_{L_k^4(\mathbb{R}^d)}\Big]^2 \ge \mathfrak{C}_3(t,s,k,h)(1-\varepsilon^2)\|f\|^2_{L_k^2(\mathbb{R}^d)}. \tag{4.8}$$

Proof. Since $f \in L_k^2(\mathbb{R}^d)$ is ε-bandlimited on T, it follows

$$\|\mathbb{1}_T \Phi_h^D(f)\|^2_{L^2_{\mu_k}(\mathbb{R}_+^{d+1})} = C_h \|f\|^2_{L_k^2(\mathbb{R}^d)} - \|\mathbb{1}_{T^c} \Phi_h^D(f)\|^2_{L^2_{\mu_k}(\mathbb{R}_+^{d+1})} \ge (1-\varepsilon^2)\|f\|^2_{L_k^2(\mathbb{R}^d)}. \tag{4.9}$$

In view of (4.9), the inequalities (4.6) and (4.7) follow from the first and the second local inequalities in Theorem 4.2, respectively, while (4.8) follows from (4.5). □

Corollary 4.2. *Let* $1 < p \leqslant 2$ *and* $t > 0$*. Then for all* $f \in L_k^2(\mathbb{R}^d)$*, we have*

$$\|\Phi_h^D(f)\|_{L_{\mu_k}^{\frac{(2\gamma+d)p'}{2\gamma+d-2p't}, p'}(\mathbb{R}_+^{d+1})} \leqslant C_1(t,k,h) \Big[\| \, \|y\|^t f\|_{L_k^2(\mathbb{R}^d)} + \| \, \|y\|^t f\|_{L_k^{2p}(\mathbb{R}^d)}\Big], \quad \text{for } t < \frac{d+2\gamma}{2p'}$$

$$\|\Phi_h^D(f)\|_{L_{\mu_k}^{\infty, p'}(\mathbb{R}_+^{d+1})} \leqslant C_2(t,k,h) \|f\|^{1-\frac{d+2\gamma}{2tp'}}_{L_k^{2p}(\mathbb{R}^d)} \| \, \|y\|^t f\|^{\frac{d+2\gamma}{2tp'}}_{L_k^{2p}(\mathbb{R}^d)}, \quad \text{for } t > \frac{d+2\gamma}{p'}$$

$$\|\Phi_h^D(f)\|_{L_{\mu_k}^{2p', p'}(\mathbb{R}_+^{d+1})} \leqslant C_3(t,k,h) \Big[\|f\|^{\frac12}_{L_k^2(\mathbb{R}^d)} \| \, \|y\|^t f\|^{\frac12}_{L_k^2(\mathbb{R}^d)} + \|f\|^{\frac12}_{L_k^{2p}(\mathbb{R}^d)} \| \, \|y\|^t f\|^{\frac12}_{L_k^{2p}(\mathbb{R}^d)}\Big], \quad \text{for } t = \frac{d+2\gamma}{2p'}$$

where $L_{\mu_k}^{p,q}(\mathbb{R}_+^{d+1})$ *denotes the Lorentz space corresponding to the norm*

$$\|g\|_{L_{\mu_k}^{p,q}(\mathbb{R}_+^{d+1})} := \sup_{T \subset \mathbb{R}_+^{d+1},\, 0 < \mu_k(T) < \infty} (\mu_k(T))^{\frac1p - \frac1q} \|\mathbb{1}_T g\|_{L^q_{\mu_k}(\mathbb{R}_+^{d+1})}, \tag{4.10}$$

and $C_j(t,k,h)$*, for* $j = 1,2,3$*, are the constants given in Theorem 4.2.*

Theorem 4.3. *Let $s, t > 0$ and $1 < p \leqslant 2$. Then for all $f \in L^2_k(\mathbb{R}^d)$, we have*

$$\|\Phi_h^D(f)\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)} \leqslant \begin{cases} C_1(s,t,k,h)\Big[\| \|y\|^s f\|_{L^2_k(\mathbb{R}^d)} + \| \|y\|^s f\|_{L^{2p}_k(\mathbb{R}^d)}\Big]^{\frac{t}{4s+t}} \\ \qquad \Big\| \|(\frac{1}{b}, y)\|^t \Phi_h^D(f)\Big\|^{\frac{4s}{4s+t}}_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}, & \text{for } s < \frac{d+2\gamma}{2p'}, \\ C_2(s,t,k,h)\Big[\|f\|^{1-\frac{d+2\gamma}{2sp'}}_{L^{2p}_k(\mathbb{R}^d)} \Big\| \|y\|^s f\Big\|^{\frac{d+2\gamma}{2sp'}}_{L^{2p}_k(\mathbb{R}^d)}\Big]^{\frac{tp'}{4\gamma+2d+tp'}} \\ \qquad \Big\| \|(\frac{1}{b}, y)\|^t \Phi_h^D(f)\Big\|^{\frac{4\gamma+2d}{4\gamma+2d+tp'}}_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}, & \text{for } s > \frac{d+2\gamma}{2p'}, \\ C_3(s,t,k,h)\Big[\|f\|^{\frac{1}{2}}_{L^2_k(\mathbb{R}^d)} \Big\|\|y\|^t f\Big\|^{\frac{1}{2}}_{L^2_k(\mathbb{R}^d)} + \\ \qquad \|f\|^{\frac{1}{2}}_{L^{2p}_k(\mathbb{R}^d)} \Big\| \|y\|^s f\Big\|^{\frac{1}{2}}_{L^{2p}_k(\mathbb{R}^d)}\Big]^{\frac{tp'}{2\gamma+d+tp'}} \Big\| \|(\frac{1}{b}, y)\|^t \Phi_h^D(f)\Big\|^{\frac{2\gamma+d}{2\gamma+d+tp'}}_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}, & \text{for } s = \frac{d+2\gamma}{2p'}, \end{cases}$$

where

$$C_1(s,t,k,h) = \Big[(\tfrac{t}{4s})^{\frac{4s}{4s+t}} + (\tfrac{4s}{t})^{\frac{t}{4s+t}}\Big]^{\frac{1}{p'}} (C_1(s,k,h))^{\frac{t}{4s+t}} \Big(\frac{d_k(\Gamma(\frac{2\gamma+d}{2}))^2}{4(2\gamma+d)\Gamma(2\gamma+d)}\Big)^{\frac{2st}{(4s+t)(2\gamma+d)}},$$

$$C_2(s,t,k,h) = \Big[(\tfrac{tp'}{4\gamma+2d})^{\frac{4\gamma+2d}{4\gamma+2d+tp'}} + (\tfrac{4\gamma+2d}{tp'})^{\frac{tp'}{4\gamma+2d+tp'}}\Big]^{\frac{1}{p'}} \Big(\frac{d_k\big(\Gamma(\frac{2\gamma+d}{2})\big)^2 \big(C_2(s,k,h)\big)^{p'}}{4(2\gamma+d)\Gamma(2\gamma+d)}\Big)^{\frac{t}{4\gamma+2d+tp'}},$$

$$C_3(s,t,k,h) = \Big[(\tfrac{tp'}{2\gamma+d})^{\frac{2\gamma+d}{2\gamma+d+tp'}} + (\tfrac{2\gamma+d}{tp'})^{\frac{tp'}{2\gamma+d+tp'}}\Big]^{\frac{1}{p'}} \Big(\frac{d_k\big(\Gamma(\frac{2\gamma+d}{2})\big)^2 \big(C_3(s,k,h)\big)^{p'}}{4(2\gamma+d)\Gamma(2\gamma+d)}\Big)^{\frac{t}{2(2\gamma+d+tp')}},$$

and $C_j(s,k,h)$, for $j = 1, 2, 3$, are the constants given in Theorem 4.2.

Proof. (i) Let $0 < s < \frac{d+2\gamma}{2p'}$, $t > 0$ and $r > 0$. Then

$$\|\Phi_h^D(f)\|^{p'}_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)} \leqslant \|\mathbb{1}_{V_r}\Phi_h^D(f)\|^{p'}_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)} + \|\mathbb{1}_{V_r^c}\Phi_h^D(f)\|^{p'}_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}. \tag{4.11}$$

Here V_r denotes the subset of $\mathbb{R}^{d+1}_+$ given by

$$V_r := \Big\{(b,x) \in \mathbb{R}^{d+1}_+ : \|(\frac{1}{b}, x)\| < r\Big\}.$$

From Theorem 4.2, we have

$$\|\mathbb{1}_{V_r}\Phi_h^D(f)\|^{p'}_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)} \leqslant (C_1(s,k,h))^{p'} \left(\frac{d_k\big(\Gamma(\frac{2\gamma+d}{2})\big)^2}{4(2\gamma+d)\Gamma(2\gamma+d)}\right)^{\frac{2sp'}{2\gamma+d}} r^{4sp'}\Big[\| \|y\|^s f\|_{L^2_k(\mathbb{R}^d)} + \| \|y\|^s f\|_{L^{2p}_k(\mathbb{R}^d)}\Big]^{p'}. \tag{4.12}$$

Moreover, it is easy to see that

$$\Big\|\mathbb{1}_{V_r^c}\Phi_h^D(f)\Big\|^{p'}_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)} \leqslant r^{-tp'} \Big\| \|(\frac{1}{b}, y)\|^t \Phi_h^D(f)\Big\|^{p'}_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}. \tag{4.13}$$

Combining the relations (4.11), (4.12) and (4.13), we get

$$\|\Phi_h^D(f)\|^{p'}_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)} \leqslant (C_1(s,k,h))^{p'} \left(\frac{d_k(\Gamma(\frac{2\gamma+d}{2}))^2}{4(2\gamma+d)\Gamma(2\gamma+d)}\right)^{\frac{2sp'}{2\gamma+d}} r^{4sp'}\Big[\Big\| \|y\|^s f\Big\|_{L^2_k(\mathbb{R}^d)} + \Big\| \|y\|^s f\Big\|_{L^{2p}_k(\mathbb{R}^d)}\Big]^{p'} + r^{-tp'} \Big\| \|(\frac{1}{b}, y)\|^t \Phi_h^D(f)\Big\|^{p'}_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}.$$

We choose

$$r=\Big(\frac{t}{4s(C_1(s,k,h))^{p'}\left(\frac{d_k\left(\Gamma(\frac{2\gamma+d}{2})\right)^2}{4(2\gamma+d)\Gamma(2\gamma+d)}\right)^{\frac{2sp'}{2\gamma+d}}}\Big)^{\frac{1}{(4s+t)p'}}\Big(\frac{\left\|\,\|(\frac{1}{b},y)\|^t\Phi_h^D(f)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}}{\left\|\,\|y\|^s f\right\|_{L^2_k(\mathbb{R}^d)}+\left\|\,\|y\|^s f\right\|_{L^{2p}_k(\mathbb{R}^d)}}\Big)^{\frac{1}{4s+t}}$$

to obtain the first inequality.

(ii) Let $s>\frac{d+2\gamma}{2p'}$, $t>0$ and $r>0$. From Theorem 4.2 we have

$$\left\|\mathbb{1}_{V_r}\Phi_h^D(f)\right\|^{p'}_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}\leqslant\frac{d_k(\Gamma(\frac{2\gamma+d}{2}))^2(C_2(s,k,h))^{p'}}{4(2\gamma+d)\Gamma(2\gamma+d)}r^{4\gamma+2d}\|f\|^{p'-\frac{d+2\gamma}{2s}}_{L^{2p}_k(\mathbb{R}^d)}\,\|\,\|y\|^s f\|^{\frac{d+2\gamma}{2s}}_{L^{2p}_k(\mathbb{R}^d)}. \tag{4.14}$$

Combining the relations (4.11), (4.13) and (4.14), we get

$$\begin{aligned}\|\Phi_h^D(f)\|^{p'}_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}&\leqslant\frac{d_k\left(\Gamma(\frac{2\gamma+d}{2})\right)^2\left(C_2(s,k,h)\right)^{p'}}{4(2\gamma+d)\Gamma(2\gamma+d)}r^{4\gamma+2d}\|f\|^{p'-\frac{d+2\gamma}{2s}}_{L^{2p}_k(\mathbb{R}^d)}\,\|\,\|y\|^s f\|^{\frac{d+2\gamma}{2s}}_{L^{2p}_k(\mathbb{R}^d)}\\&+\;r^{-tp'}\left\|\,\left\|(\tfrac{1}{b},y)\right\|^t\Phi_h^D(f)\right\|^{p'}_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}.\end{aligned}$$

We choose

$$r=\Big(\frac{2tp'\Gamma(2\gamma+d)}{d_k(\Gamma(\frac{2\gamma+d}{2}))^2(C_2(s,k,h))^{p'}}\Big)^{\frac{1}{4\gamma+2d+tp'}}\Big(\frac{\left\|\,\|(\frac{1}{b},y)\|^t\Phi_h^D(f)\right\|^{p'}_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}}{\|f\|^{p'-\frac{d+2\gamma}{2s}}_{L^{2p}_k(\mathbb{R}^d)}\left\|\,\|y\|^s f\right\|^{\frac{d+2\gamma}{2s}}_{L^{2p}_k(\mathbb{R}^d)}}\Big)^{\frac{1}{4\gamma+2d+tp'}}$$

to obtain the second inequality.

(iii) Let $s=\frac{d+2\gamma}{2p'}$, $s>0$ and $r>0$. From Theorem 4.2 we have

$$\begin{aligned}\|\mathbb{1}_{V_r}\Phi_h^D(f)\|^{p'}_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}&\leqslant\Big(\frac{d_k(\Gamma(\frac{2\gamma+d}{2}))^2(C_3(s,k,h))^{p'}}{4(2\gamma+d)\Gamma(2\gamma+d)}\Big)^{\frac{1}{2}}r^{2\gamma+d}\\&\Big[\|f\|^{\frac{1}{2}}_{L^2_k(\mathbb{R}^d)}\,\|\,\|y\|^s f\|^{\frac{1}{2}}_{L^2_k(\mathbb{R}^d)}+\|f\|^{\frac{1}{2}}_{L^{2p}_k(\mathbb{R}^d)}\,\|\,\|y\|^s f\|^{\frac{1}{2}}_{L^{2p}_k(\mathbb{R}^d)}\Big]^{p'}.\end{aligned} \tag{4.15}$$

Combining the relations (4.11), (4.13) and (4.15), we get

$$\begin{aligned}\|\Phi_h^D(f)\|^{p'}_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}&\leqslant\Big(\frac{d_k(\Gamma(\frac{2\gamma+d}{2}))^2(C_3(s,k,h))^{p'}}{4(2\gamma+d)\Gamma(2\gamma+d)}\Big)^{\frac{1}{2}}r^{2\gamma+d}\\&\Big[\|f\|^{\frac{1}{2}}_{L^2_k(\mathbb{R}^d)}\,\|\,\|y\|^s f\|^{\frac{1}{2}}_{L^2_k(\mathbb{R}^d)}+\|f\|^{\frac{1}{2}}_{L^{2p}_k(\mathbb{R}^d)}\,\|\,\|y\|^s f\|^{\frac{1}{2}}_{L^{2p}_k(\mathbb{R}^d)}\Big]^{p'}+r^{-tp'}\left\|\,\|(\frac{1}{b},y)\|^t\Phi_h^D(f)\right\|^{p'}_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}.\end{aligned}$$

We choose

$$r=C(d,s,t,p',k)\Big(\frac{\left\|\,\|(\frac{1}{b},y)\|^t\Phi_h^D(f)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{d+1}_+)}}{\|f\|^{\frac{1}{2}}_{L^2_k(\mathbb{R}^d)}\,\|\,\|y\|^s f\|^{\frac{1}{2}}_{L^2_k(\mathbb{R}^d)}+\|f\|^{\frac{1}{2}}_{L^{2p}_k(\mathbb{R}^d)}\,\|\,\|y\|^s f\|^{\frac{1}{2}}_{L^{2p}_k(\mathbb{R}^d)}}\Big)^{\frac{p'}{2\gamma+d+tp'}},$$

where

$$C(d,s,t,p',k)=\Big(\frac{2tp'}{\Gamma(\frac{2\gamma+d}{2})(C_3(s,k,h))^{\frac{p'}{2}}}\sqrt{\frac{\Gamma(2\gamma+d)}{d_k(2\gamma+d)}}\Big)^{\frac{1}{2\gamma+d+tp'}}$$

to obtain the third inequality. □

Corollary 4.3. *Let $s, t > 0$. Then for all $f \in L^2_k(\mathbb{R}^d)$, we have*

$$\|f\|_{L^2_k(\mathbb{R}^d)} \leqslant \begin{cases} \frac{C_1(s,t,k,h)}{\sqrt{C_h}}\Big[\big\| \|y\|^s f\big\|_{L^2_k(\mathbb{R}^d)} + \big\| \|y\|^s f\big\|_{L^4_k(\mathbb{R}^d)}\Big]^{\frac{t}{4s+t}} \big\| \|(\frac{1}{b}, y)\|^t \Phi^D_h(f)\big\|^{\frac{4s}{4s+t}}_{L^2_{\mu_k}(\mathbb{R}^{d+1}_+)}, & \text{for } 0 < s < \frac{2\gamma+d}{4} \\ \frac{C_2(s,t,k,h)}{\sqrt{C_h}}\Big(\|f\|^{1-\frac{d+2\gamma}{4s}}_{L^4_k(\mathbb{R}^d)} \big\| \|y\|^s f\big\|^{\frac{d+2\gamma}{4s}}_{L^4_k(\mathbb{R}^d)}\Big)^{\frac{t}{2\gamma+d+t}} \big\| \|(\frac{1}{b}, y)\|^t \Phi^D_h(f)\big\|^{\frac{2\gamma+d}{2\gamma+d+t}}_{L^2_{\mu_k}(\mathbb{R}^{d+1}_+)}, & \text{for } s > \frac{2\gamma+d}{4} \\ \frac{C_3(s,t,k,h)}{\sqrt{C_h}}\Big[\|f\|^{\frac{1}{2}}_{L^2_k(\mathbb{R}^d)} \big\|\|y\|^t f\big\|^{\frac{1}{2}}_{L^2_k(\mathbb{R}^d)} + \|f\|^{\frac{1}{2}}_{L^4_k(\mathbb{R}^d)}\big\| \|y\|^s f\big\|^{\frac{1}{2}}_{L^4_k(\mathbb{R}^d)}\Big]^{\frac{2t}{2\gamma+d+2t}} & \\ \qquad \big\| \|(\frac{1}{b}, y)\|^t \Phi^D_h(f)\big\|^{\frac{2\gamma+d}{2\gamma+d+2t}}_{L^2_{\mu_k}(\mathbb{R}^{d+1}_+)}, & \text{for } s = \frac{2\gamma+d}{4}. \end{cases}$$

Remark 4.2. *We note that we have studied these types of uncertainty principles and others for some integral transforms as the Dunkl Gabor transform, the (k, a)-generalized wavelet transform, the k-Hankel Gabor transform, the q-Dunkl wavelet transform. the q-Bessel Gabor transform and others integral transforms. These studies have given some papers. We cite as examples [33, 34, 35, 36, 37].*

5 Open Problem

In the present paper, we have successfully studied new quantitative uncertainty principles associated with the Dunkl wavelet transforms. The obtained results have a novelty and contribution to the literature. It is our hope that this work motivate the researchers to study the qualitative uncertainty principles as Hardy's, Morgan's, Beurling's and Miyachi's uncertainty principles associated with the Dunkl wavelet transform.

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