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Ulam stability for nonlinear fractional differential equations involving two fractional derivatives

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Abstract

In this article, we define and study the Ulam stability for nonlinear fractional differential equations involving two fractional derivatives. Some examples are presented to illustrate the main results.

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1 Introduction and preliminaries

Differential equations involving fractional derivatives have played a central role in engineering science and applied mathematics see [8, 9, 15, 16, 20]. Several articles and books on the differential equations using fractional calculus have appeared recently, see [1, 5, 9, 18, 23] and references therein. Many scholars investigated the existence and uniqueness for solutions of some differential equations of arbitrary order, for example, see [6, 10, 12, 23, 24]. Moreover, the study of Ulam type stability problems play a fundamental role in the theory of differential equations. Since then, many works dealing with the theoretical development of Ulam's type stability theory of boundary value problems for fractional differential equations. For more details, the reader can address the following works [4, 7, 14, 19, 21]. Recently, many authors presented some interestting results on Ulam stability for fractional differential equations

with different fractional derivative, reader can see [2, 3, 6, 11, 13, 18, 25] and references cited therein.
In the present work, we discuss the existence and the Ulam stabilities as well as the generalized Ulam-Hyers stabilities for the following fractional problem:

$$\begin{cases} D^{\alpha}\left(D^{\beta}+\lambda\right)u(t)-\varphi\left(t,u(t),D^{\alpha+\beta-1}u(t)\right)=0, 0<\alpha,\beta<1, t\in[0,T], \\ u(0)-\int_0^T\psi(s)\,u(s)\,ds=0,\ u(T)-\theta=0,\ \theta>0, \end{cases} \tag{1}$$

where D^{ϑ} denote the Caputo fractional derivatives of order $\vartheta, \vartheta\in\{\alpha,\beta,\alpha+\beta-1\}$ and $\varphi:[0,T]\times\mathbb{R}^2\to\mathbb{R}$, is continuous function on $[0,T]$, λ is real constant. The operator D^{ϑ} is the fractional derivative in the sense of Caputo, defined by

$$\begin{aligned} D^{\vartheta}\varphi(t) &= \frac{1}{\Gamma(n-\vartheta)}\int_0^t (t-s)^{n-\vartheta-1}\varphi^{(n)}(s)\,ds \\ &= J^{n-\vartheta}\varphi^{(n)}(t), t>0,\ n-1<\vartheta<n,\ n\in N^*, \end{aligned}$$

and the Riemann-Liouville fractional integral of order $\vartheta>0$, defined by

$$J^{\vartheta}\varphi(t)=\frac{1}{\Gamma(\vartheta)}\int_0^t (t-s)^{\vartheta-1}\varphi(s)\,ds, t>0,$$

where $\Gamma(\vartheta)=\int_0^{\infty}e^{-u}x^{\vartheta-1}dx$.

We need the The following lemmas [15, 17]:

Lemma 1.1 *Let* $r,s>0$, $\varphi\in L_1([a,b])$. *Then* $J^rJ^s\varphi(t)=J^{r+s}\varphi(t)$, $D^sJ^s\varphi(t)=\varphi(t)$, $t\in[a,b]$.

Lemma 1.2 *Let* $s>r>0$, $\varphi\in L_1([a,b])$. *Then* $D^rJ^s\varphi(t)=J^{s-r}\varphi(t)$, $t\in[a,b]$.

Also we need the following lemmas [15]:

Lemma 1.3 *For* $\vartheta>0$, *the general solution of the fractional differential equation* $D^{\vartheta}u(t)=0$ *is given by*

$$u(t)=\sum_{i=0}^{n-1}c_it^i,$$

where $c_i\in\mathbb{R}$, $i=0,1,2,..,n-1$, $n=[\vartheta]+1$.

Lemma 1.4 *Let* $\vartheta>0$. *Then*

$$J^{\vartheta}D^{\vartheta}u(t)=u(t)+\sum_{i=0}^{n-1}c_it^i,$$

for some $c_i\in\mathbb{R}$, $i=0,1,2,...,n-1$, $n=[\vartheta]+1$.

Let us now define the space $W = \left\{u : u \in C\left([0,T], \mathbb{R}\right), D^{\alpha+\beta-1}u \in C\left([0,T], \mathbb{R}\right)\right\}$ equipped, with the norm $\|u\|_W = \|u\| + \left\|D^{\alpha+\beta-1}u\right\|$, where

$$\|u\| = \sup_{t\in[0,T]} |u(t)| \text{ and } \left\|D^{\alpha+\beta-1}u\right\| = \sup_{t\in[0,T]} \left|D^{\alpha+\beta-1}u(t)\right|.$$

It is clear that $(W, \|u\|_W)$ is a Banach space.

In what follows, we present four types of the Ulam stability for the problem (1).

Definition 1.5 *The fractional boundary value problem (1) is Ulam-Hyers stable if there exists a real number $d_\varphi > 0$ such that for each $\sigma > 0$ and for each solution $v \in W$ of the inequality*

$$\left|D^{\alpha}\left(D^{\beta}+\lambda\right)v(t) - f\left(t, v(t), D^{\alpha+\beta-1}v(t)\right)\right| \le \sigma, \ t \in [0,T], \tag{2}$$

there exists a solution $u \in W$ of fractional boundary value problem (1) with

$$|v(t) - u(t)| \le d_\varphi \sigma, \ t \in [0,T].$$

Definition 1.6 *The fractional boundary value problem (1) is generalized Ulam-Hyers stable if there exists $h_\varphi \in C(\mathbb{R}_+, \mathbb{R}_+), \omega_\varphi(0) = 0$, such that for each solution $v \in W$ of the inequality (2) there exists a solution $u \in W$ of the fractional boundary value problem (1) with*

$$|v(t) - u(t)| \le h_\varphi(\sigma), \ t \in [0,T].$$

Definition 1.7 *The fractional boundary value problem (1) is Ulam-Hyers-Rassias stable with respect to $g \in W$ if there exists a real number $d_\varphi > 0$ such that for each $\sigma > 0$ and for each solution $v \in W$ of the inequality*

$$\left|D^{\alpha}\left(D^{\beta}+\lambda\right)v(t) - f\left(t, v(t), D^{\alpha+\beta-1}v(t)\right)\right| \le \sigma g(t), \ t \in [0,T], \tag{3}$$

there exists a solution $u \in W$ of problem (1) with

$$|v(t) - u(t)| \le d_\varphi \sigma g(t), \ t \in [0,T].$$

Definition 1.8 *The fractional boundary value problem (1) is generalized Ulam-Hyers-Rassias stable with respect to $g \in W$ if there exists a real number $d_{\varphi,g} > 0$ such that for each solution $v \in W$ of the inequality*

$$\left|D^{\alpha}\left(D^{\beta}+\lambda\right)v(t) - f\left(t, v(t), D^{\alpha+\beta-1}v(t)\right)\right| \le g(t), \ t \in [0,T], \tag{4}$$

there exists a solution $u \in W$ of problem (1) with

$$|v(t) - u(t)| \le d_{\varphi,g}\, g(t), \ t \in [0,T].$$

A function $v \in W$ is a solution of the inequality (2) if and only if there exists a function : $[0, T] \to \mathbb{R}$ such that

(1) : $|(t)| \leq \sigma, t \in [0, T]$.

(2) : $D^{\alpha}\left(D^{\beta}+\lambda\right) v(t)=f\left(t, v(t), D^{\alpha+\beta-1} v(t)\right)+(t), t \in[0, T]$.

Clearly,

(i) : Definition 1.5 $\Rightarrow$ Definition 1.6.

(ii) : Definition 1.7 $\Rightarrow$ Definition 1.8.

2 Main results

Lemma 2.1 *For a given* $\delta \in C([0, T], \mathbb{R})$, *the solution of the boundary value problem*

$$\begin{cases} D^{\alpha}\left(D^{\beta}+\lambda\right) u(t)=\delta(t), \ t \in[0, T], \\ x(0)=\int_{0}^{T} \psi(s) u(s) d s, \ x(T)=\theta, \ \theta>0, \end{cases} \tag{5}$$

is given by

$$\begin{aligned} u(t) = & \int_{0}^{t} \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \delta(s) d s-\lambda \int_{0}^{t} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} u(s) d s \\ & +\frac{t^{\beta}}{T^{\beta}}\left[\theta-\int_{0}^{T} \frac{(T-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \delta(s) d s+\lambda \int_{0}^{T} \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} u(s) d s\right] \\ & +\left(1-\frac{t^{\beta}}{T^{\beta}}\right) \int_{0}^{T} \psi(s) u(s) d s . \end{aligned} \tag{6}$$

By applying Lemma 1.3 and Lemma 1.4, the general solution of (5) is written as

$$u(t)=J^{\alpha+\beta} \delta(t)-\lambda J^{\beta} u(t)+\frac{c_{0}}{\Gamma(\beta+1)} t^{\beta}+c_{1}, \tag{7}$$

where c_0 and c_1 are arbitrary constants. By the boundary condition $u(0)= \int_{0}^{T} \psi(s) u(s) d s$, we conclude that $c_{1}=\int_{0}^{1} \psi(s) u(s) d s$.

Using the boundary condition $u(T)=\theta$, we obtain that

$$c_{0}=\frac{\Gamma(\beta+1)}{T^{\beta}}\left[\theta-\int_{0}^{T} \psi(s) u(s) d s-J^{\alpha+\beta} \delta(T)+\lambda J^{\beta} u(T)\right].$$

Substituting the value of c_0 and c_1 in (7), we obtain the solution (6).

In view of Lemma 2.1, we can transform the problem (1) into an equivalent fixed point problem $\phi u = u$, where the operator $\phi : W \to W$ is defined by:

$$\begin{aligned} \phi u(t) &= \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \varphi\left(s, u(s), D^{\alpha+\beta-1}u(s)\right) ds - \lambda \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} u(s)\, ds \\ &+ \frac{t^\beta}{T^\beta}\left[\theta - \int_0^T \frac{(T-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \varphi\left(s, u(s), D^{\alpha+\beta-1}u(s)\right) ds \right. \quad (8) \\ &\left. + \lambda \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} u(s)\, ds \right] - \left(\frac{t^\beta}{T^\beta} - 1\right) \int_0^T \psi(s)\, u(s)\, ds. \end{aligned}$$

Observe that the existence of a fixed point for the operator ϕ implies the existence of a solution for the problem (1).

For convenience we introduce the notations:

$$\begin{aligned} \Lambda_1 &:= \frac{2\omega T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{2|\lambda| T^\beta}{\Gamma(\beta+1)} + 2TL_\psi, \\ \Lambda_2 &:= \frac{2NT^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + |\theta|, \quad (9) \end{aligned}$$

and

$$\begin{aligned} \Delta_1 &:= \frac{\omega T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \frac{\omega\beta T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda| T^{\beta-1}}{\Gamma(\beta)} + \frac{\beta|\lambda| T^{\beta-1}}{\Gamma(\beta+1)} + \beta L_\psi, \\ \Delta_2 &:= \frac{NT^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \frac{N\beta T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta+1)} + \frac{\beta}{T}|\theta|. \quad (10) \end{aligned}$$

2.1 Existence and uniqueness

The first results are based on Banach's fixed point theorem. We prove the following theorem:

Theorem 2.2 *Let $\varphi : [0,T] \times \mathbb{R} \to \mathbb{R}$, is continuous function satisfying the hypothesis*

(H_1) there existe nonnegative constants ω such that for all $t \in [0,1]$ and all $u_i,\ v_i \in \mathbb{R}\ (i = 1,2)$, we have

$$|\varphi(t, u_1, u_2) - \varphi(t, v_1, v_2)| \le \omega\left(|u_1 - v_1| + |u_2 - v_2|\right).$$

Then the fractional boundary value problem (1) has a unique solution provided by $\Lambda_1 + \frac{\Delta_1}{\Gamma(3-\alpha-\beta)} < 1$, where Λ_1 and Δ_1 are defined by (9) and (10), respectively.

Let us fix $\sup_{t\in[0,T]}\varphi(t,0,0)=N<\infty$ and define

$$r\geq\frac{1}{1-\left(\Lambda_1+\frac{1}{\Gamma(3-\alpha-\beta)}\Delta_1\right)}\left(\Lambda_2+\frac{1}{\Gamma(3-\alpha-\beta)}\Delta_2\right),$$

where $\Lambda_i,\Delta_i,i=1,2$, are given by (9) and (10), respectively. We show that $\phi B_r\subset B_r$, where ϕ defined by (8) and $B_r=\{u\in W:\|u\|_W\leq r\}$.

For $u\in B_r$, we find the following estimate based on the hypothesis (H_1) :

$$\left|\varphi\left(t,u(t),D^{\alpha+\beta-1}u(t)\right)\right|\leq\left(\left|\varphi\left(t,u(t),D^{\alpha+\beta-1}u(t)\right)-\varphi(t,0,0)\right|+|\varphi(t,0,0)|\right)$$

$$\leq\ \omega\left(|u(t)|+\left|D^{\alpha+\beta-1}u(t)\right|\right)+N\leq\omega\left(\|u\|+\left\|D^{\alpha+\beta-1}u\right\|\right)+N\leq\omega\|u\|_W+N\leq\omega r+N.$$

Using this estimate, we get

$$\begin{aligned}|\phi u(t)| &\leq \int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left|\varphi\left(s,u(s),D^{\alpha+\beta-1}u(s)\right)\right|ds+|\lambda|\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}u(s)\,ds\\ &\quad+\frac{t^\beta}{T^\beta}\left[|\theta|+\int_0^T\frac{(T-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left|\varphi\left(s,u(s),D^{\alpha+\beta-1}u(s)\right)\right|ds\right.\\ &\quad\left.+|\lambda|\int_0^T\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}|u(s)|\,ds\right]+\left(\frac{t^\beta}{T^\beta}-1\right)\int_0^T|\psi(s)|\,|u(s)|\,ds.\\ &\leq 2\left[\frac{\omega T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{|\lambda|T^\beta}{\Gamma(\beta+1)}+TL_\psi\right]r+\frac{2NT^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+|\theta|\\ &= \Lambda_1 r+\Lambda_2\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\left|\phi(u)'(t)\right|\\ \leq\ &\int_0^t\frac{(t-s)^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}\left|\varphi\left(s,u(s),D^{\alpha+\beta-1}u(s)\right)\right|ds+|\lambda|\int_0^t\frac{(t-s)^{\beta-2}}{\Gamma(\beta-1)}|u(s)|\,ds\\ &+\frac{\beta t^{\beta-1}}{T^\beta}\left[|\theta|+\int_0^T\frac{(T-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left|\varphi\left(s,u(s),D^{\alpha+\beta-1}u(s)\right)\right|ds\right.\\ &\left.+|\lambda|\int_0^T\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}|u(s)|\,ds\right]+\frac{\beta t^{\beta-1}}{T^\beta}\int_0^T|\psi(s)|\,|u(s)|\,ds\\ \leq\ &\left[\left(\frac{T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}+\frac{\beta T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta+1)}\right)\omega+\frac{|\lambda|T^{\beta-1}}{\Gamma(\beta)}+\frac{\beta|\lambda|T^{\beta-1}}{\Gamma(\beta+1)}+\beta L_\psi\right]r\\ &+\left(\frac{T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}+\frac{\beta T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta+1)}\right)N+\frac{\beta}{T}|\theta|\\ =\ &\Delta_1 r+\Delta_2,\end{aligned}$$

which implies that

$$\left|D^{\alpha+\beta-1}\phi u(t)\right| \leq \int_0^t \frac{(t-s)^{1-(\alpha+\beta)}}{\Gamma(2-\alpha-\beta)} \left|\phi(u)'(t)\right| ds \leq \frac{\Delta_1 r+\Delta_2}{\Gamma(3-\alpha-\beta)}.$$

Thus

$$\begin{aligned} \|\phi u\|_W &= \|\phi u\| + \left\|D^{\alpha+\beta-1}\phi u\right\| \\ &\leq \left(\Lambda_1 + \frac{\Delta_1}{\Gamma(3-\alpha-\beta)}\right) r + \Lambda_2 + \frac{\Delta_2}{\Gamma(3-\alpha-\beta)} \leq r, \end{aligned} \tag{11}$$

which implies that $\phi B_r \subset B_r$. Now for $u, v \in B_r$ and for all $t \in [0,T]$, we obtain

$$\begin{aligned} &\left|\phi u(t) - \phi v(t)\right| \\ \leq& \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \left|\varphi\left(s, u(s), D^{\alpha+\beta-1}u(s)\right) - \varphi\left(s, v(s), D^{\alpha+\beta-1}v(s)\right)\right| ds \\ &+\frac{t^\beta}{T^\beta}\int_0^T \frac{(T-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \left|\varphi\left(s, u(s), D^{\alpha+\beta-1}u(s)\right) - \varphi\left(s, v(s), D^{\alpha+\beta-1}v(s)\right)\right| ds \\ &+|\lambda| \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} |u(s)-v(s)|\, ds + \frac{|\lambda|\, t^\beta}{T^\beta}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} |u(s)-v(s)|\, ds \\ &+\left|\frac{t^\beta}{T^\beta}-1\right| \int_0^T |\psi(s)|\, |u(s)-v(s)|\, ds \\ \leq& \frac{2\omega T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left(\|u-v\| + \left\|D^{\alpha+\beta-1}u - D^{\alpha+\beta-1}v\right\|\right) + \left(\frac{2|\lambda|\, T^\beta}{\Gamma(\beta+1)} + 2L_\psi\right)\|u-v\| \\ =& \Lambda_1 \|u-v\|_W . \end{aligned}$$

Also we have

$$\left|\phi(u)'(t) - \phi(v)'(t)\right| \leq \Delta_1 \|u-v\|_W .$$

Thus we obtain

$$\begin{aligned} \left|D^{\alpha+\beta-1}\phi u(t) - D^{\alpha+\beta-1}\phi v(t)\right| &\leq \int_0^t \frac{(t-s)^{1-(\alpha+\beta)}}{\Gamma(2-\alpha-\beta)} \left|\phi(u)'(t) - \phi(v)'(t)\right| ds \\ &\leq \frac{\Delta_1}{\Gamma(3-\alpha-\beta)} \|u-v\|_W . \end{aligned} \tag{12}$$

From the above inequalities, we get

$$\begin{aligned} \|\phi u - \phi v\|_W &\leq \|\phi u - \phi v\| + \left\|D^{\alpha+\beta-1}\phi u - D^{\alpha+\beta-1}\phi v\right\| \\ &\leq \left(\Lambda_1 + \frac{\Delta_1}{\Gamma(3-\alpha-\beta)}\right)\|u-v\|_W . \end{aligned} \tag{13}$$

which shows that ϕ is a contraction in view of the assumptions $\Lambda_1 + \frac{\Delta_1}{\Gamma(3-\alpha-\beta)} < 1$. Hence, by Banach's fixed point theorem, the operator ϕ has a unique fixed point which corresponds to the unique solution of problem (1). This completes the proof.

2.2 Ulam-Hyers stability

In what follows, we will study Ulam's type stability of the fractional boundary value problem (1).

Theorem 2.3 *Assume that $\varphi : [0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying (H_1). If*

$$\omega\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+|\lambda|\frac{T^{\beta}}{\Gamma(\beta+1)}<1, \tag{14}$$

then the fractional boundary value problem (1) is Ulam-Hyers stable and consequently, generalized Ulam-Hyers stable.

Let $v\in W$ be a solution of the inequality (2), i.e.

$$\left|D^{\alpha}\left(D^{\beta}+\lambda\right)v(t)-f\left(t,v(t),D^{\alpha+\beta-1}v(t)\right)\right|\leq\sigma, t\in[0,T],$$

and let us denote by $u\in W$ the unique solution of the problem

$$\begin{cases} D^{\alpha}\left(D^{\beta}+\lambda\right)u(t)=\varphi\left(t,u(t),D^{\alpha+\beta-1}u(t)\right), t\in[0,T], 0<\alpha,\beta<1, \\ u(0)=v(0),\ u(T)=v(T). \end{cases}$$

By using Lemma 11, we have

$$u(t)=J^{\alpha+\beta}\delta(t)-\lambda J^{\beta}u(t)+\frac{c_0}{\Gamma(\beta+1)}t^{\beta}+c_1$$

and by integration of the inequality (2), we obtain

$$\begin{aligned}\left|v(t)-J^{\alpha+\beta}\delta_v(t)-\lambda J^{\beta}v(t)-\frac{b_0}{\Gamma(\beta+1)}t^{\beta}-b_1\right| &\leq \frac{\sigma t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\ &\leq \frac{\sigma T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}.\end{aligned}$$

On the other hand, if $u(0)=v(0)$ and $u(T)=v(T)$, then

$$c_0=b_0 \text{ and } c_1=b_1.$$

For any $t\in[0,T]$, we have

$$\begin{aligned} v(t)-u(t) &= v(t)-J^{\alpha+\beta}\delta_u(t)-\lambda J^{\beta}u(t)-\frac{b_0}{\Gamma(\beta+1)}t^{\beta}-b_1 \\ &\quad +J^{\alpha+\beta}\left(\delta_v(t)-\delta_u(t)\right)-\lambda J^{\beta}\left(v(t)-u(t)\right),\end{aligned}$$

where,

$$\delta_u(t) = \delta\left(t, u(t), D^{\alpha+\beta-1}u(t)\right) \text{ and } \delta_v(t) = \varphi\left(t, v(t), D^{\alpha+\beta-1}v(t)\right),$$

then

$$\begin{aligned}
&J^{\alpha+\beta}\left(\delta_v(t) - \delta_u(t)\right) \\
&= J^{\alpha+\beta}[\varphi\left(t, u(t), D^{\alpha+\beta-1}u(t)\right) - \varphi\left(t, v(t), D^{\alpha+\beta-1}v(t)\right)] \\
&= \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}[\varphi\left(s, u(s), D^{\alpha+\beta-1}u(s)\right) - \varphi\left(s, v(s), D^{\alpha+\beta-1}v(s)\right)].
\end{aligned}$$

Using (H_1), we get

$$\left|J^{\alpha+\beta}\left(\delta_v(t) - \delta_u(t)\right)\right| \leq \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}\left\|u(s) - v(s)\right\|_W ds.$$

This yields that

$$\begin{aligned}
|v(t) - u(t)| \leq{}& \left|v(t) - J^{\alpha+\beta}\delta_u(t) - \lambda J^{\beta}u(t) - \frac{b_0}{\Gamma(\beta+1)}t^{\beta} - b_1\right| \\
&+\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}\left(\|u(s) - v(s)\|\right. \\
&+ \left.\left\|D^{\alpha+\beta-1}u(s) - D^{\alpha+\beta-1}v(s)\right\|\right) ds \\
&+\frac{|\lambda|}{\Gamma(\beta)}\int_0^t (t-s)^{\beta}\|u(s) - v(s)\| ds
\end{aligned}$$

which implies that

$$|v(t) - u(t)| \leq \frac{\varepsilon T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \left[\frac{\omega T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda| T^{\beta}}{\Gamma(\beta+1)}\right]\|v(s) - u(s)\|_W.$$

Hence

$$\|v(s) - u(s)\|_W \left(1 - \left[\frac{\omega T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda| T^{\beta}}{\Gamma(\beta+1)}\right]\right) \leq \frac{\sigma T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}.$$

Then, for each $t \in [0, T]$

$$|u(t) - v(t)| \leq \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)\left(1 - \left[\frac{\omega T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda| T^{\beta}}{\Gamma(\beta+1)}\right]\right)}\sigma := d\sigma. \tag{15}$$

Therefore, the fractional boundary value problem (1) is Ulam-Hyers stable. By taking $h(\sigma) = d\sigma, h(0) = 0$ yields that the fractional boundary value problem (1) generalized Ulam-Hyers stable.

Theorem 2.4 *Let $\varphi : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function and suppose that (H_1) and (14) holds. In addition, the following hypothesis holds*

(H_2) : There exists an function $g \in C([0,T], \mathbb{R}_+)$ and there exists $\eta_g > 0$ such that for any $t \in [0,T]$

$$\frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} g(s) ds \le \eta_g g(t). \tag{16}$$

Then the fractional boundary value problem (1) is Ulam-Hyers-Rassias stable.

Let $v \in W$ be a solution of the inequality (4), i.e.

$$\left| D^{\alpha}\left(D^{\beta}+\lambda\right) v(t) - \varphi\left(t, v(t), D^{\alpha+\beta-1} v(t)\right)\right| \le g(t), \ t \in [0,T], \tag{17}$$

and let us denote by $u \in W$ the unique solution of the problem

$$\begin{cases} D^{\alpha}\left(D^{\beta}+\lambda\right) u(t) = \varphi\left(t, u(t), D^{\alpha+\beta-1} u(t)\right), t \in [0,T], 0<\alpha,\beta<1, \\ u(0) = v(0), \ u(T) = v(T). \end{cases}$$

Thanks to Lemma 10, we obtain

$$u(t) = J^{\alpha+\beta}\delta(t) - \lambda J^{\beta} u(t) + \frac{c_0}{\Gamma(\beta+1)} t^{\beta} + c_1,$$

and by integration of the inequality (17), we obtain

$$\begin{aligned} &\left| u(t) - J^{\alpha+\beta}\delta_v(t) - \lambda J^{\beta} v(t) - \frac{b_0}{\Gamma(\beta+1)} t^{\beta} - b_1 \right| \\ \le\ & \frac{\sigma}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} g(s) ds \le \sigma\eta_g g(t). \end{aligned}$$

In a similar way to that of the proof of Theorem 12, using (H_1), we have

$$\begin{aligned} |v(t) - u(t)| \le\ & \left| v(t) - J^{\alpha+\beta}\delta_u(t) - \lambda J^{\beta} u(t) - \frac{b_0}{\Gamma(\beta+1)} t^{\beta} - b_1 \right| \\ & + \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} \left(\|u(s) - v(s)\| \right. \\ & \left. + \left\| D^{\alpha+\beta-1} u(s) - D^{\alpha+\beta-1} v(s) \right\| \right) ds \\ & + \frac{|\lambda|}{\Gamma(\beta)} \int_0^t (t-s)^{\beta} \|u(s) - v(s)\| ds \end{aligned}$$

Using (H_2), we can write

$$|v(t) - u(t)| \le \left[\frac{\omega T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda| T^{\beta}}{\Gamma(\beta+1)} \right] \|v(s) - u(s)\|_W,$$

which implies that

$$\|v(s)-u(s)\|\left(1-\left[\frac{\omega T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{|\lambda|\,T^{\beta}}{\Gamma(\beta+1)}\right]\right)\leq\sigma\eta_g g(t).$$

For any $t\in[0,T]$, we have

$$|u(t)-v(t)|\leq\left(\frac{\eta_g}{1-\left[\frac{\omega T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{|\lambda|T^{\beta}}{\Gamma(\beta+1)}\right]}\right)\sigma g(t):d\sigma g(t). \qquad (18)$$

Then, the fractional boundary value problem (1) is Ulam-Hyers-Rassias stable.

3 Examples

To illustrate our main results, we treat the following examples.

Example 3.1 *Let us consider the following fractional boundary value problem*

$$\begin{cases} D^{\frac{3}{4}}\left(D^{\frac{2}{3}}+\frac{1}{15}\right)u(t)=\frac{\sqrt{\pi}e^{-\pi t}\cos(\pi t)|u(t)|}{\left(15\sqrt{\pi}+7e^{t}\right)(1+|u(t)|)}+\frac{\sqrt{\pi}e^{-\pi t}\cos(\pi t)\sin\left|D^{\frac{5}{12}}u(t)\right|}{\left(15\sqrt{\pi}+7e^{t}\right)}+\ln\left(1+t^{2}\right),\ t\in[0,1], \\ x(0)=\int_0^1\frac{3\ln(1+s)}{35(1+s^2)}ds,\ x(1)=\frac{2}{3}. \end{cases} \qquad (19)$$

Set

$$\varphi(t,u,v)=\frac{\sqrt{\pi}e^{-\pi t}\cos(\pi t)\,|u|}{\left(15\sqrt{\pi}+7e^{t}\right)^{2}(1+|u|)}+\frac{\sqrt{\pi}e^{-\pi t}\cos(\pi t)\sin|v|}{\left(15\sqrt{\pi}+7e^{t}\right)^{2}}+\ln\left(1+t^{2}\right),u,v\in\mathbb{R},t\in[0,1].$$

For any $(u_1,v_1),(u_2,v_2)\in\mathbb{R}^2$ *and* $t\in[0,1]$, *we have*

$$|\varphi(t,u_1,v_1)-\varphi(t,u_2,v_2)|\leq\frac{\sqrt{\pi}}{\left(15\sqrt{\pi}+7e\right)^{2}}\left(|u_1-u_2|+|v_1-v_2|\right).$$

Hence the condition (H_1) *holds with* $\omega=\frac{\sqrt{\pi}}{\left(15\sqrt{\pi}+7e\right)^{2}}$.

For $\alpha=\frac{3}{4},\beta=\frac{2}{3},\lambda=\frac{1}{15},T=1,\psi(s)=\frac{3\ln(1+s)}{35(1+s^2)}$ *and* $\theta=\frac{2}{3}$, *we have*

$$\Lambda_1=0.208\,47,\Delta_1=0.119\,68.$$

It follows then that

$$\Lambda_1+\frac{\Delta_1}{\Gamma(3-\alpha-\beta)}=0.342\,68<1.$$

Hence by Theorem 2.2, the fractional boundary value problem (19) has a unique solution.

Also, we have

$$\omega \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + |\lambda| \frac{T^{\beta}}{\Gamma(\beta+1)} = 7.4527 \times 10^{-2} < 1.$$

Then, all the hypotheses of Theorem 2.3 are satisfied. Thus, by the conclusion of Theorem 2.3, problem (19) is Ulam-Hyers stable.

Example 3.2 *As a second illustrative example, let us take*

$$\begin{cases} D^{\frac{2}{5}}\left(D^{\frac{3}{4}} + \frac{1}{10}\right) u(t) = \frac{\sin(t)}{32\pi} |u(t)| + \frac{\left|D^{\frac{3}{20}} u(t)\right|}{32\pi + \left|D^{\frac{3}{20}} u(t)\right|} + \frac{3+\sinh e^{t^2}}{2\pi}, \ t \in [0,1], \\ x(0) = \int_0^1 \frac{e^{-\pi s}}{25\sqrt{2+\pi s^2}} u(s)\, ds, \ x(1) = \sqrt{3} \end{cases} \tag{20}$$

where $\alpha = \frac{2}{5}, \beta = \frac{3}{4}, \lambda = \frac{1}{10}, T = 1, \psi(s) = \frac{e^{-\pi s}}{25\sqrt{2+\pi s^2}}, \theta = \sqrt{3}$ *and* $\varphi(t,x,y) = \frac{\sin(t)}{32\pi} |u| + \frac{|v|}{32\pi + |v|} + \frac{3+\sinh e^{t^2}}{2\pi}$.

For any $(u_1, v_1), (u_2, v_2) \in \mathbb{R}^2$ *and* $t \in [0,1]$*, we have*

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \leq \frac{1}{32\pi} \left(|u_1 - u_2| + |v_1 - v_2|\right).$$

Hence condition (H_1) *is satisfied with* $\omega = \frac{1}{32\pi}$.

Then

$$\Lambda_1 = 0.29272, \ \Delta_1 = 0.20204,$$

and

$$\Lambda_1 + \frac{\Delta_1}{\Gamma(3-\alpha-\beta)} = 0.50638 < 1.$$

Let $g(t) = \xi t^3, \xi \in \mathbb{R}$*. We have*

$$\frac{1}{\Gamma(\frac{2}{5}+\frac{3}{4})} \int_0^t (t-s)^{\frac{2}{5}+\frac{3}{4}-1} g(s) ds \leq \frac{6\xi}{\Gamma\left(\frac{103}{20}\right)} t^3 = \eta_g g(t).$$

Thus condition (H_2) *is satisfied with* $g(t) = \xi t^3$ *and* $\eta_g = \frac{6\xi}{\Gamma\left(\frac{103}{20}\right)}$*. It follows from Theorem 2.2 that the fractional problem (20) as a unique solution on* $[0,1]$*, and from Theorem 2.4 problem (20) is Ulam-Hyers-Rassias stable.*

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