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Results On Multiplicative Generalized $(\alpha, \alpha)-$Derivations

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Abstract

Let R be a ring and I a nonzero ideal of R. A mapping $F : R \to R$ is called a multiplicative generalized $(\alpha, \alpha)-$derivation if there exists a mapping $d : R \to R$ such that $F(xy) = F(x)\alpha(y) + \alpha(x)d(y)$, for all $x, y \in R$ where $\alpha : R \to R$ is an automorphism. In the present paper, we shall prove that $[\alpha(x), d(x)] = 0$, for all $x \in I$ if any one of the following holds: i) $F([x, y]) = 0$, ii) $F(x \circ y) = 0$, iii) $F([x, y]) = \pm\alpha([x, y])$, iv) $F(x \circ y) = \pm\alpha(x \circ y)$, v) $F([x, y]) = \pm\alpha(x \circ y)$, vi) $F(x \circ y) = \pm\alpha([x, y])$, vii) $F([x, y]) = \pm\alpha([F(x), y])$, viii) $F(x \circ y) = \pm\alpha(F(x) \circ y)$, for all $x, y \in I$.

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1 Introduction

Let R will be an associative ring with center Z. For any $x, y \in R$ the symbol $[x, y]$ represents commutator $xy - yx$ and the Jordan product $xoy = xy + yx$. Recall that a ring R is prime if for $x, y \in R$, $xRy = \{0\}$ implies either $x = 0$ or $y = 0$ and R is semiprime if for $x \in R$, $xRx = \{0\}$ implies $x = 0$.

An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. Let $F : R \to R$ be map associated with another map $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. If F is additive and d is a derivation of R, then F is called a generalized derivation of R a concept introduced by M. Bresar [1]. Obviously, every derivation is a generalized derivation of R.

In [2], the notion of multiplicative derivation was introduced by Daif motivated by Martindale [8]. $d : R \to R$ is called a multiplicative derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. These maps are not additive. In [6], Goldman and Semrl gave the complete description of these maps. Let $R = C[0, 1]$ be a ring of all continuous (real or complex valued) functions and define a mapping $d : R \to R$ such as

$$d(f)(x) = \left\{ \begin{array}{ll} f(x)\log|f(x)|, & f(x) \neq 0 \\ 0, & \text{otherwise} \end{array} \right\}.$$

It is clear that d is multiplicative derivation, but d is not additive. Inspired by the definition multiplicative derivation, the notion of multiplicative generalized derivation was extended by Daif and Tamman El-Sayiad in [4] as follows:

$F : R \to R$ is called a multiplicative generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y+xd(y)$, for all $x, y \in R$. Dhara and Ali gave a slight generalization of this definition taking g is any mapping (not necessarily an additive mapping or a derivation) in [5]. Hence, one may observe that the concept of multiplicative generalized derivations includes the concept of derivations, generalized derivations and the left multipliers (i.e., $F(xy) = F(x)y$ for all $x, y \in R$). So, it should be interesting to extend some results concerning these notions to multiplicative generalized derivations. Every generalized derivation is a multiplicative generalized derivation. But the converse is not ture in general. The following example justifies this:

[5, Example 1.1] Let S be any ring and

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in S \right\}.$$

Define the maps d and $F : R \to R$ as follows: $d\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & bc \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then it is straightforward to verify that F is a multiplicative generalized derivation associated with a multiplicative derivation d, but F is not a generalized derivation of R.

Let α and β be two automorphisms of R. An additive mapping $d : R \to R$ is called $(\alpha, \beta)-$derivation if if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ holds for all

$x, y \in R$. Inspired by the definition of $(\alpha, \beta)-$derivation, the notion of generalized $(\alpha, \beta)-$derivation was extended as follows: Let α, β be two automorphisms of R. An additive mapping $F : R \to R$ is called a generalized $(\alpha, \beta)-$derivation if there exists a $(\alpha, \beta)-$derivation $d : R \to R$ such that $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$. There are some applications of $(\alpha, \beta)-$derivations which can help to develop an approach to deformations of Lie algebras and Banach algebra theory. In [10], the authors have generalized the concept of a generalized $(\alpha, \beta)-$derivation to multiplicative generalized $(\alpha, \beta)-$derivation and they gave the following example:

[10, Example 1.1] Let

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}.$$

Define the maps $F, d, \alpha, \beta : R \to R$ as follows:

$$d \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, F \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & bc \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\alpha \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & -b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \beta \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}.$$

Then it is straightforward to verify that F is a multiplicative generalized $(\alpha, \beta)-$derivation associated with a mapping d, but F is neither a generalized derivation nor a multiplicative generalized derivation of R.

Many papers in literature have investigated the commutativity of prime and semiprime rings satisfying certain functional identities involving derivations or generalized derivations. In [3], Daif and Bell proved that R is semiprime ring, U is a nonzero ideal of R and d is a derivation of R such that $d([x, y]) = \pm[x, y]$, for all $x, y \in U$, then R contains a nonzero central ideal. This theorem considered for generalized derivations by Quadri et al. in [9] and extended by Gölbaşı when F is a multiplicative generalized derivation of R in [7].

In the present paper, our aim is to investigate some identities with multiplicative generalized $(\alpha, \alpha)-$derivation on a nonzero ideal of semiprime ring. In many articles about generalized $(\alpha, \alpha)-$derivation, d is taken as a nonzero $(\alpha, \alpha)-$derivation. In this study, differently, we have received d as a nonzero mapping.

2 Results

Throughout the paper, R will be semiprime ring and I be a nonzero ideal of R and F a generalized $(\alpha, \alpha)-$multiplicative derivation of R with associated a nonzero mapping g of R.

Also, we will make some extensive use of the basic commutator identities:
$[x, yz] = y[x, z] + [x, y]z$
$[xy, z] = [x, z]y + x[y, z]$
$xo(yz) = (xoy)z - y[x, z] = y(xoz) + [x, y]z$
$(xy)oz = x(yoz) - [x, z]y = (xoz)y + x[y, z]$.
We remind some well known results which will be useful in our proofs:
Fact : Let R be a semiprime ring, then
i) The center of R contains no nonzero nilpotent elements.
ii) If P is a nonzero prime ideal of R and $a, b \in R$ such that $aRb \subseteq P$, then either $a \in P$ or $b \in P$.
iii) The center of a nonzero one sided ideal is contained in the center of R. In particular, any commutative one sided ideal is contained in the center of R.
[11, Lemma 2.1] Let R be a semiprime ring, I a nonzero ideal of R and $a \in R$ such that $axa = 0$ for all $x \in I$, then $a = 0$.
Let R be a semiprime ring, I a nonzero ideal of R, α an automorphism on R and F be a multiplicative generalized $(\alpha, \alpha)-$derivation associated with a nonzero mapping d of R. If $F([x, y]) = 0$, for all $x, y \in I$, then $[\alpha(x), d(x)] = 0$, for all $x \in I$.
By the hypothesis, we have

$$F([x, y]) = 0, \quad \text{for all } x, y \in I. \tag{1}$$

Replacing y by yx in (1) and using this relation, we get

$$\alpha([x, y])d(x) = 0, \quad \text{for all } x, y \in I. \tag{2}$$

Using α is an automorphism of R, we obtain that

$$[\alpha(x), z]d(x) = 0, \text{ for all } x \in I, z \in J \tag{3}$$

where $J = \alpha(I)$, a nonzero ideal of R.
Replacing z by $rz, r \in R$ in (3) and using this equation, we get

$$[\alpha(x), r]zd(x) = 0, \text{ for all } x \in I, z \in J, r \in R. \tag{4}$$

Right multiplying (4) by $\alpha(x)$, we get

$$[\alpha(x), r]zd(x)\alpha(x) = 0, \text{ for all } x \in I, z \in J, r \in R. \tag{5}$$

Replacing z by $z\alpha(x)$ in (4), we get

$$[\alpha(x), r]z\alpha(x)d(x) = 0, \text{ for all } x \in I, z \in J, r \in R. \tag{6}$$

Subtracting (5) of (6), we get

$$[\alpha(x), r]z[\alpha(x), d(x)] = 0, \text{ for all } x \in I, z \in J, r \in R. \tag{7}$$

Replacing $d(x)$ by r in (7) and using Lemma 1, we get the required result.

Let R be a semiprime ring, I a nonzero ideal of R, α an automorphism on R and F be a multiplicative generalized $(\alpha,\alpha)-$derivation associated with a nonzero mapping d of R. If $F(xoy)=0$, for all $x,y\in I$, then $[\alpha(x),d(x)]=0$, for all $x\in I$.

By our hypothesis, we get

$$F(xoy)=0, \quad \text{for all } x,y\in I. \tag{8}$$

Writing yx for y in (8) and using (8), we obtain that

$$\alpha(xoy)d(x)=0, \text{ for all } x,y\in I. \tag{9}$$

Substituting $ry, r\in R$ for y in (9) and using this equation, α an automorphism, we arrive at

$$[\alpha(x), r]\alpha(y)d(x)=0, \text{ for all } x,y\in I, r\in R. \tag{10}$$

We can write this equation such as

$$[\alpha(x), r]zd(x)=0, \text{ for all } x\in I, z\in J, r\in R.$$

where $J=\alpha(I)$, a nonzero ideal of R. Using the same arguments after (4) in the proof of Theorem 1, we get the required result.

Let R be a semiprime ring, I a nonzero ideal of R, α an automorphism on R and F be a multiplicative generalized $(\alpha,\alpha)-$derivation associated with a nonzero mapping d of R. If $F([x,y])=\pm\alpha([x,y])$, for all $x,y\in I$, then $[\alpha(x),d(x)]=0$, for all $x\in I$.

By our hypothesis, we have

$$F([x,y])=\pm\alpha([x,y]), \quad \text{for all } x,y\in I. \tag{11}$$

Replacing y by yx in (11) and using this equation, we arrive that

$$\alpha([x,y])d(x)=0, \text{ for all } x,y\in I.$$

Using the same arguments after (2) in the proof of Theorem 1, we get the required result.

Let R be a semiprime ring, I a nonzero ideal of R, α an automorphism on R and F be a multiplicative generalized $(\alpha,\alpha)-$derivation associated with a nonzere mapping d of R. If $F(xoy)=\pm\alpha(xoy)$, for all $x,y\in I$, then $[\alpha(x),d(x)]=0$, for all $x\in I$.

By our hypothesis, we have

$$F(xoy)=\pm\alpha(xoy), \text{ for all } x,y\in I. \tag{12}$$

Replacing y by yx in (12) and using this, we arrive that

$$\alpha(xoy)d(x) = 0, \text{ for all } x, y \in I.$$

Using the same arguments after (9) in the proof of Theorem 2, we conclude the required result.

Let R be a semiprime ring, I a nonzero ideal of R, α an automorphism on R and F be a multiplicative generalized (α,α) −derivation associated with a nonzero mapping d of R . If $F([x,y]) = \pm\alpha(xoy)$, for all $x, y \in I$, then $[\alpha(x), d(x)] = 0$, for all $x \in I$.

By our hypothesis, we have

$$F([x,y]) = \pm\alpha(xoy), \quad \text{for all } x, y \in I. \tag{13}$$

Replacing y by yx in (13) and using this, we get

$$\alpha\left([x,y]\right)d(x) = 0, \quad \text{for all } x, y \in I.$$

This equation is same as (2) of Theorem 1, now arguing in the similar manner, we get the required result.

Let R be a semiprime ring, I a nonzero ideal of R, α an automorphism on R and F be a multiplicative generalized (α,α) −derivation associated with a nonzero mapping d of R. If $F(xoy) = \pm\alpha\left([x,y]\right)$, for all $x, y \in I$, then $[\alpha(x), d(x)] = 0$, for all $x \in I$.

By the hypothesis, we have

$$F(xoy) = \pm\alpha\left([x,y]\right), \quad \text{for all } x, y \in I. \tag{14}$$

Replacing y by yx in (14) and using this, we get

$$\alpha(xoy)d(x) = 0, \quad \text{for all } x, y \in I.$$

This equation is same as (9) in the proof of Theorem 2. Hence, using the same arguments in there, we get the required result.

Let R be a semiprime ring, I a nonzero ideal of R, α an automorphism on R and F be a multiplicative generalized (α,α) −derivation associated with a nonzero mapping d of R. If $F([x,y]) = \pm\alpha\left([F(x),y]\right)$, for all $x, y \in I$, then $[\alpha(x), d(x)] = 0$, for all $x \in I$.

By our hypothesis, we get

$$F([x,y]) = \pm\alpha\left([F(x),y]\right) \quad \text{for all } x, y \in I. \tag{15}$$

Replacing y by yx, $x \in I$ in (15) and using this equation, we arrive at

$$\alpha\left([x,y]\right)d(x) = \pm\alpha\left(y\right)\alpha\left([F(x),x]\right), \quad \text{for all } x, y \in I. \tag{16}$$

Writing $ry, r \in R$ instead of y in (16) and using this, we have

$$[\alpha(x), r]\alpha(y)\, d(x) = 0, \text{ for all } x, y \in I, r \in R \tag{17}$$

and so

$$[\alpha(x), r]zd(x) = 0, \text{ for all } x \in I, z \in J, r \in R,$$

where $J = \alpha(I)$, a nonzero ideal of R. Using the same arguments after (4) in the proof of Theorem 1, we get the required result.

Let R be a semiprime ring, I a nonzero ideal of R, α an automorphism on R and F be a multiplicative generalized $(\alpha, \alpha)-$derivation associated with a nonzero mapping d of R. If $F(xoy) = \pm\alpha(F(x)oy)$, for all $x, y \in I$, then $[\alpha(x), d(x)] = 0$, for all $x \in I$.

By our hypothesis, we get

$$F(xoy) = \pm\alpha(F(x)oy), \quad \text{for all } x, y \in I. \tag{18}$$

Writing yx for y in (18) and using this, we obtain that

$$\alpha(xoy)d(x) = \mp\alpha(y)\,\alpha([F(x), x]), \text{ for all } x, y \in I. \tag{19}$$

Substituting $ry, r \in R$ for y in (19) and using this equation, we find that

$$\alpha([x, r])\,\alpha(y)\, d(x) = 0, \text{ for all } x, y \in I, r \in R$$

and so

$$[\alpha(x), r]\alpha(y)\, d(x) = 0, \text{ for all } x, y \in I, r \in R.$$

This equation is same as (10) in the proof of Theorem 2. Hence, using the same arguments, we get the required result.

3 Open Problem

How to generalize these theorems for a nonzero Lie ideals of semiprime rings? Are the results remain valid if we suppose that α is an endomorphism?

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