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Characterization of Multiplicative (Generalized)-Derivation in Prime and Semiprime Near Rings

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Abstract

A mapping $\mathcal{F} : \mathcal{N} \to \mathcal{N}$ is said to be a right multiplicative (generalized)- derivation (resp. left multiplicative (generalized)-derivation) on a near ring $\mathcal{N}$ if there exists a map $f : \mathcal{N} \to \mathcal{N}$ such that $\mathcal{F}(xy) = \mathcal{F}(x)y + xf(y)$ (resp. $\mathcal{F}(xy) = f(x)y + x\mathcal{F}(y)$) holds for all $x, y \in \mathcal{N}$. The main purpose of this article is to give a characterization of these maps by proving that $\mathcal{F}$ is a left or right multiplier and f is commuting on $\mathcal{U}$ if one of the following holds: $(i)\mathcal{F}(xy) = \pm xy$; $(ii)\mathcal{F}(x)\mathcal{F}(y) = \pm xy$; $(iii)\mathcal{F}(x)\mathcal{F}(y) = \pm yx$ $(iv)\mathcal{F}(x)y = x\mathcal{F}(y)$; $(v)\mathcal{F}([x, y]) = \pm x^m(x \circ y)x^n$; $(vi)\mathcal{F}(x \circ y) = \pm x^m[x, y]x^n$; $(vii)\mathcal{F}([x, y]) = \pm x^m[x, y]x^n$ and $(viii)\mathcal{F}(x \circ y) = \pm x^m(x \circ y)x^n$, where $m \geq 0$; $n \geq 0$ and $x, y \in \mathcal{U}$, a nonzero semigroup ideal of $\mathcal{N}$. Moreover, we give some exmaples to demonstrate the restrictions imposed on the hypothesis of results which are not superfluous.

Keywords: *Prime near ring, Semiprime near-rings, Multiplicative (generalized)-derivations, Commutivity.*

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1 Introduction

Throughout the paper, $\mathcal{N}$ denotes a zero-symmetric left near ring with multiplicative centre $\mathcal{Z}$. For $x, y \in \mathcal{N}$, the symbol $[x, y]$ will denote the commutator $xy - yx$ and the symbol $x \circ y$ will denote the anticommutator $xy + yx$. A near ring $\mathcal{N}$ is said to be prime if $x, y \in \mathcal{N}$, $x\mathcal{N}y = \{0\}$ implies $x = 0$ or $y = 0$ and $\mathcal{N}$ is called semiprime if $x\mathcal{N}x = \{0\}$ implies $x = 0$. A nonempty subset $\mathcal{U}$ of $\mathcal{N}$ is called a semigroup right (resp. semigroup left) ideal of $\mathcal{N}$ if $\mathcal{U}\mathcal{N} \subseteq \mathcal{U}$ (resp. $\mathcal{N}\mathcal{U} \subseteq \mathcal{U}$) and if $\mathcal{U}$ is both a semigroup right ideal and a semigroup left ideal, it is called a semigroup ideal. Let S be a non empty subset of $\mathcal{N}$. A mapping $g : \mathcal{N} \to \mathcal{N}$ is called commuting on S if $[g(x), x] = 0$ for all $x \in S$.
The notion of multiplicative derivation in rings was introduced by Daif [8] and it was motivated by the work of Martindale [12]. A mapping (not necessarily additive) $f : \mathcal{R} \to \mathcal{R}$ is called a multiplicative derivation on a ring $\mathcal{R}$ if $f(xy) = f(x)y + xf(y)$ for all $x, y \in \mathcal{R}$. Further, Goldman and Semrl [7] gave the complete description of these mappings. Daif and Tammam El-Sayaid [9] extended multiplicative derivations to multiplicative (generalized) derivations as follows: A mapping $\mathcal{F} : \mathcal{R} \to \mathcal{R}$ is called a multiplicative generalized derivation on a ring $\mathcal{R}$ if there exists a multiplicative derivation $f : \mathcal{R} \to \mathcal{R}$ such that $\mathcal{F}(xy) = \mathcal{F}(x)y + xf(y)$ for all $x, y \in \mathcal{R}$. Recently, Dhara and Ali [3] gave a more precise definition of multiplicative (generalized)- derivation by taking f as any map.

In 1987, the study of derivations in near rings was initiated by H.E. Bell and G. Mason [5]. An additive mapping $f : \mathcal{N} \to \mathcal{N}$ is said to be a derivation on a near ring $\mathcal{N}$ if $f(xy) = f(x)y + xf(y)$ for all $x, y \in \mathcal{N}$. A mapping (not necessarily additive) is called a multiplicative derivation on a near ring $\mathcal{N}$ if $f(xy) = f(x)y + xf(y)$ for all $x, y \in \mathcal{N}$. Motivated by the definition of multiplicative (generalized)- derivation in rings, we define: A mapping $\mathcal{F} : \mathcal{N} \to \mathcal{N}$ is called a right multiplicative (generalized)- derivation if there exists a map $f : \mathcal{N} \to \mathcal{N}$ such that $\mathcal{F}(xy) = \mathcal{F}(x)y + xf(y)$ for all $x, y \in \mathcal{N}$ and $\mathcal{F} : \mathcal{N} \to \mathcal{N}$ is called a left multiplicative (generalized)- derivation if there exists a map $f : \mathcal{N} \to \mathcal{N}$ such that $\mathcal{F}(xy) = f(x)y + x\mathcal{F}(y)$ for all $x, y \in \mathcal{N}$. Every generalized derivation is a multiplicative (generalized)- derivation but converse need not be true in general.

M. Ashraf et al. [2] proved that a prime ring R must be commutative, if R satisfies any one of the following conditions: (i) $f(xy) = \pm xy$, (ii) $f(xy) = \pm yx$, (iii) $f(x)f(y) = \pm xy$, (iv) $f(x)f(y) = \pm yx$, where f is a generalized derivation of R and I is a nonzero two-sided ideal of R. Recently, E. Koç and Ö. Gölbaşı have this results for multiplicative generalized derivations associated with derivation on semiprime ring in [10]. In this article, the derivation here will be removed for the conditions given above. Instead of the derivation will be given proof for any map.

In [4], M.N. Daif and H.E. Bell proved that if R is a semiprime ring, U is a nonzero ideal of R and d is a derivation of R such that $d([x,y]) = \pm[x,y]$, for all $x, y \in U$, then $U \subset Z$. It was extend this results concerning semiprime rings to a multiplicative generalized derivations by E. Koç and Ö. Gölbaşı in [11]. These conditions have been generalized here and have been proved in the semiprime near ring.

Example 1.1 Let S be a zero-symmetric left near ring. Let us consider

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} \mid x, y, z \in S \right\}.$$

Then $\mathcal{N}$ is a zero-symmetric left near ring with regard to matrix addition and matrix multiplication. Define mappings $\mathcal{F}, f : \mathcal{N} \to \mathcal{N}$ by

$$\mathcal{F}\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & yx & 0 \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} \quad \text{and } f\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It can be verified that $\mathcal{F}$ is a right multiplicative (generalized)- derivation associated with map f but $\mathcal{F}$ is not a left multiplicative (generalized)- derivation associated with map f. However $\mathcal{F}$ is not a generalized derivation.

Example 1.2 Let S be a zero-symmetric left near ring. Let us consider

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} \mid x, y, z \in S \right\}.$$

Then $\mathcal{N}$ is a zero-symmetric left near ring with regard to matrix addition and matrix multiplication. Define mappings $\mathcal{F}, f : \mathcal{N} \to \mathcal{N}$ by

$$\mathcal{F}\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & zx & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and } f\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & y^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is straightforward to check that $\mathcal{F}$ is a right as well as left multiplicative (generalized)- derivation associated with map f. However $\mathcal{F}$ is not a generalized derivation.

2 Main results

We begin with the following lemma:

Lemma 2.1 [6, Lemma 1.3 and Lemma 1.4] Let $\mathcal{N}$ be 3-prime near ring and $\mathcal{U}$ be a nonzero semigroup ideal of $\mathcal{N}$.

(i) If $x \in \mathcal{N}$ and $x\mathcal{U} = \{0\}$ or $\mathcal{U}x = \{0\}$, then $x = 0$.

(ii) If $x, y \in \mathcal{N}$ and $x\mathcal{U}y = \{0\}$, then $x = 0$ or $y = 0$.

Theorem 2.2 *Let $\mathcal{N}$ be a semiprime near ring and $\mathcal{U}$ be a semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a right multiplicative (generalized)- derivation $\mathcal{F}$ associated with a map f such that $\mathcal{F}(xy) = xy$ for all $x, y \in \mathcal{U}$, then $\mathcal{F}$ is a multiplicative left multiplier and f is commuting on $\mathcal{U}$.*

Proof. By hypothesis

$$\mathcal{F}(xy) = xy \quad \text{for all} \;\; x, y \in \mathcal{U}. \tag{1}$$

Replacing y by yz in (1), we get

$$\mathcal{F}(xyz) = xyz$$

$$\mathcal{F}(xy)z + xyf(z) = xyz.$$

Using hypothesis, we have

$$xyf(z) = 0 \quad \text{for all} \;\; x, y, z \in \mathcal{U}. \tag{2}$$

Substituting $(f(z)r)x$ for y, where $r \in N$ in (2), we obtain

$$xf(z)rxf(z) = 0 \quad \text{for all} \;\; x, z \in \mathcal{U} \quad \text{and} \;\; r \in \mathcal{N}.$$

That is

$$xf(z)\mathcal{N}xf(z) = \{0\} \quad \text{for all} \;\; x, z \in \mathcal{U}.$$

Since $\mathcal{N}$ is a semiprime, we have

$$xf(z) = 0 \quad \text{for all} \;\; x, z \in \mathcal{U}. \tag{3}$$

Replacing x by $(f(z)x)s$, $s \in \mathcal{N}$ in (3) and then right multiplying by x, we get

$$f(z)xsf(z)x = 0 \quad \text{for all} \;\; x, z \in \mathcal{U} \quad \text{and} \;\; s \in \mathcal{N}.$$

That is

$$f(z)x\mathcal{N}f(z)x = \{0\} \quad \text{for all} \;\; x, z \in \mathcal{U}.$$

By semiprimeness of $\mathcal{N}$, we get

$$f(z)x = 0 \quad \text{for all} \;\; x, z \in \mathcal{U}. \tag{4}$$

From (3) and (4), we obtain $[f(z), x] = 0$ for all $x, z \in \mathcal{U}$. Taking $z = x$, we have $[f(x), x] = 0$ for all $x \in \mathcal{U}$. Therefore, f is commuting on $\mathcal{U}$.

Since $xf(z) = 0$ for all $x, z \in \mathcal{U}$, then $\mathcal{F}(xy) = \mathcal{F}(x)y$ for all $x, y \in \mathcal{U}$.

Theorem 2.3 *Let $\mathcal{N}$ be a semiprime near ring and $\mathcal{U}$ be a semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a right multiplicative (generalized)- derivation $\mathcal{F}$ associated with a map f such that $\mathcal{F}(x)\mathcal{F}(y) = xy$ for all $x, y \in \mathcal{U}$, then $\mathcal{F}$ is a multiplicative left multiplier and f is commuting on $\mathcal{U}$.*

Proof. By hypothesis

$$\mathcal{F}(x)\mathcal{F}(y) = xy \quad \text{for all} \;\; x, y \in \mathcal{U}. \tag{5}$$

Replacing y by yz in (5), we have

$$\mathcal{F}(x)(\mathcal{F}(y)z + yf(z)) = xyz \quad \text{for all} \;\; x, y, z \in \mathcal{U},$$

$$\mathcal{F}(x)\mathcal{F}(y)z + \mathcal{F}(x)yf(z) = xyz.$$

Using the hypothesis, we get

$$\mathcal{F}(x)yf(z) = 0 \quad \text{for all} \;\; x, y, z \in \mathcal{U}. \tag{6}$$

Replacing y by $\mathcal{F}(u)y$, $u \in \mathcal{U}$ and using the hypothesis, we obtain

$$xuyf(z) = 0 \quad \text{for all} \;\; x, y, z, u \in \mathcal{U}. \tag{7}$$

Replacing x by $uy(f(z)r)$, $r \in \mathcal{N}$ in (7), we have

$$uyf(z)\mathcal{N}uyf(z) = \{0\} \quad \text{for all} \;\; u, y, z \in \mathcal{U}.$$

By semiprimeness of $\mathcal{N}$, we get

$$uyf(z) = 0 \quad \text{for all} \;\; u, y, z \in \mathcal{U}.$$

Substituting $(f(z)s)u$, $s \in \mathcal{N}$ for y in the above equation, we find that

$$uf(z)suf(z) = 0 \quad \text{for all} \;\; u, z \in \mathcal{U} \;\; \text{and} \;\; s \in \mathcal{N}.$$

That is, $uf(z)\mathcal{N}uf(z) = \{0\}$ for all $u, z \in \mathcal{U}$. Semiprimeness of $\mathcal{N}$ yields that

$$uf(z) = 0 \quad \text{for all} \;\; u, z \in \mathcal{U}. \tag{8}$$

Replacing u by $(f(z)u)t$, $t \in \mathcal{N}$in (8), we have

$$f(z)utf(z) = 0.$$

Right multiplying by u in above expression, we obtain

$$f(z)u\mathcal{N}f(z)u = \{0\}.$$

By semiprimeness of $\mathcal{N}$, we get

$$f(z)u = 0. \tag{9}$$

Combining (8) and (9), we have $[f(z), u] = 0$ for all $u, z \in \mathcal{U}$. In particular for $z = u = x$, we find that $[f(x), x] = 0$ for all $x \in \mathcal{U}$. Therefore f is commuting on $\mathcal{U}$.
Since $uf(z) = 0$ for all $u, z \in \mathcal{U}$, then $\mathcal{F}(xy) = \mathcal{F}(x)y$ for all $x, y \in \mathcal{U}$.

Theorem 2.4 *Let $\mathcal{N}$ be a semiprime near ring and $\mathcal{U}$ be a semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a right multiplicative (generalized)- derivation $\mathcal{F}$ associated with a map f such that $\mathcal{F}(x)\mathcal{F}(y) = yx$ for all $x, y \in \mathcal{U}$, then f is commuting on $\mathcal{U}$.*

Proof. By hypothesis

$$\mathcal{F}(x)\mathcal{F}(y) = yx \quad \text{for all } x, y \in \mathcal{U}. \tag{10}$$

Replacing y by yx in(10), we have

$$\mathcal{F}(x)(\mathcal{F}(y)x + yf(x)) = yx^2,$$

$$\mathcal{F}(x)\mathcal{F}(y)x + \mathcal{F}(x)yf(x) = yx^2.$$

Using hypothesis, we get

$$\mathcal{F}(x)yf(x) = 0 \quad \text{for all } x, y \in \mathcal{U}. \tag{11}$$

Replacing y by $\mathcal{F}(z)y$ in (11), we obtain

$$\mathcal{F}(x)\mathcal{F}(z)yf(x) = 0 \quad \text{for all } x, y, z \in \mathcal{U}$$

which implies that

$$zxyf(x) = 0 \quad \text{for all } x, y, z \in \mathcal{U}. \tag{12}$$

Replacing y by $(f(x)r)zx$, $r \in \mathcal{N}$ in (12), we get

$$zxf(x)\mathcal{N}zxf(x) = \{0\}.$$

By semiprimeness of $\mathcal{N}$, we have

$$zxf(x) = 0 \quad \text{for all} \;\; x, z \in \mathcal{U}. \tag{13}$$

Substituting $x(f(x)s)$, $s \in \mathcal{N}$ for z in (13) and using the semiprimeness of $\mathcal{N}$, we get

$$xf(x) = 0 \quad \text{for all} \;\; x \in \mathcal{U}. \tag{14}$$

Replacing z by $f(x)z$ and y by yr in (12), we obtain

$$f(x)zxyrf(x) = 0 \quad \text{for all} \;\; x, y, z \in \mathcal{U}, \; r \in \mathcal{N}.$$

Right multiplying by zxy in above expression and using the semiprimeness of $\mathcal{N}$, we get

$$f(x)zxy = 0. \tag{15}$$

Replacing z by $xy(tf(x))$, $t \in \mathcal{N}$ in (15), we find that

$$f(x)xytf(x)xy = 0 \quad \text{for all} \;\; x, y \in \mathcal{U} \quad \text{and} \quad t \in \mathcal{N}.$$

By semiprimeness of $\mathcal{N}$, we have $f(x)xy = 0$. Replacing y by $u(f(x)x)$, $u \in \mathcal{N}$ and using the semiprimeness of $\mathcal{N}$, we obtain

$$f(x)x = 0 \quad \text{for all} \;\; x \in \mathcal{U}. \tag{16}$$

From (14) and (16), we get $[f(x), x] = 0 \quad \text{for all} \;\; x \in \mathcal{U}$.

Theorem 2.5 *Let $\mathcal{N}$ be a semiprime near ring and $\mathcal{U}$ be a semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a right multiplicative (generalized)- derivation $\mathcal{F}$ associated with a map f such that $\mathcal{F}(x)y = x\mathcal{F}(y)$ for all $x, y \in \mathcal{U}$, then $\mathcal{F}$ is a multiplier and f is commuting on $\mathcal{U}$.*

Proof. By hypothesis

$$\mathcal{F}(x)y = x\mathcal{F}(y) \quad \text{for all} \;\; x, y \in \mathcal{U}. \tag{17}$$

Replacing y by yz in (17), we get

$$\mathcal{F}(x)yz = x\mathcal{F}(yz) \quad \text{for all} \;\; x, y, z \in \mathcal{U}.$$

Using the hypothesis, we have

$$x\mathcal{F}(y)z = x\mathcal{F}(y)z + xyf(z)$$

which implies that

$$xyf(z) = 0 \quad \text{for all} \;\; x, y, z \in \mathcal{U}. \tag{18}$$

Replacing x by $yf(z)r$, $r \in \mathcal{N}$ in (18), we obtain

$$yf(z)\mathcal{N}yf(z) = \{0\} \quad \text{for all } \; y, z \in \mathcal{U}.$$

Since $\mathcal{N}$ is semiprime, we have

$$yf(z) = 0 \quad \text{for all } \; y, z \in \mathcal{U}. \tag{19}$$

Replacing y by $f(z)ys$, $s \in \mathcal{N}$ in (19) and then right multiplying by y, we get

$$f(z)ysf(z)y = 0 \quad \text{for all } \; y, z \in \mathcal{U} \quad \text{and } \; s \in \mathcal{N}.$$

By semiprimeness of $\mathcal{N}$, we find

$$f(z)y = 0 \quad \text{for all } \; y, z \in \mathcal{U}. \tag{20}$$

From (19) and (20), we obtain $[f(z), y] = 0$ for all $y, z \in \mathcal{U}$. In particular, $[f(x), x] = 0$ for all $x \in \mathcal{U}$. Since $yf(z) = 0$ for all $y, z \in \mathcal{U}$, then $\mathcal{F}(xy) = \mathcal{F}(x)y$ for all $x, y \in \mathcal{U}$ and by hypothesis, we have $\mathcal{F}(xy) = \mathcal{F}(x)y = x\mathcal{F}(y)$ for all $x, y \in \mathcal{U}$. Therefore $\mathcal{F}$ is a multiplier on $\mathcal{U}$.

Theorem 2.6 *Let $\mathcal{N}$ be a 3-prime near ring and $\mathcal{U}$ be a nonzero right semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a left multiplicative (generalized)- derivation $\mathcal{F}$ associated with a map f such that $\mathcal{F}([x, y]) = \pm x^m(x \circ y)x^n$ for all $x, y \in \mathcal{U}$, where $m \geq 0$ and $n \geq 0$ are non negative integers, then $\mathcal{F}$ is a multiplicative right multiplier or f is commuting on $\mathcal{U}$.*

Proof. By hypothesis

$$\mathcal{F}([x, y]) = \pm x^m(x \circ y)x^n \quad \text{for all } \; x, y \in \mathcal{U}. \tag{21}$$

Replacing y by xy in (21), we get

$$\mathcal{F}(x[x, y]) = \pm x^{m+1}(x \circ y)x^n,$$

$$f(x)[x, y] + x\mathcal{F}([x, y]) = \pm x^{m+1}(x \circ y)x^n \quad \text{for all } \; x, y \in \mathcal{U}.$$

Using hypothesis, we have

$$f(x)[x, y] = 0 \quad \text{for all } \; x, y \in \mathcal{U}.$$

which implies that

$$f(x)xy = f(x)yx \quad \text{for all } \; x, y \in \mathcal{U}. \tag{22}$$

Replacing y by $yf(x)$ in (22) and using it again, we obtain

$$f(x)y[f(x), x] = 0 \quad \text{for all } \; x, y \in \mathcal{U}.$$

By Lemma 2.1(ii), we have $f(x) = 0$ or $[f(x), x] = 0$ for all $x \in \mathcal{U}$. If $f(x) = 0$ for all $x \in \mathcal{U}$, then $\mathcal{F}(xy) = x\mathcal{F}(y)$ for all $x, y \in \mathcal{U}$. Later case yields f is commuting on $\mathcal{U}$.

Theorem 2.7 *Let $\mathcal{N}$ be a 3-prime near ring and $\mathcal{U}$ be a nonzero right semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a left multiplicative (generalized)- derivation $\mathcal{F}$ associated with a map f such that $\mathcal{F}(x \circ y) = \pm x^m[x, y]x^n$ for all $x, y \in \mathcal{U}$, where $m \geq 0$ and $n \geq 0$ are non negative integers, then $\mathcal{F}$ is a multiplicative right multiplier f is commuting on $\mathcal{U}$.*

Proof. By hypothesis

$$\mathcal{F}(x \circ y) = \pm x^m[x, y]x^n \quad \text{for all} \ \ x, y \in \mathcal{U}. \tag{23}$$

Substituting xy for y in (23), we get

$$\mathcal{F}(x(x \circ y)) = \pm x^{m+1}[x, y]x^n,$$

$$f(x)(x \circ y) + x\mathcal{F}(x \circ y) = \pm x^{m+1}[x, y]x^n \quad \text{for all} \ \ x, y \in \mathcal{U}.$$

Using hypothesis, we have

$$f(x)(x \circ y) = 0 \quad \text{for all} \ \ x, y \in \mathcal{U}$$

which implies that

$$f(x)xy = -f(x)yx \quad \text{for all} \ \ x, y \in \mathcal{U}. \tag{24}$$

Replacing y by $yf(x)$ in (24) and using it again, we obtain

$$f(x)y[f(x), x] = 0 \quad \text{for all} \ \ x, y \in \mathcal{U}. \tag{25}$$

By Lemma 2.1(ii), we have $f(x) = 0$ or $[f(x), x] = 0$ for all $x \in \mathcal{U}$. If $f(x) = 0$ for all $x \in \mathcal{U}$, then $\mathcal{F}(xy) = x\mathcal{F}(y)$ for all $x, y \in \mathcal{U}$. Later case yields that f is commuting on $\mathcal{U}$.

Theorem 2.8 *Let $\mathcal{N}$ be a 3-prime near ring and $\mathcal{U}$ be a nonzero right semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a left multiplicative (generalized)- derivation $\mathcal{F}$ associated with a map f such that $\mathcal{F}([x, y]) = \pm x^m[x, y]x^n$ for all $x, y \in \mathcal{U}$, where $m \geq 0$ and $n \geq 0$ are non negative integers, then $\mathcal{F}$ is a multiplicative right multiplier f is commuting on $\mathcal{U}$.*

Proof. By hypothesis

$$\mathcal{F}([x, y]) = \pm x^m[x, y]x^n \quad \text{for all} \ \ x, y \in \mathcal{U}. \tag{26}$$

Replacing y by xy in (26), we get

$$\mathcal{F}(x[x, y]) = \pm x^{m+1}[x, y]x^n,$$

$$f(x)[x, y] + x\mathcal{F}([x, y]) = \pm x^{m+1}[x, y]x^n \quad \text{for all} \ \ x, y \in \mathcal{U}.$$

Using hypothesis, we have

$$f(x)[x, y] = 0 \quad \text{for all} \ \ x, y \in \mathcal{U}.$$

Arguing in the similar manner as we have done in Theorem 2.6, we get the result.

Theorem 2.9 *Let $\mathcal{N}$ be a 3-prime near ring and $\mathcal{U}$ be a nonzero right semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a left multiplicative (generalized)- derivation $\mathcal{F}$ associated with a map f such that $\mathcal{F}(x \circ y) = \pm x^m(x \circ y)x^n$ for all $x, y \in \mathcal{N}$, where $m \geq 0$ and $n \geq 0$ are non negative integers, then $\mathcal{F}$ is a multiplicative right multiplier f is commuting on $\mathcal{U}$.*

Proof. By hypothesis

$$\mathcal{F}(x \circ y) = \pm x^m(x \circ y)x^n \quad \text{for all } \ x, y \in \mathcal{U}. \tag{27}$$

Replacing y by xy in (27), we get

$$\mathcal{F}(x(x \circ y)) = \pm x^{m+1}(x \circ y)x^n,$$

$$f(x)(x \circ y) + x\mathcal{F}(x \circ y) = \pm x^{m+1}(x \circ y)x^n \quad \text{for all } \ x, y \in \mathcal{U}.$$

Using hypothesis, we have

$$f(x)(x \circ y) = 0 \quad \text{for all } \ x, y \in \mathcal{U}.$$

Arguing in the similar manner as we have done in Theorem 2.7, we get the required result.

The following example shows that the semiprimeness hypothesis in Theorems 2.2, 2.3, 2.4 and 2.5 cannot be omitted.

Example 2.10 Let S be a zero-symmetric left near ring. Let us consider

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z \in S \right\}.$$

It can be easily seen that $\mathcal{N}$ is a zero-symmetric left near ring with regard to matrix addition and matrix multiplication but $\mathcal{N}$ is not semiprime. If we set

$$\mathcal{U} = \left\{ \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mid y, z \in S \right\},$$

then clearly $\mathcal{U}$ is a semigroup ideal of $\mathcal{N}$. Define mappings $\mathcal{F}, f : \mathcal{N} \to \mathcal{N}$ by

$$\mathcal{F}\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & zy \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad f\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & y & x^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is straightforward to check that $\mathcal{F}$ is a right multiplicative (generalized)-derivation associated with map f satisfying $\mathcal{F}(xy) = \pm xy$; $\mathcal{F}(x)\mathcal{F}(y) = \pm xy$; $\mathcal{F}(x)\mathcal{F}(y) = \pm yx$ and $\mathcal{F}(x)y = x\mathcal{F}(y)$ for all $x, y \in \mathcal{U}$ but $[f(x), x] \neq 0$ for all $x \in \mathcal{U}$.

The following example demonstrates that the 3-primeness hypothesis in Theorems 2.6, 2.7, 2.8 and 2.9 is not superfluous.

Example 2.11 Let S be a zero-symmetric left near ring. Let us consider

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z \in S \right\}.$$

It is easy to verify that $\mathcal{N}$ is a non 3-prime zero-symmetric left near ring with regard to matrix addition and matrix multiplication. If we set

$$\mathcal{U} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x, y \in S \right\},$$

then clearly $\mathcal{U}$ is a semigroup ideal of $\mathcal{N}$. Define mappings $\mathcal{F}, f : \mathcal{N} \to \mathcal{N}$ by

$$\mathcal{F}\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & zy \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad f\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x^2 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\mathcal{F}$ is a left multiplicative (generalized)- derivation associated with map f satisfying $\mathcal{F}([x,y]) = \pm(x \circ y)$; $\mathcal{F}(x \circ y) = \pm[x,y]$; $\mathcal{F}([x,y]) = \pm[x,y]$ and $\mathcal{F}(x \circ y) = \pm(x \circ y)$ for all $x, y \in \mathcal{U}$ but $[f(x), x] \neq 0$ for all $x \in \mathcal{U}$.

3 Open Problem

Our hypotheses are dealt with on the prime and semiprime near ring. Considering all hypotheses on the semiprime near ring gives more general results. In addition, if the commuting map is handled on the semiprime ring, many articles on this subject are transformed into different results.

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