Int. J. Open Problems Compt. Math., Vol. 13, No. 4, December 2020
ISSN 1998-6262; Copyright ©ICSRS Publication, 2020
www.i-csrs.org

Upper and Lower $^*g\alpha$-continuous functions

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Received 20 August 2020; Accepted 15 October 2020

(Communicated by M.Venkatachalam)

Abstract

In this paper we introduce $^*g\alpha$-complete accumulation points and utilizing the same, we derive some characterizations of $^*G\alpha O$-compact spaces. We also derive the properties of $^*G\alpha O$-compact spaces by utilizing 1-lower(resp. 1-upper) $^*g\alpha$-continuous multifunctions.

Keywords: *$^*g\alpha$-closed sets, $^*g\alpha$-open sets,$^*g\alpha O$-compact spaces.*

1 Introduction

R.Devi et.al[2] introduced the concepts of $^*G\alpha$-closed sets in topological spaces. Recently M.Caldas et.al[4] introduced the concepts of GO-compact spaces,1-lower, 1-upper g-continuous functions and g-continuous multifunctions by utilizing the g-closed[5] sets in topological spaces.

A space X is $^*G\alpha O$-compact if every $^*g\alpha$-open cover of X has a finite subcover. Since every open set is $^*g\alpha$-open, implies that every $^*G\alpha O$-compact space compact.

The collection of all $^*g\alpha$-closed (resp. $^*g\alpha$-open) subsets of X is denoted by $^*G\alpha C(X)$ (resp. $^*G\alpha O(X)$). We define $^*G\alpha C(X,x) = \{V \in^* G\alpha C(X)|x \in V\}$. We similarly define $^*G\alpha O(X,x)$. Let x be a point of X and U be a subset of X, then U is called a $^*g\alpha$-neighborhood of x in X[3] if there exists a $^*g\alpha$-open set V of X such that $x \in V \subset U$.

In this paper we introduce $^*g\alpha$-complete accumulation points and utilizing the same, we derive some characterizations of $^*G\alpha O$-compact spaces. By introducing the notion of 1-lower(rep. 1-upper) $^*g\alpha$-continuous, we investigate some of the properties of $^*G\alpha O$-compact spaces. We also derive the properties of $^*G\alpha O$-compact spaces by utilizing 1-lower(resp. 1-upper) $^*g\alpha$-continuous multifunctions.

Throught the present paper (X,τ) and (Y,σ)(or simply X and Y) denote the topological spaces. Let A be a subset of X. We denote the closure(resp. $^*g\alpha$-closure) and interior(resp. $^*g\alpha$-interior) of a set by $cl(A)(resp.^*g\alpha - cl(A))$ and $int(A)(resp.^*g\alpha - int(A))$.

2 Preliminaries

Definition 2.1. *A subset A of a space (X,τ) is called*

(i). α-open set [7] if $A \subseteq int(cl(int(A)))$ and an α-closed set if $cl(int(cl(A))) \subseteq A$,

(ii). a generalized α-closed (briefly $g\alpha$-closed) set [6] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in (X,τ),

(iii). a $^*g\alpha$-closed set [2] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $g\alpha$-open in (X,τ) and

(iv). a $^*g\alpha$-open set [2] if $U \subseteq int(A)$ whenever $U \subseteq A$ and U is $g\alpha$-closed in (X,τ).

Definition 2.2. *(i). A net $\xi = \{x_\alpha | \alpha \in \Lambda\}^* g\alpha$-accumulates at a point $x \in X$ if the net is frequently in every $U \in^* g\alpha O(X,x)$, i.e for each $U \in^* g\alpha O(X,x)$ and for each $\alpha_0 \in \lambda$, there is some $\alpha \geq \alpha_0$ such that $x_\alpha \in U$.*

(ii). A net $\xi^* g\alpha$-converges to a point x of X f it is eventually in every $U \in^* g\alpha O(X,x)$.

(iii). The filter base $\theta = \{F_\alpha | \alpha \in \Gamma\}^* g\alpha$-accumulates at a point $x \in X$ if $x \in \bigcap^*_{\alpha\in\Gamma} g\alpha - cl(F_\alpha)$.

(iv). A subset $S \subset X$, a $^*g\alpha$-cover of S is a family of $^*g\alpha$-open subsets U_α of X for each $\alpha \in I$ such that $S \subset U_{\alpha\in I} U_\alpha$.

(v). A filterbase $\theta = \{F_\alpha | \alpha \in \Gamma\}^* g\alpha$-converges to a point x in X if for each $U \in^* g\alpha O(X,x)$, there exists an F_α in θ such that $F_\alpha \subset U$.

Definition 2.3. *A function $f : X \rightarrow Y$ is said to be $^*g\alpha$-continuous[2] if the inverse image of each open set is $^*g\alpha$-open in X.*

Recall that a multifunction(also called multivalued function[1]) F on a set X into a set Y, is denoted by $F : X \rightarrow Y$ a relation on X into Y, i.e $F \subset X \times Y$.

Let $F : X \rightarrow Y$ be a multifunction. The upper and lower inverse of a set V of Y are denoted by $F^+(V)$ and $F^-(V)$.

$F^+(V) = \{x \in X | F(x) \subset V\}$

$F^-(V) = \{x \in X | F(x) \cap V \neq \phi\}$

3 Main Results

In the following sub section we introduce the definitions of $^*g\alpha$-complete accumulation points and $^*g\alpha$-adherent points and the results are derived.

3.1 Properties of $^*g\alpha$-compact spaces

Definition 3.1. *A point x in a space X is said to be a $^*g\alpha$-complete accumulation point of a subset S of X if $Card(S \cap U) = Card(S)$ for each $U \in {}^*G\alpha O(X, x)$, where $Card(S)$ denotes the cardinality of S.*

Definition 3.2. *In a topological space X, a point x is said to be a $^*g\alpha$-adherent point of a filterbase Θ on X if it lies in the $^*g\alpha$-closure of all sets of Θ.*

Theorem 3.3. *If a space X is $^*g\alpha$-compact then each infinite subset of X has a $^*g\alpha$-complete accumulation point.*

Proof. *Let the space X be $^*g\alpha$-compact and S an infinite subset of X. Let K be the set of points x in X which are not $^*g\alpha$-complete accumulation points of S. Now it is obvious that for each point x in K, we find $U(x) \in {}^*G\alpha O(X, x)$ such that $Card(S \cap U(x)) \neq Card(S)$. If K is the whole space X, then $\Theta = \{U(x)|x \in X\}$ is a $^*g\alpha$-cover of X. By hyphothesis, X is $^*g\alpha$-compact, so there exists a finite subcover $\psi = \{U(x_i)\}$, where $i = 1, 2, 3, .., n$ such that $S \subset \bigcup \{U(x_i) \cap S|i = 1, 2, 3, ..., n\}$. Then $Card(S) = max \{Card(U(x_i) \cap S)|i = 1, 2, 3, ..., n\}$ which is not satisfied with our assumption. This implies that S has a $^*g\alpha$-complete accumulation point.*

Theorem 3.4. *For a space X the following statements are equivalent:*

(i). X is $^*g\alpha$-compact.

(ii). Every net in X with a well-ordered directed set as its domain $^*g\alpha$-accumulates to some point of X.

Proof
(i)⇒(ii): *Suppose that* X *is* $^*g\alpha$*-compact and* $\xi = \{x_\alpha | \alpha \in \wedge\}$ *a net with a well-ordered directed set* $\wedge$ *as domain. Assume that* ξ *has no* $^*g\alpha$*-adherent point in* X*. Then for each point* x *in* X*, there exists* $V(x) \in {}^*G\alpha O(X,x)$ *and an* $\alpha(x) \in \wedge$ *such that* $V(x) \cap \{x_\alpha | \alpha \geq \alpha(x)\} = \phi$*. This implies that* $\{x_\alpha | \alpha \geq \alpha(x)\}$ *is a subset of* $X \backslash V(x)$*. Then the collection* $C = \{V(x) | x \in X\}$ *is a* $^*g\alpha$*-cover of X. By hyphothesis,* X *is* $^*g\alpha$*-compact and so* C *has a finite subfamily* $\{V(x_i)\}$*, where* $i = 1, 2, 3, .., n$ *such that* $X = \bigcup\{V(x_i)\}$*. Suppose that the corresponding elements of* $\wedge$ *be* $\{\alpha(x_i)\}$*, where* $i = 1, 2, 3, .., n$*. Since* $\wedge$ *is well ordered and* $\{\alpha(x_i)\}$ *exists. Suppose it is* $\{\alpha(x_l)\}$*. Then for* $\gamma \geq \{\alpha(x_l)\}$*, we have* $\{x_\delta | \delta \geq \gamma\} \subset \cap(X \backslash V(x_i)) = X \backslash \cup V(x_i) = \phi$ *which is impossible. This shows that* ξ *has atleast one* $^*g\alpha$*-adherent point in* X*.*
(ii)⇒(i): *Now it is enough to prove that each infinite subset has a* $^*g\alpha$*-complete accumulation point by using Theorem 3.3. Suppose that* $S \subset X$ *is an infinite subset of* X*. According to Zorn's Lemma, the infinite set* S *can be well-ordered. This means that we can assume* S *to be a net with a domain which is a well-ordered index set. It follows that* S *has a* $^*g\alpha$*-adherent point* z*. Therefore* z *is a* $^*g\alpha$*-complete accumulation point of* S*. This follows that* X *is* $^*g\alpha$*-compact.*

In the following sub section we introduce the definitions of upper $^*g\alpha$-continuous functions and lower $^*g\alpha$-continuous functions and their properties were derived.

3.2 Upper and Lower $^*g\alpha$-continuous functions

Definition 3.5. *A multifunction* $F : X \to Y$ *is said to be lower(resp. upper)* $^*g\alpha$*-continuous if* $X \backslash F^-(S)$*(resp.* $F^-(S)$*) is* $^*g\alpha$*-closed in* X *for each open(resp. closed) set* S *in* Y*.*

For the following two lemmas we shall assume that if $^*g\alpha\text{-}cl(A) = A$, then A is $^*g\alpha$-closed.

Lemma 3.6. *For a multifunction* $F : X \to Y$*, the following statements are equivalent:*

(i) F *is lower* $^*g\alpha$*-continuous.*

(ii) If $x \in F^-(U)$ *for a point* x *in* X *and an open set* $U \subset Y$*, then* $V \subset F^-(U)$ *for some* $V \in {}^*G\alpha O(X)$*.*

(iii) If $x \notin F^+(D)$ *for a point* x *in* X *and a closed set* $D \subset Y$*, then* $F^+(D) \subset K$ *for some* $^*g\alpha$*-closed set* K *with* $x \notin K$*.*

(iv) $F^-(U) \in {}^*G\alpha O(X)$ *for each open set* $U \subset Y$*.*

Proof.
(i) ⇒ (iv): *Let U be any open set in Y. By (i), $X - F^-(U)$ is $^*g\alpha$-closed in X and hence $F^-(U) \in {}^*G\alpha O(X)$.*
(iv) ⇒ (ii): *Let U be any open set of Y and $x \in F^-(U)$, by (iv), $F^-(U) \in {}^*G\alpha O(X)$. Put $V = F^-(U)$. Then $V \in {}^*G\alpha O(X)$ and $V \subset F^-(U)$.*
(ii) ⇒ (iii): *Let D be closed in Y and $x \notin F^+(D)$. Then $Y - D$ is open in Y and $x \in X - F^+(D) = F^-(X - D)$. Therefore, there exists $V \in {}^*G\alpha O(X)$ such that $V \subset F^-(U)$. Now, put $K = X - V$, then $x \notin K$, K is $^*g\alpha$-closed and $K = X - V \supset X - F^-(Y - D) = F^+(D)$.*
(iii) ⇒ (i): *We show that $F^+(H)$ is $^*g\alpha$-closed for any closed set H of Y. Let H be any closed set and $x \notin F^+(H)$. By (iii), there exists a $^*g\alpha$-closed set K such that $x \notin K$ and $F^+(H) \subset K$; hence $F^+(H) \subset {}^*g\alpha\text{-}cl(F^+(H)) \subset K$. Since $x \notin K$, we have $x \notin {}^*g\alpha\text{-}cl(F^+(H))$. This implies that $^*g\alpha\text{-}cl(F^+(H)) \subset F^+(H)$. In general, we have $F^+(H) \subset {}^*g\alpha\text{-}cl(F^+(H))$ and hence $F^+(H) = {}^*g\alpha\text{-}cl(F^+(H)$ is $^*g\alpha$-closed for any closed set H of Y.*

Lemma 3.7. *For a multifunction $F : X \to Y$, the following statements are equivalent:*

(i) *F is upper $^*g\alpha$-continuous.*

(ii) *If $x \in F^+(V)$ for a point x in X and an open set $V \subset Y$, then $F(U) \subset V$ for some $U \in {}^*G\alpha O(X)$.*

(iii) *If $x \notin F^-(D)$ for a point x in X and a closed set $D \subset Y$, then $F^-(D) \subset K$ for some $^*g\alpha$-closed set K with $x \notin K$.*

(iv) *$F^+(U) \in {}^*G\alpha O(X)$ for each open set $U \subset Y$.*

Proof.
(i) ⇒ (iv): *Let U be any open set in Y. Then, $Y - U$ is closed. By (i), $F^-(Y - U) = X - F^+(U)$ is $^*g\alpha$-closed in X and hence $F^+(U) \in {}^*G\alpha O(X)$.*
(iv) ⇒ (ii): *Let V be any open set of Y and $x \in F^+(V)$. By (iv), $F^+(V) \in {}^*G\alpha O(X)$. Put $U = F^+(V)$. Then $U \in {}^*G\alpha O(X)$ and $F(U) \subset V$.*
(ii) ⇒ (iii): *Let D be closed in Y and $x \notin F^-(D)$. Then $Y - D$ is open in Y and $x \in X - F^-(D) = F^+(Y - D)$. By (ii), there exists $U \in {}^*G\alpha O(X)$ such that $F(U) \subset Y - D$. Now, put $K = X - U$, then $x \notin K$, K is $^*g\alpha$-closed and $K = X - U \supset X - F^+(Y - D) = F^-(D)$.*
(iii) ⇒ (i): *We show that $F^-(H)$ is $^*g\alpha$-closed for any closed set H of Y. Let H be any closed set and $x \notin F^-(H)$. By (iii), there exists a $^*g\alpha$-closed set K such that $x \notin K$ and $F^-(H) \subset K$; hence $F^-(H) \subset {}^*g\alpha\text{-}cl(F^-(H)) \subset K$. Since $x \notin K$, we have $x \notin {}^*g\alpha\text{-}cl(F^-(H))$. This implies that $^*g\alpha\text{-}cl(F^-(H)) \subset F^-(H)$. In general, we have $F^-(H) \subset {}^*g\alpha\text{-}cl(F^-(H))$ and hence $F^-(H) = {}^*g\alpha\text{-}cl(F^-(H)$ is $^*g\alpha$-closed for any closed set H of Y.*

Lemma 3.8. *The following statements are equivalent for a space X:*

(i) X is $^*g\alpha$-compact.

(ii) Every lower $^*g\alpha$-continuous multifunction from X into the closed sets of a space assumes a minimal value with respect to the set inclusion reation.

Proof.
(i) ⇒ (ii): *Suppose that F is lower $^*g\alpha$-continuous multifunction from X into the closed subsets of a space Y. We denote the poset of all closed subsets of Y with the set inclusion relation "$\subseteq$" by $\wedge$. Now we show that $F : X \to \wedge$ is a lower $^*g\alpha$-continuous function. We will show that $N = F^-(\{S \subset Y | S \in \wedge \text{ and } S \subseteq C\})$ is $^*g\alpha$-closed in X for each closed set C of Y. Let $z \notin N$, then $F(z) \neq S$ for every closed set S of Y. It is obvious that $z \in F^-(Y\backslash C)$, where $Y\backslash C$ is open in Y. By Lemma 3.2(ii), we have $W \subset F^-(Y\backslash C)$ for some $W \in {}^*G\alpha O(z)$. Hence $F(w) \cap (Y\backslash C) \neq \phi$ for each w in W. So for each w in W, $F(w)\backslash C \neq \phi$. Consequently, $F(w)\backslash S \neq \phi$ for every closed subset S of Y for which $S \subseteq C$. We consider that $W \cap N = \phi$. This means that N is $^*g\alpha$-closed. Thus we observe that F assumes a minimal value.*
(ii) ⇒ (i): *Suppose that X is not $^*g\alpha$-compact. It follows that we have a net $\{x_i | i \in \wedge\}$, where $\wedge$ is a well ordered set with no $^*g\alpha$-accumulation points. We give $\wedge$ the order topology. Let $M_j = {}^*g\alpha\text{-}cl\,\{x_i | i \geq j\}$ for every j in $\wedge$. We establish a multifunction $F : X \to \wedge$ where $F(x) = \{i \in \wedge | i \geq j_x\}$, j_x is the first element of all those j's for which $x \notin M_j$. Since $\wedge$ has the order topology, $F(x)$ is closed. By the fact that $\{j_x | x \in X\}$ has no greatest element in $\wedge$, then F does not assume any minimal value with respect to set inclusion. We now show that $F^-(U) \in {}^*G\alpha O(X)$ for every open set U in $\wedge$. If $U = \wedge$, then $F^-(U) = X$ which is $^*g\alpha$-open. Suppose that $U \subset \wedge$ and $Z \in F^-(U)$. It follows that $F(z) \cap U \neq \phi$. Suppose $j \in F(z) \cap U$. This means that $j \in U$ and $j \in F(z) = \{i \in \wedge | i \geq j_x\}$. Therefore $M_j \geq M_{j_x}$. Since $z \notin M_{j_x}$, then $z \notin M_j$. There exists $W \in {}^*G\alpha O(z)$ such that $W \cap \{x_i | i \in \wedge\} = \phi$. This means that $W \cap M_j = \phi$. Let $w \in W$. Since $W \cap M_j = \phi$, it follows that $w \notin M_j$ and since j_w is the first element for which $w \notin M_j$, then $j_w \leq j$. Therefore $j \in \{i \in \wedge | i \geq j_w\} = F(w)$. By the fact that $j \in U$, then $j \in F(w) \cap U$. It follows that $F(w) \cap U \neq \phi$ and therefore $w \in F^-(U)$. So we have $W \subset F^-(U)$ and thus $z \in W \subset F^-(U)$. Therefore $F^-(U)$ is $^*g\alpha$-open. This shows that F is lower $^*g\alpha$-continuous which contradicts the hyphothesis. So the space X is $^*g\alpha$-compact.*

Theorem 3.9. *Suppose that $F : X \to Y$ is a multifunction from a $^*g\alpha$-compact domain X into itself. Let $F(S)$ be $^*g\alpha$-closed for S being a $^*g\alpha$-closed set in X. If $F(x) \neq \phi$ for every point $x \in X$, then there exists a non-empty $^*g\alpha$-closed set C of X such that $F(C) = C$.*
Proof. *Let $\wedge = \{S \subset X | S \neq \phi, S \in {}^*G\alpha C(X) \text{ and } F(S) \subset S\}$. It is evident*

that x belongs to $\wedge$. Therefore $\wedge \neq \phi$ and also it is partially ordered by set inclusion. Suppose that $\{S_\gamma\}$ is a chain in $\wedge$. Then $F(S_\gamma) \subset S$ for each γ. By the fact that the domain is $^*g\alpha$-compact, $S = \cap_\gamma S_\gamma \neq \phi$ and also $S \in {}^*G\alpha C(X)$. Moreover, $F(S) \subset F(S_\gamma) \subset S_\gamma$ for each γ. It follows that $F(S) \subset S_\gamma$. Hence $S \in \wedge$ and $S = inf\,\{S_\gamma\}$. It follows from Zorn's lemma that $\wedge$ has a minimal element C. Therefore $C \in {}^*G\alpha C(X)$ and $F(C) \subset C$. Since C is the minimal element of $\wedge$, we have $F(C) = C$.

4 Conclusion

In this paper we have introduced $^*g\alpha$-complete accumulation points and utilizing the same, we derived some characterizations of $^*G\alpha O$-compact spaces. We also derived the properties of $^*G\alpha O$-compact spaces by utilizing 1-lower(resp. 1-upper) $^*g\alpha$-continuous multifunctions.

5 Open Problem

As a motivation of this work, it can be extended to study the concepts of various types of connected spaces, normal spaces and Hausdorff spaces by using these types of multi functions defined by using the existing sets.

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