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On The Connections Between Padovan Numbers and Padovan p-Numbers

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Abstract

In this paper, we define the Padovan-Padovan p-sequence which are obtained from the characteristic polynomials of Padovan and Padovan p-sequences. Then using the roots of characteristic polynomial of the Padovan-Padovan p-sequence, we produce the Binet formula for the Padovan-Padovan p-numbers. In addition, we provide a new combinatorial representation of the Padovan-Padovan p-numbers by the aid of the nth power of the generating matrix of Padovan-Padovan p-sequence. Furthermore, we obtain an exponential representation of the Padovan-Padovan p-numbers and we develop relationships between the Padovan-Padovan p-numbers and their permanent, determinant, and sums of certain matrices. Finally, we give some determinantal and permanental representations of the Padovan-Padovan p-numbers by using various Hessenberg matrices.

Keywords: *Matrix, Padovan sequence, Padovan p-sequence, Representation.*

AMS Classification: *11K31; 11C20; 15A15.*

1 Introduction

It is well-known that the Padovan sequence $\{P(n)\}$ is defined by the following recurrence relation:

$$P(n) = P(n-2) + P(n-3)$$

for $n \geq 3$, with initial conditions $P(0) = P(1) = P(2) = 1$.

The Padovan p-numbers $\{Pap(n)\}$ for any given $p\,(p = 2, 3, 4, \ldots)$ is defined [5] by the following homogeneous linear recurrence relation:

$$Pap(n + p + 2) = Pap(n + p) + Pap(n) \tag{1}$$

for $n \geq 1$, with initial conditions $Pap(1) = Pap(2) = \cdots = Pap(p) = 0$, $Pap(p + 1) = 1$ and $Pap(p + 2) = 0$. When $p = 1$ in (1), the Padovan p-numbers $\{Pap(n)\}$ is reduced to the usual Padovan sequence $\{P(n)\}$.

It is easy to see that the characteristic polynomials of Padovan sequence and Padovan p-sequence are $h_1(x) = x^3 - x - 1$ and $h_2(x) = x^{p+2} - x^p - 1$, respectively. We use these in the next section.

A lower Hessenberg matrix $H_n = (h_{ij})$ is an $n \times n$ matrix, where $h_{ij} = 0$, whenever $j > i + 1$ and $a_{j,j+1} \neq 0$ for some j. This paper we will refer to the following lower Hessenberg matrix, for $n \geq 2$

$$H_n = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{bmatrix}.$$

Let the $(n + k)$th term of a sequence be defined recursively by a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1}$$

in which $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [13], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix method as follows:

Let the matrix A be defined by

$$A = [a_{i,j}]_{k\times k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & & c_{k-2} & c_{k-1} \end{bmatrix},$$

then

$$A^{n}\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}$$

for $n \geq 0$.

Many of the numbers obtained by using homogeneous linear recurrence relations and their miscellaneous properties have been studied by some authors; see for example, [1, 4, 8, 9, 11, 12, 20, 21]. In [6, 7, 10, 14, 15, 16, 17, 22, 23, 24, 25], the authors defined some linear recurrence sequences and gave their various properties by matrix methods. In the present paper, we discuss connections between the Padovan and Padovan p-numbers. Firstly, we define the Padovan-Padovan p-sequence and then we study recurrence relation among this sequence, Padovan and Padovan p-sequences. In addition, we obtain their generating matrices, Binet formulas, permanental, determinantal, combinatorial, exponential representations, and we derive a formula for the sums of Padovan-Padovan p-numbers. Finally, we give some of the determinantal and permanental representations of Padovan-Padovan p-numbers by using various Hessenberg matrices.

2 The Main Results

We define the Padovan-Padovan p-sequence $\left\{Pa_n^{P,p}\right\}$ by the following homogeneous linear recurrence relation for any given $p\,(4,5,6,\ldots)$ and $n \geq 0$

$$Pa_{n+p+5}^{P,p} = 2Pa_{n+p+3}^{P,p} + Pa_{n+p+2}^{P,p} - Pa_{n+p+1}^{P,p} - Pa_{n+p}^{P,p} + Pa_{n+3}^{P,p} - Pa_{n+1}^{P,p} - Pa_{n}^{P,p} \quad (2)$$

with initial conditions $Pa_0^{P,p} = \cdots = Pa_{p+3}^{P,p} = 0$ and $Pa_{p+4}^{P,p} = 1$.

By the recurrence relation (2), we have

$$\begin{bmatrix} Pa_{n+p+5}^{P,p} \\ Pa_{n+p+4}^{P,p} \\ Pa_{n+p+3}^{P,p} \\ \vdots \\ Pa_{n+1}^{P,p} \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & -1 & -1 & 0 & \cdots & 0 & 1 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} Pa_{n+p+4}^{P,p} \\ Pa_{n+p+3}^{P,p} \\ Pa_{n+p+2}^{P,p} \\ \vdots \\ Pa_{n}^{P,p} \end{bmatrix}$$

for the Padovan-Padovan p-sequence $\left\{Pa_n^{P,p}\right\}$. We can write the following companion matrix:

$$C_p = \left[\begin{array}{cccccccccccc}
0 & 2 & 1 & -1 & -1 & 0 & \cdots & 0 & 1 & 0 & -1 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0
\end{array}\right]_{(p+5)\times(p+5)}.$$

The companion matrix $C_p = [c_{i,j}]_{(p+5)\times(p+5)}$ is said to be the Padovan-Padovan p-matrix. For more details on the companion type matrices, see [18, 19]. It can be readily established by mathematical induction that for $p \geq 4$ and $n \geq p+4$

$$(C_p)^n = \left[\begin{array}{ccc}
Pa^{P,p}_{n+p+4} & Pa^{P,p}_{n+p+5} & -Pa^{P,p}_{n+p+4} + Pa^{P,p}_{n+p+3} + Pap\,(n+p+1) + Pap\,(n+1) \\
Pa^{P,p}_{n+p+3} & Pa^{P,p}_{n+p+4} & -Pa^{P,p}_{n+p+3} + Pa^{P,p}_{n+p+2} + Pap\,(n+p) + Pap\,(n) \\
Pa^{P,p}_{n+p+2} & Pa^{P,p}_{n+p+3} & -Pa^{P,p}_{n+p+2} + Pa^{P,p}_{n+p+1} + Pap\,(n+p-1) + Pap\,(n-1) \\
\vdots & \vdots & \vdots \\
Pa^{P,p}_{n+1} & Pa^{P,p}_{n+2} & -Pa^{P,p}_{n+1} + Pa^{P,p}_{n} + Pap\,(n-2) + Pap\,(n-p-2) \\
Pa^{P,p}_{n} & Pa^{P,p}_{n+1} & -Pa^{P,p}_{n} + Pa^{P,p}_{n-1} + Pap\,(n-3) + Pap\,(n-p-3)
\end{array}\right.$$

$$\left.\begin{array}{ccccc}
-Pa^{P,p}_{n+p+5} + Pap\,(n+p+4) & -Pa^{P,p}_{n+p+3} + Pap\,(n+3) & Pap\,(n+4) & \cdots & \\
-Pa^{P,p}_{n+p+4} + Pap\,(n+p+3) & -Pa^{P,p}_{n+p+2} + Pap\,(n+2) & Pap\,(n+3) & \cdots & \\
-Pa^{P,p}_{n+p+3} + Pap\,(n+p+2) & -Pa^{P,p}_{n+p+1} + Pap\,(n+1) & Pap\,(n+2) & \cdots & C_p^* \\
\vdots & \vdots & \vdots & & \\
-Pa^{P,p}_{n+2} + Pap\,(n+1) & -Pa^{P,p}_{n} + Pap\,(n-p) & Pap\,(n-p+1) & \cdots & \\
-Pa^{P,p}_{n+1} + Pap\,(n) & -Pa^{P,p}_{n-1} + Pap\,(n-p-1) & Pap\,(n-p) & \cdots &
\end{array}\right],$$

where

$$C_p^* = \left[\begin{array}{cccc}
Pap\,(n+p) & -Pa^{P,p}_{n+p+4} + Pap\,(n+p+1) & -Pa^{P,p}_{n+p+5} + Pap\,(n+p+2) & -Pa^{P,p}_{n+p+3} \\
Pap\,(n+p-1) & -Pa^{P,p}_{n+p+3} + Pap\,(n+p) & -Pa^{P,p}_{n+p+4} + Pap\,(n+p+1) & -Pa^{P,p}_{n+p+2} \\
Pap\,(n+p-2) & -Pa^{P,p}_{n+p+2} + Pap\,(n+p-1) & -Pa^{P,p}_{n+p+3} + Pap\,(n+p) & -Pa^{P,p}_{n+p+1} \\
\vdots & \vdots & \vdots & \vdots \\
Pap\,(n-3) & -Pa^{P,p}_{n+1} + Pap\,(n-2) & -Pa^{P,p}_{n+2} + Pap\,(n-1) & -Pa^{P,p}_{n} \\
Pap\,(n-4) & -Pa^{P,p}_{n} + Pap\,(n-3) & -Pa^{P,p}_{n+1} + Pap\,(n-2) & -Pa^{P,p}_{n-1}
\end{array}\right].$$

We easily derive that $\det C_p = (-1)^{p+1}$. In [5], Deveci and Karaduman gave a Binet formula for the Padovan p-numbers by matrix method. Now we concentrate on finding another Binet formula for the Padovan-Padovan p-numbers by the aid of the matrix $(C_p)^n$.

Lemma 2.1. *The characteristic equation of all the Padovan-Padovan p-numbers* $x^{p+5} - 2x^{p+3} - x^{p+2} + x^{p+1} + x^p - x^3 + x + 1 = 0$ *does not have multiple roots for* $p \geq 4$.

Proof. It is clear that $x^{p+5} - 2x^{p+3} - x^{p+2} + x^{p+1} + x^p - x^3 + x + 1 = \left(x^{p+2} - x^p - 1\right)\left(x^3 - x - 1\right)$. In [5], it was shown that the equation $x^{p+2} - x^p - 1 = 0$ does not have multiple roots for $p \geq 2$. It is easy to see that the roots of the equation $x^3 - x - 1 = 0$ are

$$\alpha = \frac{1}{3}\sqrt[3]{\frac{27}{2} - \frac{3\sqrt{69}}{2}} + \frac{\sqrt[3]{\frac{1}{2}\left(9 + \sqrt{69}\right)}}{3^{\frac{2}{3}}},$$

$$\beta = -\frac{1}{6}\left(1 - i\sqrt{3}\right)\sqrt[3]{\frac{27}{2} - \frac{3\sqrt{69}}{2}} - \frac{\left(1 + i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}\left(9 + \sqrt{69}\right)}}{2 \times 3^{\frac{2}{3}}}$$

and

$$\gamma = -\frac{1}{6}\left(1 + i\sqrt{3}\right)\sqrt[3]{\frac{27}{2} - \frac{3\sqrt{69}}{2}} - \frac{\left(1 - i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}\left(9 + \sqrt{69}\right)}}{2 \times 3^{\frac{2}{3}}}$$

Since $(\alpha)^{p+2} - (\alpha)^p - 1 \neq 0$, $(\beta)^{p+2} - (\beta)^p - 1 \neq 0$ and $(\gamma)^{p+2} - (\gamma)^p - 1 \neq 0$, the equation $x^{p+5} - 2x^{p+3} - x^{p+2} + x^{p+1} + x^p - x^3 + x + 1 = 0$ does not have multiple roots for $p \geq 4$. □

If $x_1, x_2, \ldots, x_{p+5}$ are roots of the equation $x^{p+5} - 2x^{p+3} - x^{p+2} + x^{p+1} + x^p - x^3 + x + 1 = 0$, then by Lemma 2.1, it is known that $x_1, x_2, \ldots, x_{p+5}$ are distinct. Let V_p be a $(p+5) \times (p+5)$ Vandermonde matrix as follows:

$$V_p = \begin{bmatrix} (x_1)^{p+4} & (x_2)^{p+4} & \cdots & (x_{p+5})^{p+4} \\ (x_1)^{p+3} & (x_2)^{p+3} & \cdots & (x_{p+5})^{p+3} \\ \vdots & \vdots & & \vdots \\ x_1 & x_2 & \cdots & x_{p+5} \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Assume that $V_p(i,j)$ is a $(p+5) \times (p+5)$ matrix derived from the Vandermonde matrix V_p by replacing the j^{th} column of V_p by $W_p(i)$, where, $W_p(i)$ is a $(p+5) \times 1$ matrix as follows:

$$W_p(i) = \begin{bmatrix} (x_1)^{n+p+5-i} \\ (x_2)^{n+p+5-i} \\ \vdots \\ (x_{p+3})^{n+p+5-i} \end{bmatrix}$$

Theorem 2.2. *Let p be a positive integer such that $p \geq 4$ and let $(C_p)^n = c_{i,j}^{(p,n)}$ for $n \geq 1$, then*

$$c_{i,j}^{(p,n)} = \frac{\det V_p(i,j)}{\det V_p}.$$

Proof. Since the equation $x^{p+5} - 2x^{p+3} - x^{p+2} + x^{p+1} + x^p - x^3 + x + 1 = 0$ does not have multiple roots for $p \geq 4$, the eigenvalues of the Padovan-Padovan p-matrix C_p are distinct. Then, it is clear that C_p is diagonalizable. Let $A_p = diag(x_1, x_2, \ldots, x_{p+5})$, then we may write $C_p V_p = V_p A_p$. Since the matrix V_p is invertible, we obtain the equation $(V_p)^{-1} C_p V_p = A_p$. Therefore, C_p is similar to A_p; hence, $(C_p)^n V_p = V_p (A_p)^n$ for $n \geq 1$. So we have the following linear system of equations:

$$\begin{cases} c_{i,1}^{(p,n)}(x_1)^{p+2} + c_{i,2}^{(p,n)}(x_1)^{p+1} + \cdots + c_{i,p+5}^{(p,n)} = (x_1)^{n+p+5-i} \\ c_{i,1}^{(p,n)}(x_2)^{p+2} + c_{i,2}^{(p,n)}(x_2)^{p+1} + \cdots + c_{i,p+5}^{(p,n)} = (x_2)^{n+p+5-i} \\ \vdots \\ c_{i,1}^{(p,n)}(x_{p+5})^{p+2} + c_{i,2}^{(p,n)}(x_{p+5})^{p+1} + \cdots + c_{i,p+5}^{(p,n)} = (x_{p+5})^{n+p+5-i} \end{cases}.$$

Then we conclude that

$$c_{i,j}^{(p,n)} = \frac{\det V_p(i,j)}{\det V_p}$$

for each $i, j = 1, 2, \ldots, p+5$. □

Thus by Theorem 2.2 and the matrix $(C_p)^n$, we have the following useful result for the Padovan-Padovan p-numbers.

Corollary 2.3. *Let p be a positive integer such that $p \geq 4$ and let $Pa_n^{P,p}$ be the nth element of Padovan-Padovan p-sequence, then*

$$Pa_n^{P,p} = \frac{\det V_p(p+5,1)}{\det V_p}$$

and

$$Pa_n^{P,p} = -\frac{\det V_p(p+4,p+5)}{\det V_p}$$

for $n \geq 1$.

The generating function of the Padovan-Padovan p-sequence $\{Pa_n^{P,p}\}$ is given by

$$h(x) = \frac{x^{p+4}}{1 - 2x^2 - x^3 + x^4 + x^5 - x^{p+2} + x^{p+4} + x^{p+5}},$$

where $p \geq 4$. Then we can give an exponential representation for the Padovan-Padovan p-numbers by the aid of the generating function with the following Theorem.

Theorem 2.4. *The Padovan-Padovan p-sequence* $\{Pa_n^{P,p}\}$ *have the following exponential representation:*

$$h(x) = x^{p+4} \exp\left(\sum_{i=1}^{\infty} \frac{(x^2)^i}{i} \left(2 + x - x^2 - x^3 + x^p - x^{p+2} - x^{p+3}\right)^i\right),$$

where $p \geq 3$.

Proof. Since

$$\ln h(x) = \ln x^{p+4} - \ln\left(1 - 2x^2 - x^3 + x^4 + x^5 - x^{p+2} + x^{p+4} + x^{p+5}\right)$$

and

$$\begin{aligned} -\ln\left(1 - 2x^2 - x^3 + x^4 + x^5 - x^{p+2} + x^{p+4} + x^{p+5}\right) = & -[-x^2\left(2 + x - x^2 - x^3 + x^p - x^{p+2} - x^{p+3}\right) - \\ & \frac{1}{2}x^4\left(2 + x - x^2 - x^3 + x^p - x^{p+2} - x^{p+3}\right)^2 - \cdots \\ & -\frac{1}{i}x^{2i}\left(2 + x - x^2 - x^3 + x^p - x^{p+2} - x^{p+3}\right)^i - \cdots] \end{aligned}$$

it is clear that

$$h(x) = x^{p+4} \exp\left(\sum_{i=1}^{\infty} \frac{(x^2)^i}{i} \left(2 + x - x^2 - x^3 + x^p - x^{p+2} - x^{p+3}\right)^i\right)$$

Thus we have the conclusion. □

Let $K(k_1, k_2, \ldots, k_v)$ be a $v \times v$ companion matrix as follows:

$$K(k_1, k_2, \ldots, k_v) = \begin{bmatrix} k_1 & k_2 & \cdots & k_v \\ 1 & 0 & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Theorem 2.5. *(Chen and Louck [3]) The* (i, j) *entry* $k_{i,j}^{(n)}(k_1, k_2, \ldots, k_v)$ *in the matrix* $K^n(k_1, k_2, \ldots, k_v)$ *is given by the following formula:*

$$k_{i,j}^{(n)}(k_1, k_2, \ldots, k_v) = \sum_{(t_1, t_2, \ldots, t_v)} \frac{t_j + t_{j+1} + \cdots + t_v}{t_1 + t_2 + \cdots + t_v} \times \binom{t_1 + \cdots + t_v}{t_1, \ldots, t_v} k_1^{t_1} \cdots k_v^{t_v} \quad (3)$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + vt_v = n - i + j$, $\binom{t_1 + \cdots + t_v}{t_1, \ldots, t_v} = \frac{(t_1 + \cdots + t_v)!}{t_1! \cdots t_v!}$ *is a multinomial coefficient, and the coefficients in (3) are defined to be* 1 *if* $n = i - j$.

Then we can give other combinatorial representations than for the Padovan-Padovan p-numbers by the following Corollary.

Corollary 2.6. *Let $Pa_n^{P,p}$ be the nth Padovan-Padovan p-number for $n \geq 1$. Then*

i.

$$Pa_n^{P,p} = \sum_{(t_1,t_2,\ldots,t_{p+5})} \binom{t_1+t_2+\cdots+t_{p+5}}{t_1,t_2,\cdots,t_{p+5}} 2^{t_2} (-1)^{t_4+t_5+t_{p+4}+t_{p+5}}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (p+5)\, t_{p+5} = n - p - 4$.

ii.

$$Pa_n^{P,p} = - \sum_{(t_1,t_2,\ldots,t_{p+5})} \frac{t_{p+5}}{t_1+t_2+\cdots+t_{p+5}} \times \binom{t_1+t_2+\cdots+t_{p+5}}{t_1,t_2,\cdots,t_{p+5}} 2^{t_2} (-1)^{t_4+t_5+t_{p+4}+t_{p+5}}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (p+5)\, t_{p+5} = n + 1$.

Proof. If we take $i = p+5$, $j = 1$ for the case i. and $i = p+4$, $j = p+5$ for the case ii. in Theorem 2.5, then we can directly see the conclusions from $(C_p)^n$. □

Now we consider the relationship between the Padovan-Padovan p-numbers and permanent of a certain matrix which is obtained using the Padovan-Padovan p-matrix $(C_p)^n$.

Definition 2.7. *Let $M = [m_{i,j}]$ be $u \times v$ real matrix and let $r^1, r^2, \ldots, r^u$ and $t^1, t^2, \ldots, t^v$ be respectively, the row and column vectors of M. If r^α contains exactly two non-zero entries, then M is contractible on row α. Similarly, M is contractible on column is β provided t^β contains exactly two non-zero entries.*

Suppose that x_1, x_2, …,x_u are row vectors of the matrix M. If M is contractible in the k^{th} column such that $m_{i,k} \neq 0, m_{j,k} \neq 0$ and $i \neq j$, then the $(u-1) \times (v-1)$ matrix $M_{ij:k}$ obtained from M by replacing the i^{th} row with $m_{i,k}x_j + m_{j,k}x_i$ and deleting the j^{th} row. The k^{th} column is called the contraction in the k^{th} column relative to the i^{th} row and the j^{th} row.

In [2], Brualdi and Gibson obtained that $per\,(M) = per\,(N)$ if M is a real matrix of order $\alpha > 1$ and N is a contraction of M.

Now we concentrate on finding relationships among the Padovan-Padovan p-numbers and permanents of certain matrices which are obtained by using the generating matrix of Padovan-Padovan p-numbers. Let $R_{m,p}^{Pa} = [r_{i,j}]$ be the $m \times m$ super-diagonal matrix, defined by

$$r_{i,j} = \begin{cases} 2 & \text{if } i=\varepsilon \text{ and } j=\varepsilon+1 \text{ for } 1\le\varepsilon\le m-1, \\ 1 & \begin{array}{c}\text{if } i=\varepsilon \text{ and } j=\varepsilon+2 \text{ for } 1\le\varepsilon\le m-2, \\ i=\varepsilon \text{ and } j=\varepsilon+p+1 \text{ for } 1\le\varepsilon\le m-p-1 \\ \text{and} \\ i=\varepsilon+1 \text{ and } j=\varepsilon \text{ for } 1\le\varepsilon\le m-1,\end{array} \\ -1 & \begin{array}{c}\text{if } i=\varepsilon \text{ and } j=\varepsilon+3 \text{ for } 1\le\varepsilon\le m-3, \\ i=\varepsilon \text{ and } j=\varepsilon+4 \text{ for } 1\le\varepsilon\le m-4, \\ i=\varepsilon \text{ and } j=\varepsilon+p+3 \text{ for } 1\le\varepsilon\le m-p-3 \\ \text{and} \\ i=\varepsilon \text{ and } j=\varepsilon+p+4 \text{ for } 1\le\varepsilon\le m-p-4,\end{array} \\ 0 & \text{otherwise.} \end{cases}, \text{ for } m\ge p+5.$$

Then we have the following Theorem.

Theorem 2.8. *For $m \geq p+5$,*

$$perR^{Pa}_{m,p} = Pa^{P,p}_{m+p+4}.$$

Proof. We will use the induction method on m. Let us consider matrix $R^{Pa}_{m,p}$ and let the equation be hold for $m \geq p+5$, then we show that the equation holds for $m+1$. If we expand the $perR^{Pa}_{m,p}$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$\begin{aligned} perR^{Pa}_{m+1,p} &= 2perR^{Pa}_{m-1,p} + perR^{Pa}_{m-2,p} - perR^{Pa}_{m-3,p} - perR^{Pa}_{m-4,p} \\ &\quad + perR^{Pa}_{m-p-1,p} - perR^{Pa}_{m-p-3,p} - perR^{Pa}_{m-p-4,p}. \end{aligned}$$

Since

$$perR^{Pa}_{m-1,p} = Pa^{P,p}_{m+p+3},$$

$$perR^{Pa}_{m-2,p} = Pa^{P,p}_{m+p+2},$$

$$perR^{Pa}_{m-3,p} = Pa^{P,p}_{m+p+1},$$

$$perR^{Pa}_{m-4,p} = Pa^{P,p}_{m+p},$$

$$perR^{Pa}_{m-p-1,p} = Pa^{P,p}_{m+3},$$

$$perR^{Pa}_{m-p-3,p} = Pa^{P,p}_{m+1}$$

and

$$perR^{Pa}_{m-p-4,p} = Pa^{P,p}_{m},$$

we easily obtain that $perR^{Pa}_{m+1,p} = Pa^{P,p}_{m+p+5}$. So the proof is complete. □

Let $S_{m,p}^{Pa}=[s_{i,j}]$ be the $m\times m$ matrix, defined by

$$s_{i,j}=\begin{cases} 2 & \text{if } i=\varepsilon \text{ and } j=\varepsilon+1 \text{ for } 1\le\varepsilon\le m-3,\\ & \text{if } i=\varepsilon \text{ and } j=\varepsilon+1 \text{ for } \varepsilon=m-1,\\ & i=\varepsilon \text{ and } j=\varepsilon+2 \text{ for } 1\le\varepsilon\le m-3,\\ 1 & i=\varepsilon \text{ and } j=\varepsilon+p+1 \text{ for } 1\le\varepsilon\le m-p-2\\ & \text{and}\\ & i=\varepsilon+1 \text{ and } j=\varepsilon \text{ for } 1\le\varepsilon\le m-1,\\ & \text{if } i=\varepsilon \text{ and } j=\varepsilon+3 \text{ for } 1\le\varepsilon\le m-4,\\ & i=\varepsilon \text{ and } j=\varepsilon+4 \text{ for } 1\le\varepsilon\le m-5,\\ -1 & i=\varepsilon \text{ and } j=\varepsilon+p+3 \text{ for } 1\le\varepsilon\le m-p-4\\ & \text{and}\\ & i=\varepsilon \text{ and } j=\varepsilon+p+4 \text{ for } 1\le\varepsilon\le m-p-4,\\ 0 & \text{otherwise.}\end{cases}\quad, \text{ for } m\ge p+5.$$

Then we have the following Theorem.

Theorem 2.9. *For $m\ge p+5$,*

$$perS_{m,p}^{Pa}=-Pa_{m-1}^{P,p}.$$

Proof. Let us consider matrix $S_{m,p}^{Pa}$ and let the equation be hold for $m\ge p+5$, then we show that the equation holds for $m+1$. If we expand the $perS_{m,p}^{Pa}$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$\begin{aligned} perS_{m+1,p}^{Pa} &= 2perS_{m-1,p}^{Pa}+perS_{m-2,p}^{Pa}-perS_{m-3,p}^{Pa}-perS_{m-4,p}^{Pa}\\ &\quad +perS_{m-p-1,p}^{Pa}-perS_{m-p-3,p}^{Pa}-perS_{m-p-4,p}^{Pa}.\end{aligned}$$

Since

$$perS_{m-1,p}^{Pa}=-Pa_{m-2}^{P,p},$$
$$perS_{m-2,p}^{Pa}=-Pa_{m-3}^{P,p},$$

$$\begin{aligned} perS_{m-3,p}^{Pa} &= -Pa_{m-4}^{P,p},\\ perS_{m-4,p}^{Pa} &= -Pa_{m-5}^{P,p},\end{aligned}$$

$$perS_{m-p-1,p}^{Pa}=-Pa_{m-p-2}^{P,p},$$
$$perS_{m-p-3,p}^{Pa}=-Pa_{m-p-4}^{P,p}$$

and

$$perS_{m-p-4,p}^{Pa}=-Pa_{m-p-5}^{P,p},$$

we easily obtain that $perS_{m+1,p}^{Pa}=-Pa_{m}^{P,p}$. So the proof is complete. □

Assume that $T^{Pa}_{m,p} = [t_{i,j}]$ be the $m \times m$ matrix, defined by

$$T^{Pa}_{m,p} = \begin{bmatrix} 1 & \cdots & \overset{\substack{(m-p-5)\,\text{th} \\ \downarrow}}{1} & 0 & \cdots & 0 \\ 1 & & & & & \\ 0 & & & & & \\ \vdots & & & S^{Pa}_{m-1,p} & & \\ 0 & & & & & \\ 0 & & & & & \end{bmatrix}, \text{ for } m > p+5,$$

then we have the following results:

Theorem 2.10. *For $m > p + 5$,*

$$perT^{Pa}_{m,p} = \sum_{i=0}^{m-2} Pa^{P,p}_{i}.$$

Proof. If we extend $perT^{Pa}_{m,p}$ with respect to the first row, we write

$$perT^{Pa}_{m,p} = perT^{Pa}_{m-1,p} - perS^{Pa}_{m-1,p}.$$

Thus, by the results and an inductive argument, the proof is easily seen. □

A matrix M is called convertible if there is an $n \times n$ $(1, -1)$-matrix K such that $perM = \det(M \circ K)$, where $M \circ K$ denotes the Hadamard product of M and K.

Now we give relationships among the Padovan-Padovan p-numbers and determinants of certain matrices which are obtained by using the matrix $R^{Pa}_{m,p}$, $S^{Pa}_{m,p}$ and $T^{Pa}_{m,p}$. Let $m > p + 5$ and let H be the $m \times m$ matrix, defined by

$$H = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & -1 & 1 \end{bmatrix}.$$

Then we have the following useful results.

Corollary 2.11. *For $m > p + 5$,*

$$\det\left(R^{Pa}_{m,p} \circ H\right) = Pa^{P,p}_{m+p+4},$$

$$\det\left(S_{m,p}^{Pa}\circ H\right)=-Pa_{m-1}^{P,p},$$

and

$$\det\left(T_{m,p}^{Pa}\circ H\right)=\sum_{i=0}^{m-2}Pa_{i}^{P,p}.$$

Proof. Since $perR_{m,p}^{Pa}=\det\left(R_{m,p}^{Pa}\circ H\right)$, $perS_{m,p}^{Pa}=\det\left(S_{m,p}^{Pa}\circ H\right)$ and $perT_{m,p}^{Pa}$ $=\det\left(T_{m,p}^{Pa}\circ H\right)$ for $m>p+5$, by Theorem 2.8, Theorem 2.9 and Theorem 2.10, we have the conclusion. □

Now we give the sums of the Padovan-Padovan p-numbers.

Let

$$S_n=\sum_{i=0}^{n}Pa_{i}^{P,p}$$

and suppose that U_P^{Pa} and $\left(U_P^{Pa}\right)^n$ are the $(p+6)\times(p+6)$ matrices such that

$$U_P^{Pa}=\begin{bmatrix}1 & 0 & 0 & \cdots & 0 & 0 & 0\\ 1 & & & & & & \\ 0 & & & & & & \\ \vdots & & & C_p & & & \\ 0 & & & & & & \\ 0 & & & & & & \\ 0 & & & & & & \end{bmatrix}$$

and then it can be shown by induction that

$$\left(U_P^{Pa}\right)^n=\begin{bmatrix}1 & 0 & 0 & \cdots & 0 & 0 & 0\\ S_{n+p+3} & & & & & & \\ S_{n+p+2} & & & & & & \\ S_{n+p+1} & & & (C_p)^n & & & \\ \vdots & & & & & & \\ S_n & & & & & & \\ S_{n-1} & & & & & & \end{bmatrix}.$$

3 Conclusion

Using the coefficients of the Padovan-Padovan p-sequence, we define the $(p+5)\times(p+5)$ lower Hessenberg matrices $M_p^{Pa,1}$ and $M_p^{Pa,2}$ as follow:

$$M_p^{Pa,1}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 2 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 2 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0\\ -1 & 1 & 2 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0\\ -1 & -1 & 1 & 2 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & -1 & -1 & 1 & 2 & 0 & 1 & 0 & \cdots & 0 & 0 & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots & \vdots\\ 0 & \cdots & 0 & -1 & -1 & 1 & 2 & 0 & 1 & 0 & 0 & 0\\ 1 & 0 & \cdots & 0 & -1 & -1 & 1 & 2 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & \cdots & 0 & -1 & -1 & 1 & 2 & 0 & 1 & 0\\ -1 & 0 & 1 & 0 & \cdots & 0 & -1 & -1 & 1 & 2 & 0 & 1\\ -1 & -1 & 0 & 1 & 0 & \cdots & 0 & -1 & -1 & 1 & 2 & 0 \end{bmatrix} \tag{4}$$

and

$$M_p^{Pa,2}=\begin{bmatrix} 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 2 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 2 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0\\ -1 & 1 & 2 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0\\ -1 & -1 & 1 & 2 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & -1 & -1 & 1 & 2 & 0 & -1 & 0 & \cdots & 0 & 0 & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots & \vdots\\ 0 & \cdots & 0 & -1 & -1 & 1 & 2 & 0 & -1 & 0 & 0 & 0\\ 1 & 0 & \cdots & 0 & -1 & -1 & 1 & 2 & 0 & -1 & 0 & 0\\ 0 & 1 & 0 & \cdots & 0 & -1 & -1 & 1 & 2 & 0 & -1 & 0\\ -1 & 0 & 1 & 0 & \cdots & 0 & -1 & -1 & 1 & 2 & 0 & -1\\ -1 & -1 & 0 & 1 & 0 & \cdots & 0 & -1 & -1 & 1 & 2 & 0 \end{bmatrix}. \tag{5}$$

We show that its determinant and permanents produce the terms of Padovan-Padovan p-sequence. Then we have the following Corollary.

Corollary 3.1. *Suppose that the $(p+5)\times(p+5)$ lower Hessenberg matrices $M_p^{Pa,1}$ and $M_p^{Pa,2}$ have the form (4) and (5). Then, for $p\geq 4$*

$$per M_p^{Pa,1}=\det M_p^{Pa,2}=Pa_{2p+9}^{P,p}.$$

Corollary 3.2. *Suppose that the $(p+5)\times(p+5)$ lower Hessenberg matrices $M_p^{Pa,1}$ and $M_p^{Pa,2}$ have the form (4) and (5). Then, for $p\geq 4$*

$$\det M_p^{Pa,1}=per M_p^{Pa,2}.$$

4 Open Problem

In this paper, we examined the Padovan-Padovan p-sequence which are obtained from the characteristic polynomials of Padovan and Padovan p-sequences. Then, we obtained the miscellaneous properties of these sequences.

Does there exist a relationship between the terms of the Padovan and Padovan p-sequences and the terms of the considered sequence in this paper?

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