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Generalized Derivations of Prime Rings with Involution

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Abstract

In this paper we investigate the commutativity of prime rings $(R, *)$ with involution of second kind which admits a generalized derivation satisfying certain algebraic identities on ideals of R. We also give classifications of some functions. Finally, we provide examples to show that various restrictions imposed in the hypothesis of our theorems are not superfluous.

Keywords: *Prime ring, involution, commutativity, derivation, generalized derivation.*

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1 Introduction

Let R be a ring with center $Z = Z(R)$. For any $x, y \in R$, we define $[x, y]$ to be the commutator $xy - yx$ and $x \circ y$ to be the anti-commutator $xy + yx$. A ring R is called prime if for any $x, y \in R$, $xRy = (0)$ implies $x = 0$ or $y = 0$. A ring R is said to be of characteristic different from 2 (or 2-torsion free) if for any $x \in R$, $2x = 0$ implies $x = 0$. An additive mapping $d : R \longrightarrow R$ is said to be a derivation on R if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $F : R \longrightarrow R$ is called a generalized derivation on R if it is associated with a derivation $d : R \longrightarrow R$ satisfying $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. An additive map $* : R \longrightarrow R$ is said to be an involution if it satisfies the following conditions: (i) $(xy)^* = y^*x^*$; and (ii) $(x^*)^* = x$ for all $x, y \in R$. A ring with involution or a $*$-ring is a ring equipped with an involution. In a ring with involution $*$, an element x is called hermition if $x^* = x$, and skew-hermition if $x^* = -x$. The set of all hermition and skew-hermition elements denoted by $H(R)$ and $S(R)$ respectively. An involution is called of the first kind if $Z(R) \subseteq H(R)$, and of the second kind if $S(R) \cap Z(R) \neq (0)$.

In the past few decades, several authors have studied the commutativity of the ring R that admits certain specific types of derivations. For instance, in [7], E. C. Posner introduced the first result in this direction, and he proved that if R is a prime ring and d is a nonzero derivation of R satisfies $[d(x), x] \in Z(R)$ for all $x \in R$, then R must be commutative. Moreover, the initiated algebraic study of generalized derivations has been done by Hvala [5] and he extended some results from derivations to generalized derivations. In [2], M. Ashraf et. al. proved that if R is a prime ring, I is a nonzero ideal of R and R admits a generalized derivation F associated with a non zero derivation d, then R is commutative if one of the following conditions holds: $d(x) \circ F(y) = 0$, $[d(x), F(y)] = 0$, $d(x) \circ F(y) = x \circ y$, $d(x) \circ F(y) + x \circ y = 0$, $d(x) \circ F(y) - xy \in Z(R)$, $d(x) \circ F(y) + xy \in Z(R)$, $[d(x), F(y)] = [x, y]$, or $[d(x), F(y)] + [x, y] = 0$ for all $x, y \in I$. In [3], Bergen obtained some results concerning derivations in view of Lie ideals. H. Shuliang [8] extended M. Ashraf's work and he proved the results in light of Lie ideal of a prime ring. Motivated by the above, our purpose is to continue studying commutativity criteria for prime rings with involution admitting a generalized derivation associated with a derivation satisfying certain algebraic identities.

2 The Necessary Preliminaries

We shall use repeatedly the familiar commutator and anti-commutator identities:

(i) $[xy, z] = x[y, z] + [x, z]y$,

(ii) $[x, yz] = y[x, z] + [x, y]z$,

(iii) $x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$,

(iv) $(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$, for all $x, y, z \in R$.

We will make some use of the following well-known results :

Fact 1 ([1], Fact 2.5) *Let R be a prime ring with involution $*$ of the second kind such that $char(R) \neq 2$ provided with a derivation d of R. If $d(h) = 0$ for all $h \in H(R) \cap Z(R)$, then $d(x) = 0$ for all $x \in R$.*

Fact 2 ([2], Lemma 2.2) *Let R be a prime ring, and I be a nonzero ideal of R. If R admits a nonzero derivation d such that $[x, d(x)]$ is central for all $x \in I$, then R is commutative.*

Fact 3 ([6], Lemma 4) *Let y and xy be in the center of a prime ring R. If y is not zero, then x is in $Z(R)$.*

Fact 4 (8, Theorem 3.2) *Let R be a prime ring with $char(R) \neq 2$ and let A be a Lie ideal such that $a^2 \in A$ for all $a \in A$. If F is a generalized derivation associated with $d \neq 0$ satisfying $[d(x), F(y)] = 0$ for all $x, y \in A$, then $A \subseteq Z(R)$.*

Fact 5 ([2], Theorem 2.9) *Let R be a prime ring, and I be a nonzero ideal of R. If R admits a generalized derivation F associated with a derivation d such that $d(x)F(y) - xy \in Z(R)$ for all $x, y \in I$, then $d = 0$ or R is commutative.*

Fact 6 ([2], Theorem 2.11) *Let R be a 2-torsion free prime ring, and I be a nonzero ideal of R. If R admits a generalized derivation F associated with a derivation d such that $[d(x), F(y)] = [x, y]$ for all $x, y \in I$, then $d = 0$ or R is commutative.*

Fact 7 ([2], Theorem 2.5) *Let R be a 2-torsion free prime ring, and I be a nonzero ideal of R. If R admits a generalized derivation F associated with a derivation d such that $d(x) \circ F(y) = 0$ for all $x, y \in I$, then $d = 0$ or R is commutative.*

Fact 8 ([2], Theorem 2.7) *Let R be a 2-torsion free prime ring, and I be a nonzero ideal of R. If R admits a generalized derivation F associated with a derivation d such that $d(x) \circ F(y) = x \circ y$ for all $x, y \in I$, then $d = 0$ or R is commutative.*

3 Conditions with commutator

In [4] Herstein proved that if R is a prime ring of characteristic not 2 equipped with a nonzero derivation d such that $[d(x), d(y)] = 0$ for all $x, y \in R$, then R is commutative. In the following result we give an improve version of Herstein's result in a prime ring with involution of second kind as follows :

Theorem 3.1 *Let $(R, *)$ be a prime ring with involution of the second kind such that $char(R) \neq 2$, and I be an $*$-ideal. If R admits a generalized derivation F associated with a nonzero derivation d such that $[d(x), F(x^*)] = 0$ for all $x \in I$, then R is commutative.*

Proof. By the hypothesis, we have

$$[d(x), F(x^*)] = 0 \text{ for all } x \in I \tag{1}$$

A linearization of (1) forces

$$[d(x), F(y^*)] + [d(y), F(x^*)] = 0 \text{ for all } x, y \in I \tag{2}$$

That is

$$[d(x), F(y)] + [d(y^*), F(x^*)] = 0 \text{ for all } x, y \in I \tag{3}$$

Substituting yh for y in (3), where $h \in Z(R) \cap H(R) \setminus \{0\}$, it follows that

$$([d(x), y] + [y^*, F(x^*)])d(h) = 0 \tag{4}$$

By using the primeness property for R, the equation (4) yields that either $[d(x), y] + [y^*, F(x^*)] = 0$ or $d(h) = 0$. By considering the first case, we have

$$[d(x), y] + [y^*, F(x^*)] = 0 \text{ for all } x, y \in I \tag{5}$$

Replace y by yz in (5), where $z \in Z(R)$ to obtain

$$[d(x), y]z + [y^*, F(x^*)]z^* = 0 \text{ for all } x, y \in I \text{ and } z \in Z(R) \tag{6}$$

Multiply (5) by z^* from the right, and subtract the result from the equation (6) to get

$$[d(x), y](z - z^*) = 0 \text{ for all } x, y \in I \text{ and } z \in Z(R) \tag{7}$$

Since R is prime, and $*$ is of the second kind, the equation (7) gives that $[d(x), y] = 0$. Putting $y = x$ implies $[d(x), x] = 0$ for all $x \in I$. Hence, in view of Fact 2, R is commutative.
Now suppose that the other case holds, so by Fact 1, we get $d(k) = 0$ for all $k \in Z(R) \cap S(R) \setminus \{0\}$. Replacing y by yk in (3) we obtain

$$k([d(x), F(y)] - [d(y^*), F(x^*)]) = 0$$

Since the ring R is prime the involution $*$ is of the second kind, we get

$$[d(x), F(y)] - [d(y^*), F(x^*)] = 0 \text{ for all } x, y \in I \tag{8}$$

Combining (3) and (8), and since $char(R) \neq 2$, we obtain $[d(x), F(y)] = 0$ for all $x, y \in I$. Therefore, by Fact 4, R is commutative, and the proof is complete.

∎

Theorem 3.2 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind, and I be an $*$-ideal. If R admits a generalized derivation F associated with a nonzero derivation d such that $[d(x), F(x^*)] = [x, x^*]$ (or $[d(x), F(x^*)] = -[x, x^*]$) for all $x \in I$, then R is commutative.*

Proof. By the hypothesis, we have

$$[d(x), F(x^*)] = [x, x^*] \text{ for all } x \in I \tag{9}$$

Linearizing (9), we get

$$[d(x), F(y^*)] + [d(y), F(x^*)] = [x, y^*] + [y, x^*] \text{ for all } x, y \in I \tag{10}$$

This can be written as

$$[d(x), F(y)] + [d(y^*), F(x^*)] = [x, y] + [y^*, x^*] \text{ for all } x, y \in I \tag{11}$$

Put $y = yh$ in (11), where $h \in Z(R) \cap H(R) \setminus \{0\}$ to obtain

$$([d(x), y] + [y^*, F(x^*)])d(h) = 0 \tag{12}$$

In light of primeness, we can see that either $[d(x), y] + [y^*, F(x^*)] = 0$ or $d(h) = 0$. Assume that

$$[d(x), y] + [y^*, F(x^*)] = 0 \text{ for all } x, y \in I \tag{13}$$

Replacing y by yz, where $z \in Z(R)$, we get

$$[d(x), y]z + [y^*, F(x^*)]z^* = 0 \text{ for all } x, y \in I \text{ and } z \in Z(R) \tag{14}$$

Right multiplying (13) by z^*, we find that

$$[d(x), y]z^* + [y^*, F(x^*)]z^* = 0 \text{ for all } x, y \in I \text{ and } z \in Z(R) \tag{15}$$

Subtract (15) from (14) to obtain

$$[d(x), y](z - z^*) = 0 \text{ for all } x, y \in I \text{ and } z \in Z(R) \tag{16}$$

Since R is prime and the involution $*$ is of the second kind, the equation (16) yields that $[d(x), y] = 0$. By taking $y = x$, we conclude that $[d(x), x] = 0$ for all $x \in I$, and thus R is commutative by Fact 2.
Now consider the second case of (12) which is $d(h) = 0$ for all $h \in Z(R) \cap H(R) \setminus \{0\}$. By Fact 1, $d(k) = 0$ for all $k \in Z(R) \cap S(R) \setminus \{0\}$. Writing yk instead of y in (11), we obtain

$$([d(x), F(y)] - [d(y^*), F(x^*)] - [x, y] + [y^*, x^*])k = 0 \tag{17}$$

Because of the primeness hypothesis for R, and $S(R) \cap Z(R) \neq (0)$, the equation (17) implies that

$$[d(x), F(y)] - [d(y^*), F(x^*)] - [x, y] + [y^*, x^*] = 0 \text{ for all } x, y \in I \tag{18}$$

Comparing (11) and (18), and since R is 2-torsion free ring, we arrive at $[d(x), F(y)] = [x, y]$ for all $x, y \in I$. Hence, in view of Fact 6, R is commutative.
Proceeding on the same lines with necessary variations, we can prove the theorem when the condition $[d(x), F(x^*)] = -[x, x^*]$ holds for all $x \in I$. ∎

4 Conditions with anti-commutator

This section is devoted to finding out if commutativity still holds when the commutator in the conditions of the preceding section is replaced by anti-commutator.

Theorem 4.1 *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind, and I be an $*$-ideal. If R admits a generalized derivation F associated with a nonzero derivation d such that $d(x) \circ F(x^*) = 0$ for all $x \in I$, then R is commutative.*

Proof. Assume that

$$d(x) \circ F(x^*) = 0 \text{ for all } x \in I \tag{19}$$

A linearization of (19) yields that

$$d(x) \circ F(y^*) + d(y) \circ F(x^*) = 0 \text{ for all } x, y \in I \tag{20}$$

That is

$$d(x) \circ F(y) + d(y^*) \circ F(x^*) = 0 \text{ for all } x, y \in I \tag{21}$$

Replacing y by yh in (21), where $h \in Z(R) \cap H(R) \setminus \{0\}$, we get

$$(d(x) \circ y + y^* \circ F(x^*))d(h) = 0 \tag{22}$$

Invoking the primeness property for R, the equation (22) forces that either $d(x) \circ y + y^* \circ F(x^*) = 0$ or $d(h) = 0$. Suppose we have

$$d(x) \circ y + y^* \circ F(x^*) = 0 \text{ for all } x, y \in I \tag{23}$$

Replace y by yk in (23), where $k \in Z(R) \cap S(R) \setminus \{0\}$, this implies $(d(x) \circ y - y^* \circ F(x^*))k = 0$. Since R is prime ring and $S(R) \cap Z(R) \neq (0)$, we obtain $d(x) \circ y - y^* \circ F(x^*) = 0$. Comparing this equation with (23), and since R is 2-torsion free, we get

$$d(x) \circ y = 0 \text{ for all } x, y \in I \tag{24}$$

This leads to $yd(x) = -d(x)y$ for all $x, y \in I$. Putting $y = ry$ in (24), where $r \in R$, we get $d(x)ry - rd(x)y = 0$ which implies that $[d(x), r]y = 0$. Thus, $[d(x), r]RI = (0)$. Since R is prime, and $I \neq (0)$, we obtain $[d(x), r] = 0$. Taking $r = x$, we arrive at $[d(x), x] = 0$ for all $x \in I$. Hence, in light of Fact 2, R is commutative.

Therefore, consider the case when $d(h) = 0$ for all $h \in Z(R) \cap H(R) \setminus \{0\}$. Then by Fact 1, we get $d(k) = 0$ for all $k \in Z(R) \cap S(R) \setminus \{0\}$. Substituting yk for y in (21) we get

$$k(d(x) \circ F(y) - d(y^*) \circ F(x^*)) = 0$$

Using the primeness hypothesis together with the fact that $*$ is the second kind, we get

$$d(x) \circ F(y) - d(y^*) \circ F(x^*) = 0 \text{ for all } x, y \in I \tag{25}$$

Adding (21) to (25), and by using the 2-torsion freeness property, we arrive at $d(x) \circ F(y) = 0$ for all $x, y \in I$. Hence, by Fact 7 we conclude that R is commutative. ∎

Theorem 4.2 *Let $(R, *)$ be a 2-torsion prime ring with involution of the second kind, and I be an $*$-ideal. If R admits a generalized derivation F associated with a nonzero derivation d such that $d(x) \circ F(x^*) = x \circ x^*$ (or $d(x) \circ F(x^*) = -(x \circ x^*)$) for all $x \in I$, then R is commutative.*

Proof. Suppose we have

$$d(x) \circ F(x^*) = x \circ x^* \text{ for all } x \in I \tag{26}$$

Linearize (26) to get

$$d(x) \circ F(y^*) + d(y) \circ F(x^*) = x \circ y^* + y \circ x^* \text{ for all } x, y \in I \tag{27}$$

Thus,

$$d(x) \circ F(y) + d(y^*) \circ F(x^*) = x \circ y + y^* \circ x^* \text{ for all } x, y \in I \tag{28}$$

Writing yh instead of y in (28), where $h \in Z(R) \cap H(R) \setminus \{0\}$, we arrive at

$$(d(x) \circ y + y^* \circ F(x^*))d(h) = 0 \tag{29}$$

In light of primeness, the equation (29) yields that either $d(x) \circ y + y^* \circ F(x^*) = 0$ or $d(h) = 0$.
Assume we have

$$d(x) \circ y + y^* \circ F(x^*) = 0 \text{ for all } x, y \in I \tag{30}$$

Replacing y by yk in (30), where $k \in Z(R) \cap S(R) \setminus \{0\}$, we obtain

$$(d(x) \circ y - y^* \circ F(x^*))k = 0. \tag{31}$$

Since R is prime, and $k \neq 0$ (as the involution $*$ is of the second kind), we obtain

$$d(x) \circ y - y^* \circ F(x^*) = 0 \text{ for all } x, y \in I \tag{32}$$

Combining (32) and (30), and since the ring R is a 2-torsion free, we conclude that $d(x) \circ y = 0$ for all $x, y \in I$. This equation is the same as (24) in Theorem 4.1. Thus, by using the same techniques, we obtain that R is commutative.
Now consider the case when $d(h) = 0$, where $h \in Z(R) \cap H(R) \setminus \{0\}$. By Fact 1, we get $d(k) = 0$ for all $k \in Z(R) \cap S(R) \setminus \{0\}$. Putting $y = yk$ in (28), it follows that

$$(d(x) \circ F(y) - d(y^*) \circ F(x^*) - x \circ y + y^* \circ x^*)k = 0 \tag{33}$$

Since the ring R is prime, and the involution $*$ is of the second kind, the equation (33) yields that

$$d(x) \circ F(y) - d(y^*) \circ F(x^*) = x \circ y - y^* \circ x^* \text{ for all } x, y \in I \tag{34}$$

Adding (28) to (34), and since R is a 2-torsion free ring, we arrive at

$$d(x) \circ F(y) = x \circ y \text{ for all } x, y \in I \tag{35}$$

Therefore, by using Fact 8, we get R is commutative.
Similarly, we can prove the theorem in the case when $d(x) \circ F(x^*) = -(x \circ x^*)$ for all $x \in I$. ∎

Theorem 4.3 *Let $(R, *)$ be a 2-torsion prime ring with involution of the second kind, and I be an $*$-ideal. If R admits a generalized derivation F associated with a nonzero derivation d such that $d(x) \circ F(x^*) - xx^* \in Z(R)$ (or $d(x) \circ F(x^*) + xx^* \in Z(R)$) for all $x \in I$, then R is commutative.*

Proof. Assume that

$$d(x) \circ F(x^*) - xx^* \in Z(R) \text{ for all } x \in I \tag{36}$$

Linearizing (36), we obtain

$$d(x) \circ F(y^*) + d(y) \circ F(x^*) - xy^* - yx^* \in Z(R) \text{ for all } x, y \in I \tag{37}$$

This can be written as

$$d(x) \circ F(y) + d(y^*) \circ F(x^*) - xy - y^*x^* \in Z(R) \text{ for all } x, y \in I \tag{38}$$

Replace y by yh in (38), where $h \in Z(R) \cap H(R) \setminus \{0\}$ to get $\big(d(x)y + y^*F(x^*)\big)d(h) \in Z(R)$. Then, Fact 3 implies that either $d(x)y + y^*F(x^*) \in Z(R)$ or $d(h) = 0$. Assume that

$$d(x)y + y^*F(x^*) \in Z(R) \text{ for all } x, y \in I \tag{39}$$

Putting $y = yz$ in (39), where $z \in Z(R)$, we obtain

$$d(x)yz + z^*y^*F(x^*) \in Z(R) \text{ for all } x, y \in I \text{ and } z \in Z(R) \tag{40}$$

Multiplying (39) by z^* from the left, we get

$$z^*d(x)y + z^*y^*F(x^*) \in Z(R) \text{ for all } x, y \in I \text{ and } z \in Z(R) \tag{41}$$

Subtracting (41) from (40), we get $d(x)y(z - z^*) \in Z(R)$. Using the primeness property for R together with Fact 3, the previous equation assures that either $d(x)y \in Z(R)$ or $z - z^* = 0$. The latter case contradicts the fact that the involution $*$ is of the second kind. Therefore, $d(x)y \in Z(R)$ which implies $[d(x)y, r] = 0$ for all $x, y \in I$ and $r \in R$. That is

$$d(x)[y, r] + [d(x), r]y = 0 \text{ for all } x, y \in I \text{ and } r \in R \tag{42}$$

Substituting yt for y in (42), where $t \in R$, we get $d(x)y[t, r] = 0$ which implies that $d(x)RI[t, r] = (0)$. Since R is prime, either $d(x) = 0$ or $I[t, r] = (0)$. Suppose that $d(x) = 0$. Then $d(xt) = 0$ for all $x \in I$ and $t \in R$. Thus, $xd(t) = 0$ implies $IRd(t) = (0)$. Since R is prime, and by using the hypothesis of I and d, we get a contradiction.
Therefore, $I[t, r] = (0)$ implies $IR[t, r] = (0)$. Invoking the primeness of R, and since $I \neq (0)$, we obtain $[t, r] = 0$ for all $t, r \in R$. Hence, R is commutative.
Now consider the other case when $d(h) = 0$ for all $h \in Z(R) \cap H(R) \setminus \{0\}$. Therefore, by Fact 1, $d(k) = 0$ for all $k \in S(R) \cap Z(R)$. Putting yk instead of y in (38), we obtain $\big(d(x)F(y) - d(y^*)F(x^*) - xy + y^*x^*\big)k \in Z(R)$. Now from Fact 3, and the condition $S(R) \cap Z(R) \neq (0)$, we get

$$d(x)F(y) - d(y^*)F(x^*) - xy + y^*x^* \in Z(R) \text{ for all } x, y \in I \tag{43}$$

Adding (38) to (43), and since the ring R is 2-torsion free, we obtain $d(x)F(y) - xy \in Z(R)$ for all $x, y \in I$. Hence, by Fact 5, we get the required result. Following the same techniques as used above, we prove the theorem when $d(x)F(x^*) + xx^* \in Z(R)$ for all $x \in I$. ■

The following examples show that $*$ of the second kind is crucial in Theorems 3.1, 3.2, 4.1, 4.2, and 4.3.

Example 1.
Let $R = M_2(\mathbb{Z})$, and $*$ be the involution which is defined by: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. It is straightforward to check that R is a prime ring with involution $*$ of the first kind such that

$$[X, X^*] = 0 \text{ and } X \circ X^* = 2(ad - bc)I_2 \text{ for all } X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R.$$

Let d and F be defined as: $F\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = d\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$. Then, F is a generalized derivation with associated derivation $d = F$. Consider the ideal I to be R itself. Clearly, the conditions of Theorems 3.1 and 3.2 are satisfied. However, R is not commutative.

Example 2.
Let R be a ring which is defined as follows:
$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}$. Let I be an ideal of R which is defined by:
$I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{Z} \right\}$. Define the mappings F, d, $* : R \to R$ by
$F\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = d\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$, and $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^* = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}$.
It is straightforward to check that R is a prime ring with involution $*$ of the first kind such that $[x, x^*] = 0$ for all $x \in I$. Moreover, I is an $*$-ideal of R. Furthermore, the mapping F is a generalized derivation with associated derivation $d = F$ satisfying the conditions of Theorems 3.1, 3.2, 4.1, 4.2, and 4.3 but R is not a commutative ring.

5 Open Problem

The open problem here is to investigate the commutativity of a semiprime ring $(R, *)$ with involution of the second kind which admits a generalized derivation

F associated with a nonzero derivation d satisfying the following identities:

$$[d(x), F(x^*)] = \pm[x, x^*],$$

$$d(x) \circ F(x^*) = \pm(x \circ x^*),$$

$$d(x) \circ F(x^*) \pm xx^* \in Z(R),$$

for all x in an $*$-ideal I of R.

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