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Solvability of impulsive fractional differential equation with $\psi-$Caputo derivative

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Abstract

In this manuscript, we investigate the nonlocal impulsive fractional differential equation in $\psi-$Caputo sense. We apply the Schaefer's fixed point principle to obtain the existence of solution for the proposed impulsive fractional differential equation. The uniqueness of the solution for the proposed impulsive $\psi-$Caputo fractional differential equation is given by applying Banach's fixed point approach. A practical example illustrating the aforementioned applicability is also given.

Keywords: *Fractional $\psi-$operators; fixed point; Banach space; impulsive differential equation.*

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1 Introduction

During the past two decades, impulsive fractional differential equations (IFDEs, for short) have attracted the attention of many mathematicians, because there are many natural phenomena that are modeled in the form of IFDEs such as problems of neuro-computing, signals, fuzzy systems and control [2, 10, 14, 15, 8, 6].

Recently, in 2014, Zhang et al. [16] considered the IFDEs involving classical-Caputo fractional operator. More recently, in 2020, Zhang and Xiong [17] established the periodic solution of IFDEs.

Nowadays, a lot of definitions of fractional operator derivatives were proposed according on nonsingular kernels such as the Atangana-Baleanu [5] and conformable [12] fractional operators. Therefore, many researchers accelerated to study IFDEs from the point of view of modern definitions of fractional operators [1, 11].

Almeida in [3] introduced the $\psi-$Caputo fractional derivative. Indeed, this fractional operator is more general than Riemann-Liouville, Hadamard, Erdely Kober and Caputo operators kinds.

Let $I = [0, 1]$ and E is a Banach space. In this paper, we consider the more general IFDE involving $\psi-$Caputo operators, given by

$$\begin{cases} {}^*D_{0^+}^{\alpha,\psi} x(t) = A(t)x(t) + f(t, x(t)),\ t \in I^* = I \setminus \{t_1, t_2, ..., t_m\},\ \alpha \in (0, 1), \\ x(0) + g(x) = x_0, \\ x(t_i^+) = x(t_i^-) + y_i,\ i = 1, 2, ..., m, \end{cases} \tag{1}$$

where ${}^*D^{\alpha,\psi}$ is $\psi-$Caputo derivative operator, A is bounded operator on E, $f : I \times E \to E$, $g : PC(I, E) \to E$, $t_0 = 0 < t_1 < t_2 < ... < t_m < t_{m+1} = 1$ and

$$x(t_i^+) = \lim_{\tau\to 0^+} x(t_i + \tau) \quad , \quad x(t_i^-) = \lim_{\tau\to 0^-} x(t_i + \tau).$$

We show the existence and uniqueness of solution for the proposed Eq.(1) by using Schaefer's and Banach's fixed point theorems.

2 Preliminaries

In this section, we recall some definitions and theorems which will be used in the investigation. The following definition was considered in [7, 13].

Definition 2.1 *Consider* $I = [0, 1]$ *and let* $t_i \in I,\ i = 1, 2, ..., m$ *such that* $t_0 = 0 < t_1 < t_2 < ... < t_m < t_{m+1} = 1$. *Let* E *be the Banach space and* $C(I)$ *be the space of all continuous functions* $x : I \to E$. *It is easy to prove that* $C(I)$ *is a Banach space with the norm* $\|x\|_C = \max_{t\in I} \|x(t)\|,\ x \in C(I),\ I = [0, 1]$. *Therefore,* $PC(I)$ *is the Banach space of all of all piecewise continuous maps*

$$PC(I) = \{x : I \to E : x_{|_{I_i}} \in C(I_i),\ I_0 = [0, t_1],\ I_i = (t_i, t_{i+1}],\ i = 1, 2, ..., m,\ \textit{and}$$

$$x(t_i^+),\ x(t_i^-) \textit{are exists for all}\, i = 1, 2, ..., m\},$$

with the norm $\|x\|_{PC} = \sup_{t\in I} \|x(t)\|$.

The following two definitions were established in [3, 4], .

Definition 2.2 *Let ψ be an increasing function having a continuous derivative ψ' on $(0,\infty)$. The $\psi-$Riemann-Liouville fractional integral of order $\alpha > 0,\ \alpha \in \mathbb{R}$ for an integrable function $f : (0,\infty) \rightarrow \mathbb{R}$ is given by*

$$J_{0^+}^{\alpha,\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(\zeta)(\psi(t) - \psi(\zeta))^{\alpha-1}\, f(\zeta)\, d\zeta,$$

and the $\psi-$Riemann-Liouville fractional derivative of order $\alpha > 0,\ \alpha \in \mathbb{R}_+$ of the function f is defined as

$$D_{0^+}^{\alpha,\psi} f(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{1}{\psi'(t)} \frac{d}{dt})^n \int_0^t \psi'(\zeta)(\psi(t) - \psi(\zeta))^{n-\alpha-1}\, f(\zeta)\, d\zeta,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integral part of α.

Definition 2.3 *Let $\psi \in C^n(I,\mathbb{R})$, be an increasing function such that $\psi'(t) \neq 0$ for all $t \in I$. Let $f \in C^n(I,\mathbb{R})$, then $\psi-$Caputo fractional derivative of order $\alpha > 0,\ \alpha \in \mathbb{R}_+$ of the function f is defined as*

$$^*D_{0^+}^{\alpha,\psi} f(t) = D_{0^+}^{\alpha,\psi}[f(t) - \sum_{k=0}^{n-1} \frac{f_\psi^{[k]}(0)}{k!} (\psi(t) - \psi(0))^k],$$

where $n = [\alpha]+1$, $[\alpha]$ denotes the integral part of α and $f_\psi^{[k]}(t) = (\frac{1}{\psi'(t)} \frac{d}{dt})^k\, f(t)$.

We recall the following Lemma which was given in [4].

Lemma 2.4 *Suppose that $f : I \rightarrow \mathbb{R}$, then*
(1) if $f \in C(I)$, then $^*D_{0^+}^{\alpha,\psi}\ J_{0^+}^{\alpha,\psi} f(t) = f(t)$,
(2) if $f \in C^{n-1}(I)$, then

$$J_{0^+}^{\alpha,\psi}\ ^*D_{0^+}^{\alpha,\psi}\ f(t) = f(t) - \sum_{k=0}^{n-1} \frac{u_\psi^{[k]}(0)}{k!} (\psi(t) - \psi(0))^k.$$

Now, we recall the Gronwall inequality for fractional $\psi-$integral, which will be used later.

Theorem 2.5 *[4]. Let x be continuous, nonnegative on I, $\alpha \in (0,1)$ and d_1, d_2 are positive real numbers such that*

$$x(t) \leq d_1 + d_2 \int_0^t \psi'(\zeta)(\psi(t) - \psi(\zeta))^{\alpha-1}\, x(\zeta)\, d\zeta \quad , \quad \forall t \in I.$$

Then

$$x(t) \leq d_1\, E_\alpha(d_2\, \Gamma(\alpha))\, (\psi(t) - \psi(0))^\alpha \quad , \quad \forall t \in I,$$

where, $E_\alpha(.)$ is Mittag-Leffler function.

Finally, we recall Schaefer's fixed point theorem which was given in [9].

Theorem 2.6 ***(Schaefer's fixed point theorem)*** *Let E be a Banach space and $T : E \to E$ be a continuous and compact map. Furthermore, let*

$$\Omega = \{x \in E : x = \lambda T x,\ \lambda \in (0,1)\},$$

be a bounded set. Then T has at least one fixed point in E.

3 Main results

We will establish the solvability of Eq.(1) under the following assumptions:

(A1) The function $A : I \to E$ is continuous, bounded, *dom*(A) is constant and $\exists M_1 \in (0,\infty)$ such that $M_1 = \max\limits_{t\in I} \|A(t)\|$.

(A2) The function $f : I \times E \to E$ is continuous and $\exists L_f \in (0,\infty)$ such that:

$$\|f(t, x_1(t)) - f(t, x_2(t))\| \le L_f \|x_1 - x_2\|,$$

for all $t \in I$ and $x_1, x_2 \in E$.

(A3) The function $g : PC(I) \to E$ is continuous and $\exists M_2, L_g \in (0,\infty)$ such that $M_2 = \sup\limits_{t\in I} \|g(x)\|$ and

$$\|g(x_1) - g(x_2)\| \le L_g \|x_1 - x_2\|,$$

for all $x_1, x_2 \in PC(I)$.

First, we establish the proposed problem without impulsive and nonlocal condition. Consider the following differential equation

$$\begin{cases} {}^{*}D_{0^+}^{\alpha,\psi} x(t) = A(t)x(t) + f(t, x(t)),\ t \in I,\ \alpha \in (0,1), \\ x(0) = x_0, \end{cases} \tag{2}$$

where $x_0 \in dom(A)$. We call $x \in C(I)$ is a solution of Eq.(2) if x is $\alpha\ \psi-$Caputo differentiable and satisfies Eq.(2).

First, we give the integral representation of the Eq.(2).

Lemma 3.1 *Let the assumptions (A1) and (A2) hold. Then , the Eq.(2) is equivalent to the integral equation*

$$\begin{cases} x(t) & = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\, A(s)x(s)\, ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\, f(s, x(s))\, ds,\ t \in I. \end{cases} \tag{3}$$

Proof. From Lemma 2.4, we get

$$J_{0^+}^{\alpha,\psi}\ {}^{*}D_{0^+}^{\alpha,\psi}x(t) = J_{0^+}^{\alpha,\psi}[A(t)x(t) + f(t, x(t))].$$

So, we obtain

$$x(t) - x_0 = \frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\alpha-1}\, A(s)x(s)\, ds$$

$$+\frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\alpha-1}\, f(s, x(s))\, ds\ .\ \square$$

Now, define $T : C(I) \to C(I)$ as

$$\begin{aligned} Tx(t) = x_0 &+ \frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\alpha-1}\, A(s)x(s)\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\alpha-1}\, f(s, x(s))\, ds\ . \end{aligned} \tag{4}$$

Clearly, the solution of the Eq.(2) is the fixed point of the operator T.

Lemma 3.2 *Under the assumptions (A1)-(A2), Eq.(2) has at least one solution.*

Proof. The proof is done in 4 steps.

Step 1. T is continuous. Let $\{x_n\}$ be a sequence in $C(I)$ such that $x_n \to x \in C(I)$ as $n \to \infty$. So, for all $t \in I$, we get

$$\begin{aligned} \|Tx_n(t) - Tx(t)\| &\leq \int_0^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)}\, \|A(s)\|\|x_n(s) - x(s)\|ds \\ &+ \int_0^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)}\, \|f(s, x_n(s)) - f(s, x(s)\|ds \\ &\leq M_1 \int_0^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)}\, \|x_n(s) - x(s)\|ds \\ &+ L_f \int_0^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)}\, \|x_n(s) - x(s)\|ds \\ &\leq \frac{(M_1 + L_f)\ (\psi(1) - \psi(0))^{\alpha}}{\Gamma(\alpha + 1)}\, \|x_n - x\|_C. \end{aligned} \tag{5}$$

Thus, $\|Tx_n - Tx\|_C \to 0$ as $n \to \infty$.

Step 2. Every bounded set B_r in $C(I)$, TB_r is uniformly bounded. Let $r>0$. Define, $B_r=\{x\in C(I):\|u\|_C\leq r\}$ and let $x\in B_r$, then for all $t\in I$ we get

$$\begin{aligned}\|Tx(t)\| &\leq \|x_0\|+\int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\ \|A(s)\|\ \|x(s)\|ds\\ &+\int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\ \|f(s,x(s))\|ds\\ &\leq \|x_0\|+M_1\, r\ \int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}ds\\ &+\int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\ [L_f\|x(s)\|+\|f(s,0)\|]ds\\ &\leq \|x_0\|+\frac{(M_1\, r+L_f\, r+M^*)(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}=r^*,\end{aligned} \tag{6}$$

where $M^*=\max\limits_{t\in I}\|f(t,0)\|$. Thus, we get

$$\|Tx\|_C\leq r^*. \tag{7}$$

Hence, $T(B_r)$ is uniformly bounded in $C(I)$.
Step 3. T is equi-continuous. For all $t_1,t_2\in I$ such that $t_1<t_2$, we get

$$\begin{aligned}\|Tx(t_2)-Tx(t_1)\| \leq &\ \| \int_0^{t_2} \frac{\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\ A(s)x(s)ds\\ &-\int_0^{t_1} \frac{\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\ A(s)x(s)ds\|\\ &+\|\int_0^{t_2} \frac{\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\ f(s,x(s))ds\\ &-\int_0^{t_1} \frac{\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\ f(s,x(s))ds\|\\ &=D_1+D_2,\end{aligned} \tag{8}$$

where

$$\begin{aligned}
D_1 &= \| \int_0^{t_2} \frac{\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \, A(s)x(s)ds \\
&- \int_0^{t_1} \frac{\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \, A(s)x(s)ds\| \\
&\leq \| \int_0^{t_2} \frac{\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \, A(s)x(s)ds \\
&- \int_0^{t_1} \frac{\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \, A(s)x(s)ds\| \\
&+ \| \int_0^{t_1} \frac{\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \, A(s)x(s)ds \\
&- \int_0^{t_1} \frac{\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \, A(s)x(s)ds\| \\
&\leq \int_{t_1}^{t_2} \frac{\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \, \|A(s)\| \, \|x(s)\|ds \\
&+ \int_0^{t_1} \| \frac{\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \\
&- \frac{\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \| \|A(s)\| \, \|x(s)\|ds,
\end{aligned} \tag{9}$$

therefore, we have

$$\begin{aligned}
D_1 &\leq M_1 \, r \int_{t_1}^{t_2} \frac{\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} ds \\
&+ M_1 \, r \int_0^{t_1} [\frac{\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \\
&- \frac{\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}] ds \\
&\leq \frac{M_1 \, r \, (\psi(t_2)-\psi(t_1))^{\alpha}}{\Gamma(\alpha+1)} \\
&+ \frac{M_1 \, r}{\Gamma(\alpha+1)} [(\psi(t_1)-\psi(0))^{\alpha} - (\psi(t_2)-\psi(0))^{\alpha} \\
&+ (\psi(t_2)-\psi(t_1))^{\alpha}] \\
&\leq \frac{2M_1 \, r \, (\psi(t_2)-\psi(t_1))^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned} \tag{10}$$

Similarly, we get

$$\begin{aligned}
D_2 = &\| \int_0^{t_2} \frac{\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds \\
&- \int_0^{t_1} \frac{\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds\| \\
&\leq \| \int_0^{t_2} \frac{\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds \\
&- \int_0^{t_1} \frac{\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds\| \\
&+ \| \int_0^{t_1} \frac{\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds \\
&- \int_0^{t_1} \frac{\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds\| \\
&\leq \int_{t_1}^{t_2} \frac{\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \|f(s,x(s))\|ds \\
&+ \int_0^{t_1} \|\frac{\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \\
&- \frac{\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\|\|f(s,x(s))\|ds \\
&\leq (L_f\, r + M^*) \int_{t_1}^{t_2} \frac{\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} ds \\
&+ (L_f\, r + M^*) \int_0^{t_1} [\frac{\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \\
&- \frac{\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}] ds \\
&\leq \frac{(L_f\, r + M^*)\,(\psi(t_2)-\psi(t_1))^{\alpha}}{\Gamma(\alpha+1)} \\
&+ \frac{(L_f\, r + M^*)}{\Gamma(\alpha+1)} [(\psi(t_1)-\psi(0))^{\alpha} - (\psi(t_2)-\psi(0))^{\alpha} + (\psi(t_2)-\psi(t_1))^{\alpha}] \\
&\leq \frac{2(L_f\, r + M^*)(\psi(t_2)-\psi(t_1))^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned} \tag{11}$$

Therefore, we have

$$\|Tx(t_2) - Tx(t_1)\|_C \leq \frac{2((M_1\, r + L_f\, r + M^*)}{\Gamma(\alpha+1)} (\psi(t_2)-\psi(t_1))^{\alpha}. \tag{12}$$

Since, ψ is continuous for all $t \in I$. Thus, when $t_1 \to t_2$, $\|Tx(t_2) - Tx(t_1)\|_C \to 0$. Hence $\{Tx(t)\}$ is relatively compact. So, it is implies that $\{Tx(t)\}$ is relatively compact. Thus, T is compact operator.

Step 4.$\Upsilon = \{x \in C(I) : x = \lambda Tx,\ 0 < \lambda < 1\}$ **is bounded.** Let $x \in \Upsilon$ such that

$$x(t) = \lambda(Tx)(t),\ \lambda \in (0,1).$$

Then for each $t \in I$, we have that

$$\begin{aligned}
\|x(t)\| &= \lambda\|Tx\| \\
&= \|\lambda(x_0 + \tfrac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\ A(s)x(s)\ ds \\
&+ \tfrac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\ f(s,x(s))\ ds)\| \\
&\le \lambda(\|x_0\| + \tfrac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\ \|A(s)\|\ \|x(s)\|\ ds \\
&+ \tfrac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \|f(s,x(s))\| ds) \\
&\le \lambda(\|x_0\| + \tfrac{M_1}{\Gamma(\alpha)}\ \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\ \|x(s)\|\ ds \\
&+ \tfrac{L_f}{\Gamma(\alpha)}\ \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\ \|x(s)\|\ ds \\
&+ \tfrac{M^*}{\Gamma(\alpha)}\ \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\ ds) \\
&\le \lambda(\|x_0\| + \tfrac{M_1\ (\psi(1)-\psi(0))^\alpha}{\Gamma(\alpha+1)} \\
&+ \tfrac{M^*+L_f}{\Gamma(\alpha)}\ \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\ \|x(s)\|\ ds) \\
&\le \|x_0\| + \tfrac{M_1\ (\psi(1)-\psi(0))^\alpha}{\Gamma(\alpha+1)} \\
&+ \tfrac{M^*+L_f}{\Gamma(\alpha)}\ \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\ \|x(s)\|\ ds.
\end{aligned} \tag{13}$$

From Gronwall inequality, we get

$$\|x(t)\| \le (\|x_0\| + \tfrac{M_1\ (\psi(1)-\psi(0))^\alpha}{\Gamma(\alpha+1)})\ E_\alpha(M^* + L_f)\ \ (\psi(t)-\psi(0))^\alpha, \tag{14}$$

for all $t \in I$. Hence,

$$\|x(t)\|_C \le \|x_0\| + \tfrac{M_1\ (\psi(1)-\psi(0))^\alpha}{\Gamma(\alpha+1)})\ E_\alpha(M^* + L_f)\ \ (\psi(1)-\psi(0))^\alpha. \tag{15}$$

Thus, Υ is bounded. From Schaefer's theorem, T has at least one fixed point which is a solution to Eq.(2). □

Theorem 3.3 *Let the assumptions (A1)-(A2) hold. Then Eq.(2) has unique solution in $C(I)$, if*

$$\frac{(M_1 + L_f)(\psi(1)-\psi(0))^\alpha}{\Gamma(\alpha+1)} < 1.$$

Proof. Let $x \in B_r$ such that $r = \frac{\|x_0\| + \frac{M^*(\psi(1)-\psi(0))^\alpha}{\Gamma(\alpha+1)}}{1-(M_1+L_f)\frac{(\psi(1)-\psi(0))^\alpha}{\Gamma(\alpha+1)}}$. Then, we have

$$\begin{aligned} \|Tx(t)\| &\leq \|x_0\| + \int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \ \|A(s)\| \ \|x(s)\| ds \\ &+ \int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \ \|f(s,x(s))\| ds \\ &\leq \|x_0\| + M_1 \ r \ \int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} ds \\ &+ \int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \ [L_f \|x(s)\| + \|f(s,0)\|] ds \\ &\leq \|x_0\| + \frac{(M_1 \ r + L_f \ r + M^*)(\psi(1)-\psi(0))^\alpha}{\Gamma(\alpha+1)} = r. \end{aligned} \tag{16}$$

Hence, $TB_r \subseteq B_r$. Let $x, y \in B_r$, then

$$\begin{aligned} \|Tx(t) - Ty(t)\| &\leq \int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \ \|A(s)\| \|x(s) - y(s)\| ds \\ &+ \int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \ \|f(s,x(s)) - f(s,y(s))\| ds \\ &\leq M_1 \int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \ \|x(s) - y(s)\| ds \\ &+ L_f \ \int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \ \|x(s) - y(s)\| ds \\ &\leq \frac{(M_1 + L_f) \ (\psi(1)-\psi(0))^\alpha}{\Gamma(\alpha+1)} \ \|x-y\|_C. \end{aligned} \tag{17}$$

Thus, T is contractive. Hence, there exists a unique fixed point $x \in B_r$, which is the unique solution of Eq.(2). □

Now, we investigate the problem (1). The solution of Eq.(1) is $x \in PC(I, dom(A))$ such that x is $\psi-$Caputo differentiable,

$$^*D_{0^+}^{\alpha,\psi} x(t) = A(t)x(t) + f(t, x(t)), \ t \in I^* = I \setminus \{t_1, t_2, ..., t_m\}, \ \alpha \in (0,1),$$

and x satisfies

$$x(0) + g(x) = x_0, x(t_i^+) = x(t_i^-) + y_i, \ i = 1, 2, ..., m.$$

Let $\varrho \in (0,1)$. We consider the following problem

$$\begin{cases} ^*D_{0^+}^{\alpha,\psi} x(t) = A(t)x(t) + f(t,x(t)) \quad , \quad t \in I \quad , \quad \alpha \in (0,1) \ , \\ x(0) = x_0 - g(u) - \frac{1}{\Gamma(\alpha)} \int_0^\varrho \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1} \ A(s)x(s) \ ds \\ \qquad -\frac{1}{\Gamma(\alpha)} \int_0^\varrho \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1} \ f(s,x(s)) \ ds \ , \end{cases} \tag{18}$$

where $x_0 \in dom(A)$. Then, the solution of (18) is given as

$$\begin{array}{ll} x(t) & = x_0 - g(x) - \frac{1}{\Gamma(\alpha)} \int_0^{\varrho} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\, A(s)x(s)\, ds \\ & -\frac{1}{\Gamma(\alpha)} \int_0^{\varrho} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\, f(s,x(s))\, ds \\ & +\frac{1}{\Gamma(\alpha)} \int_0^{t} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\, A(s)x(s)\, ds \\ & +\frac{1}{\Gamma(\alpha)} \int_0^{t} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\, f(s,x(s)). \end{array} \tag{19}$$

Similar to (18), we have the following Lemma.

Lemma 3.4 *Let the assumptions (A1) and (A3) hold and $\varrho \in (0,1)$. Then solution of the following equation*

$$\begin{cases} {}^{*}D_{0^+}^{\alpha,\psi} x(t) = A(t)x(t) + f(t,x(t)),\ t \in I,\ \alpha \in (0,1) \\ x(\varrho) = x_0 - g(x), \end{cases} \tag{20}$$

is given as

$$\begin{array}{ll} x(t) & = x_0 - g(x) - \frac{1}{\Gamma(\alpha)} \int_0^{\varrho} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\, A(s)x(s)\, ds \\ & -\frac{1}{\Gamma(\alpha)} \int_0^{\varrho} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\, f(s,x(s))\, ds \\ & +\frac{1}{\Gamma(\alpha)} \int_0^{t} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\, A(s)x(s)\, ds \\ & +\frac{1}{\Gamma(\alpha)} \int_0^{t} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\, f(s,x(s))\, ds. \end{array} \tag{21}$$

Lemma 3.5 *Let $x_0, y_1, y_2, ..., y_m \in dom(A)$ and the assumptions (A1) and (A3) hold. Then $x \in PC(I)$ is a solution of Eq.(1) iff it is described as*

$$x(t) = \begin{cases} x_0 - g(x) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} A(s)x(s)ds + \\ \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\, f(s,x(s))\, ds\ ,\ t \in I_0 \\ x_0 - g(x) + y_1 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} A(s)x(s)ds + \\ \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s,x(s))ds\ ,\ t \in I_1 \\ x_0 - g(x) + y_1 + y_2 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} A(s)x(s)ds + \\ \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s,x(s))ds\ ,\ t \in I_2 \\ \vdots \\ x_0 - g(x) + \sum\limits_{i=1}^{m} y_i + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} A(s)x(s)ds + \\ \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s,x(s))ds\ ,\ t \in I_m. \end{cases} \tag{22}$$

Proof. Let x be a solution of Eq.(1). If $t \in I_0$, we get

$$\begin{array}{l} {}^{*}D_{0^+}^{\alpha,\psi} x(t) = A(t)x(t) + f(t,x(t)),\ t \in I_0,\ \alpha \in (0,1), \\ x(0) = x_0 - g(x). \end{array} \tag{23}$$

Provided that,

$$\begin{aligned} x(t) &= x_0 - g(x) + \tfrac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} A(s)x(s)ds \\ &+ \tfrac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\, f(s,x(s))\, ds, t \in I_0. \end{aligned} \tag{24}$$

Conversely, if we ψ-differentiate Eq.(24), we get Eq.(23). If $t \in I_1$, we have

$$\begin{aligned} &{}^*D_{0^+}^{\alpha,\psi} x(t) = A(t)x(t) + f(t,x(t)),\ t \in I_1,\ x(t_1^+) - x(t_1^-) = y_1, \\ &x(0) = x_0 - g(x). \end{aligned} \tag{25}$$

From Eq.(24), we drive the solution of Eq.(25) as

$$\begin{aligned} x(t_1^+) &= x_0 - g(x) + y_1 + \tfrac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} A(s)x(s)ds \\ &+ \tfrac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\, f(s,x(s))\, ds. \end{aligned} \tag{26}$$

From Lemma 3.4, we drive

$$\begin{aligned} x(t) &= x(t_1^+) - \tfrac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} A(s)x(s)ds \\ &- \tfrac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\, f(s,x(s))\, ds \\ &+ \tfrac{1}{\Gamma(\alpha)} \int_0^{t} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} A(s)x(s)ds \\ &+ \tfrac{1}{\Gamma(\alpha)} \int_0^{t} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\, f(s,x(s))\, ds. \end{aligned} \tag{27}$$

So, we establish

$$\begin{aligned} x(t\quad) &= x_0 - g(x) + y_1 + \tfrac{1}{\Gamma(\alpha)} \int_0^{t} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} A(s)x(s)ds \\ &+ \tfrac{1}{\Gamma(\alpha)} \int_0^{t} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\, f(s,x(s))\, ds. \end{aligned} \tag{28}$$

Conversely, if we ψ-differentiate Eq.(28), we obtain Eq.(25). Similarly for $t \in I_2$, we have

$$x(t_2^+) = x(t_2^-) + y_2.$$

By doing the same above steps, we have

$$\begin{aligned} x(t) &= x_0 - g(x) + y_1 + y_2 + \tfrac{1}{\Gamma(\alpha)} \int_0^{t} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} A(s)x(s)ds \\ &+ \tfrac{1}{\Gamma(\alpha)} \int_0^{t} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\, f(s,x(s))\, ds. \end{aligned} \tag{29}$$

Conversely, if we ψ-differentiate Eq.(29), we obtain

$${}^*D_{0^+}^{\alpha,\psi} x(t) = A(t)x(t) + f(t,x(t)),\ t \in I_2\ \ , x(t_2^+) - x(t_2^-) = y_2,\ x(0) = x_0 - g(x), \tag{30}$$

According to the mathematical conclusion, we get the full proof. □

Now, define $\mathcal{S} : PC(I) \to PC(I)$ as:

$$\begin{aligned}\mathcal{S}x(t\ \ ) &= x_0 - g(x) + \sum_{i=1}^{m} y_i\chi_i(t) + \tfrac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\, A(s)x(s)\, ds \\ &+\tfrac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\, f(s,x(s))\, ds\ ,\end{aligned} \tag{31}$$

where

$$\chi_j(t) = \begin{cases} 0 \ , & t \in [0,t_i] \\ 1 \ , & t \in (t_i, t_{i+1}] \end{cases} \tag{32}$$

for all $j < i$. According to the assumptions (A1)-(A3), $\mathcal{S}$ is well defined. Clearly, the solution of the Eq.(1), is the fixed point of the operator $\mathcal{S}$.

Lemma 3.6 *Let $x_0, y_1, y_2, ..., y_m \in dom(A)$. Under the assumptions (A1)-(A3), Eq.(1) has at least on solution.*

Proof. The proof is done in 4 steps.

Step 1. $\mathcal{S}$ is continuous. Let $\{x_n\}$ be a sequence in $PC(I)$ such that $x_n \to x \in PC(I)$ as $n \to \infty$. So, for all $t \in I$, we get

$$\begin{aligned}\|\mathcal{S}x_n(t) - \mathcal{S}x(t)\| &\le \|g(x_n) - g(x)\| \\ &+ \int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\ \|A(s)\|\|x_n(s) \\ &- x(s)\|ds + \int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\ \|f(s,x_n(s)) - f(s,x(s)\|ds \\ &\le L_g\|x_n - x\|_{PC} + M_1\int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\ \|x_n(s)-x(s)\|ds \\ &+ L_f\int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\ \|x_n(s)-x(s)\|ds \\ &\le (L_g + \frac{(M_1+L_f)\ (\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)})\ \|x_n - x\|_{PC}.\end{aligned} \tag{33}$$

Thus, $\|\mathcal{S}x_n - \mathcal{S}x\|_{PC} \to 0$ as $n \to \infty$.

Step 2. For every bounded set B_r in $PC(I)$, $\mathcal{S}B_r$ is uniformly bounded. Let $r > 0$. Define, $B_r = \{v \in PC(I) : \|v\|_{PC} \le r\}$ and let $x \in B_r$, then for all $t \in I$ we get

$$\begin{aligned}
\|\mathcal{S}x(t)\| &\le \|x_0\| + \|g(x)\| + \int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \|A(s)\| \, \|x(s)\| ds \\
&+ \int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \|f(s,x(s))\| ds \\
&\le \|x_0\| + M_2 + M_1 \, r \int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} ds \qquad (34) \\
&+ \int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} [L_f \|x(s)\| + \|f(s,0)\|] ds \\
&\le \|x_0\| + M_2 + \frac{(M_1 \, r + L_f \, r + M^*)(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} = R^*.
\end{aligned}$$

Thus, we get

$$\|\mathcal{S}x\|_{PC} \le R^*. \tag{35}$$

Hence, $\mathcal{S}(B_r)$ is uniformly bounded in $PC(I)$.

Step 3. $\mathcal{S}$ is equi-continuous. By doing the same steps in proofing Lemma 3.2, we can obtain that $\{\mathcal{S}x(t)\}$, $t \in I_0$ is equi-continuous. Thus, $\{\mathcal{S}x(t)\}$, $t \in I_0$ is relatively compact. Consequently, $\{\mathcal{S}x(t)\}$, $t \in I_i$, $i = 1, 2, .., m$ is equi-continuous. Thus, $\{\mathcal{S}x(t)\}$, $t \in I_i$, $i = 1, 2, ..., m$ is relatively compact. Hence $\{\mathcal{S}x(t)\}$, $t \in I_0$ is PC-type-equi-continuous [7]. Thus, $\{\mathcal{S}x(t)\}$ is PC-type- relatively compact. Hence, $\mathcal{S}$ is compact.

Step 4.$\Xi = \{x \in PC(I) : x = \lambda \mathcal{S}x, \ 0 < \lambda < 1\}$ is bounded. Let $x \in \Xi$ such that

$$x(t) = \lambda(\mathcal{S}x)(t) \quad , \lambda \in (0, 1).$$

Then for each $t \in I_0$, we have that

$$\begin{aligned}
\|x(t)\| &= \lambda\|\mathcal{S}x\| = \|\lambda(x_0 + g(x) + \tfrac{1}{\Gamma(\alpha)} \textstyle\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} A(s)x(s) \, ds \\
&+ \tfrac{1}{\Gamma(\alpha)} \textstyle\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s,x(s)) \, ds)\| \\
&\le \lambda(\|x_0\| + \|g(x)\| + \tfrac{1}{\Gamma(\alpha)} \textstyle\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \|A(s)\| \, \|x(s)\| \, ds \\
&+ \tfrac{1}{\Gamma(\alpha)} \textstyle\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \|f(s,x(s))\| ds) \\
&\le \lambda(\|x_0\| + M_2 + \tfrac{M_1}{\Gamma(\alpha)} \textstyle\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \|x(s)\| \, ds \\
&+ \tfrac{L_f}{\Gamma(\alpha)} \textstyle\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \|x(s)\| \, ds \\
&+ \tfrac{M^*}{\Gamma(\alpha)} \textstyle\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \, ds) \\
&\le \lambda(\|x_0\| + M_2 + \tfrac{M^* \, (\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} + \\
&\tfrac{M_1+L_f}{\Gamma(\alpha)} \textstyle\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \|x(s)\| \, ds) \\
&\le \|x_0\| + M_2 + \tfrac{M^* \, (\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} + \\
&\tfrac{M_1+L_f}{\Gamma(\alpha)} \textstyle\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \|x(s)\| \, ds.
\end{aligned} \tag{36}$$

From Gronwall inequality again, we get

$$\|x(t)\| \leq (\|x_0\| + M_2 + \tfrac{M^*\ (\psi(1)-\psi(0))^\alpha}{\Gamma(\alpha+1)})\ E_\alpha(M_1 + L_f)\ \ (\psi(t) - \psi(0))^\alpha, \quad (37)$$

for all $t \in I$. Hence,

$$\|x(t)\|_{PC} \leq (\|x_0\| + M_2 + \tfrac{M^*\ (\psi(1)-\psi(0))^\alpha}{\Gamma(\alpha+1)})\ E_\alpha(M_1 + L_f)\ \ (\psi(1) - \psi(0))^\alpha \quad (38)$$

Therefore, for all $t \in I_i$, $i = 1, 2, ..., m$ and $x \in Pc(I)$ we get

$$\begin{aligned}\|x(t)\| \ &\leq \|\lambda(x_0 + g(x) + \sum_{k=0}^{k=i-1} y_k + \tfrac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\ A(s)x(s)\ ds \\ &\quad + \tfrac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\ f(s, x(s))\ ds)\| \\ &\leq (\|x_0\| + M_2 + \sum_{k=0}^{i-1} \|y_k\| + \tfrac{M^*\ (\psi(1)-\psi(0))^\alpha}{\Gamma(\alpha+1)})\ E_\alpha(M_1 + L_f)\ \ (\psi(1) - \psi(0))^\alpha.\end{aligned} \quad (39)$$

Hence,

$$\|x\|_{PC} \leq (\|x_0\| + M_2 + \sum_{k=0}^{i-1} \|y_k\| + \tfrac{M^*\ (\psi(1)-\psi(0))^\alpha}{\Gamma(\alpha+1)})\ E_\alpha(M_1 + L_f)\ \ (\psi(1) - \psi(0))^\alpha. \quad (40)$$

Thus, Ξ is bounded. From Schaefer's theorem, $\mathcal{S}$ has at least one fixed point which is a solution to Eq.(1). □

Theorem 3.7 *Let $x_0, y_1, ..., y_m \in dom(A)$ and the assumptions (A1)-(A3) hold. Then Eq.(1) has unique solution in $PC(I)$, if*

$$L_g + \frac{(M_1 + L_f)(\psi(1) - \psi(0))^\alpha}{\Gamma(\alpha+1)} < 1.$$

Proof. Let $t \in I_0$. Let $x \in PC(I_0)$, then

$$\begin{aligned}\|\mathcal{S}x(t+\rho) - \mathcal{S}x(t)\| \leq \| &\int_0^{t+\rho} \frac{\psi'(s)(\psi(t+\rho) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)}\ A(s)x(s)ds \\ &- \int_0^{t} \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)}\ A(s)x(s)ds\| \\ &+ \| \int_0^{t_2} \frac{\psi'(s)(\psi(t_2) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)}\ f(s, x(s))ds \\ &- \int_0^{t} \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)}\ f(s, x(s))ds\| \\ &= G_1 + G_2,\end{aligned} \quad (41)$$

where

$$\begin{aligned}
G_1 &= \| \int_0^{t+\rho} \frac{\psi'(s)(\psi(t+\rho)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} A(s)x(s)ds \\
&- \int_0^{t} \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} A(s)x(s)ds\| \\
&\leq \| \int_0^{t+\rho} \frac{\psi'(s)(\psi(t+\rho)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} A(s)x(s)ds \\
&- \int_0^{t} \frac{\psi'(s)(\psi(t+\rho)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} A(s)x(s)ds\| \\
&+ \| \int_0^{t} \frac{\psi'(s)(\psi(t+\rho)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} A(s)x(s)ds \qquad (42) \\
&- \int_0^{t} \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} A(s)x(s)ds\| \\
&\leq \int_t^{t+\rho} \frac{\psi'(s)(\psi(t+\rho)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \|A(s)\| \, \|x(s)\|ds \\
&+ \int_0^{t} \| \frac{\psi'(s)(\psi(t+\rho)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \\
&- \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \| \|A(s)\| \, \|x(s)\|ds,
\end{aligned}$$

therefore, we have

$$\begin{aligned}
G_1 &\leq M_1 \, r \int_t^{t+\rho} \frac{\psi'(s)(\psi(t+\rho)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} ds \\
&+ M_1 \, r \int_0^{t} [\frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} - \frac{\psi'(s)(\psi(t+\rho)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}] ds \\
&\leq \frac{M_1 \, r \, (\psi(t+\rho)-\psi(t))^{\alpha}}{\Gamma(\alpha+1)} + \qquad (43) \\
& \frac{M_1 \, r}{\Gamma(\alpha+1)} [(\psi(t)-\psi(0))^{\alpha} - (\psi(t+\rho)-\psi(0))^{\alpha} + (\psi(t+\rho)-\psi(t))^{\alpha}] \\
&\leq \frac{2M_1 \, r \, (\psi(t+\rho)-\psi(t))^{\alpha}}{\Gamma(\alpha+1)}
\end{aligned}$$

and

$$
\begin{aligned}
G_2 &= \| \int_0^{t+\rho} \frac{\psi'(s)(\psi(t+\rho)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds \\
&- \int_0^{t} \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds \| \\
&\leq \| \int_0^{t+\rho} \frac{\psi'(s)(\psi(t+\rho)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds \\
&- \int_0^{t} \frac{\psi'(s)(\psi(t+\rho)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds \| \\
&+ \| \int_0^{t} \frac{\psi'(s)(\psi(t+\rho)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds \\
&- \int_0^{t} \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds \| \\
&\leq \int_t^{t+\rho} \frac{\psi'(s)(\psi(t+\rho)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \|f(s,x(s))\| ds \\
&+ \int_0^{t} \| \frac{\psi'(s)(\psi(t+\rho)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \\
&- \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \| \|f(s,x(s))\| ds \\
&\leq (L_f\, r + M^*) \int_t^{t+\rho} \frac{\psi'(s)(\psi(t+\rho)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} ds \\
&+ (L_f\, r + M^*) \int_0^{t} [\frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} - \frac{\psi'(s)(\psi(t+\rho)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}] ds \\
&\leq \frac{(L_f\, r + M^*)\ (\psi(t+\rho)-\psi(t))^{\alpha}}{\Gamma(\alpha+1)} \\
&+ \frac{(L_f\, r + M^*)}{\Gamma(\alpha+1)} [(\psi(t)-\psi(0))^{\alpha} - (\psi(t+\rho)-\psi(0))^{\alpha} + (\psi(t+\rho)-\psi(t))^{\alpha}] \\
&\leq \frac{2(L_f\, r + M^*)(\psi(t+\rho)-\psi(t))^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}
\tag{44}
$$

Therefore, we have

$$
\|\mathcal{S}x(t+\rho) - \mathcal{S}x(t)\| \leq \frac{2((M_1\, r + L_f\, r + M^*)}{\Gamma(\alpha+1)} (\psi(t+\rho)-\psi(t))^{\alpha}. \tag{45}
$$

Since, ψ is continuous for all $t \in I$. As $\rho \to 0$, then $\|\mathcal{S}x(t+\rho) - \mathcal{S}x(t)\| \to 0$. Hence, $\mathcal{S}x \in PC(I_0)$. Consequently, for all $t \in I_i$, $i = 1, 2, ..., m$, we obtain $\mathcal{S}x \in PC(I_i)$. Hence, $\mathcal{S}x \in PC(I)$, for all $x \in PC(I)$.

Let $x, y (x \neq y)$ be two solutions of Eq.(1), then

$$\begin{aligned}
\|\mathcal{S}x(t) - \mathcal{S}y(t)\| &\leq \|g(x) - g(y)\| + \int_0^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \|A(s)\| \|x(s) - y(s)\| ds \\
&+ \int_0^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \|f(s, x(s)) - f(s, y(s))\| ds \\
&\leq L_g \|x - y\|_{PC} + M_1 \int_0^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \|x(s) - y(s)\| ds \\
&+ L_f \int_0^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \|x(s) - y(s)\| ds \\
&\leq (L_g + \frac{(M_1 + L_f)\ (\psi(1) - \psi(0))^{\alpha}}{\Gamma(\alpha + 1)})\ \|x - y\|_{PC}.
\end{aligned} \tag{46}$$

Therefore,

$$\|\mathcal{S}x - \mathcal{S}y\|_{PC} \leq (L_g + \frac{(M_1 + L_f)\ (\psi(1) - \psi(0))^{\alpha}}{\Gamma(\alpha + 1)})\ \|x - y\|_{PC}.$$

Thus, $\mathcal{S}$ is contractive. Hence, there exists a unique fixed point $x \in PC(I)$, which is the unique solution of Eq.(1). $\square$

4 An application

In this section, we give the following example, which point to how to apply the abstract results in particular IFDEs. Let $E = \mathbb{R}$. Consider the following IFDE

$$\begin{cases}
{}^{*}D_{0^+}^{\frac{1}{2},\psi} x(t) = \frac{t}{20} x(t) + \frac{|x(t)|}{20(1+|x(t)|)},\ t \in [0,1] \setminus \{0.6\}\ , \\
x(0) + \int\limits_0^{0.2} x(t) dt = 1\ , \\
x(0.6^+) = x(0.6^-) + 0.3.
\end{cases} \tag{47}$$

Let $\psi(t) = \frac{t^2+t}{2}$. Here, we get $\alpha = 0.5$,

$$A(t) = \frac{t}{20},$$

$$f(t, x) = \frac{|x(t)|}{20(1 + |x(t)|)},$$

$$g(x) = \int\limits_0^{0.2} x(t) dt.$$

It is clear that all the maps A, f and g are continuous. Furthermore, we have:

$$|f(t,u_1) - f(t,u_2)| \leq \frac{1}{20}|u_1 - u_2|,$$

$$|g(u_1) - g(u_2)| \leq \frac{1}{5}|u_1 - u_2|,$$

for all $t \in [0,1]$ and $u_1, u_2 \in \mathbb{R}$. Consequently, we have $L_f = \frac{1}{20}$, $L_g = \frac{1}{5}$, $M_1 = \frac{1}{20}$. Therefore, we get

$$L_g + \frac{(M_1 + L_f)(\psi(1) - \psi(0))^\alpha}{\Gamma(\alpha+1)} = 0.31284134506 < 1$$

Thus, all the assumptions from (A1)-(A3) are satisfied. From Theorem 3.7, we conclude that the Eq.(47) has at unique solution.

5 Open Problem

How to generalize these theorems for non-instantaneous impulsive fractional differential equation involving $\psi-$Caputo derivative?

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