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# An Extension of Erlang Distribution with Properties Having Applications In Engineering and Medical-Science

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***Abstract***

***In this work an extension of Erlang distribution have been introduced, which is in fact a generalization of Erlang distribution. This generalization is derived by applying power transformation technique and it gives more flexibility to investigate complex real life data. The distinct structural properties of the formulated distribution including moments, moment generating function, skewness, kurtosis, incomplete moments, mode, median, order statistics, different measure of entropies, mean deviations, Bonferroni and Lorenz curves have been discussed. In addition expressions for survival function, hazard rate function, reverse hazard rate function and mean residual function are obtained explicitly. The behaviour of probability density function (p.d.f) and cumulative distribution function (c.d.f) are illustrated through different graphs. The estimation of the established distribution parameters are performed by maximum likelihood estimation method. Eventually the versatility of the established distribution is examined through two real life data sets related to engineering and medical science.***

**Key words:-** *Erlang distribution, power transformation technique, moments, entropies, reliability measures, maximum likelihood estimation.*
**Mathematics subject classification:** 60-XX, 62-XX, 11-KXX

# 1 Introduction

Analysing or modelling of complex real life data collected from different fields of science like bio-medicine, engineering, actuarial science and environment is challenging among researchers. In these situations an emergence of new distributions or modifications of the existing distributions become need of an hour. In the literature of statistics there exist various methods to address these problems. Power transformation of probability distributions is one of the known technique by which an extra parameter is added to the parent distribution. The resulting distributions obtained after applying different methods become richer or more flexible for analysing diverse data. Erlang distribution is one of the popular continuous probability distribution having following probability density function.

$$f(x,\alpha,\theta)=\frac{\alpha^{\theta}}{\Gamma(\theta)}x^{\theta-1}e^{-\alpha x} \quad ; x>0, \alpha>0, \theta=1,2,3,\ldots \tag{1.1}$$

The corresponding distribution function is given by

$$F(x,\alpha,\theta)=\frac{\gamma(\theta,\alpha x)}{\Gamma(\theta)} \quad ; x>0, \alpha>0, \theta=1,2,3,\ldots \tag{1.2}$$

It was established by A.K. Erlang [6] to examine the number of telephone calls which are received simultaneously to the operators of the switching stations. This distribution has since been extended for use in queuing theory, the mathematical study of waiting in lines. Erlang's distribution has also vast applications in mathematical biology and stochastic process. Due to vast applications of Erlang distribution researcher have made lot of work to improve this distribution. Haq and Dey [8] have studied the Bayes analysis of the Erlang distribution and used different informative priors. Bhattacharya and Singh [5] obtained a Bayes estimator for the Erlangian que based on two prior distributions. Khan and Jan [11] used various generalized truncated prior distribution to discuss and obtain estimates for the Erlang distribution. Sofi Mudasir at el [14] proposed and studied characterization and information measures of weighted Erlang distribution. Hesham at el [9] introduced Length-biased weighted Erlang distribution also they discussed its various properties. Sofi Mudasir at el [15] used different priors to obtain parameter estimation of weighted Erlang distribution. They used R software and expounded the performance of the distribution through real life data set. In this paper we generalize Erlang distribution by power transformation approach. This technique has attracted the attention of researchers from past decade due to which several papers has come into existence. Ghitany et al [7] established power Lindley distribution and elaborate its several structural properties. Krishnarani [12] used power transformation method on Half – Logistic distribution and studied its different characteristics. Rady et al [18] introduced power Lomax distribution, then the resulting distribution and applied it on medical related data set. Shukla et al [16] proposed and examine the performance of power Ishita distribution. A. A. Bhat et al [1] studied and formulated a new generalization of Rayleigh distribution by applying the power transformation method.

## 2 Power Erlang Distribution

Let us suppose $X$ be a random variable follows probability density function $(1.1)$, then the transformation $Y = X^{\frac{1}{\beta}}$ is said to follow power Erlang distribution denoted as $Y \sim PErD(\alpha, \beta, \theta)$ if its probability density function (p.d.f) is give as

$$f(y, \alpha, \beta, \theta) = \frac{\alpha^{\theta}\beta}{\Gamma(\theta)} y^{\beta\theta - 1} e^{-\alpha y^{\beta}} \quad ; y > 0; \alpha, \beta > 0, \theta = 1,2,3,... \qquad (2.1)$$

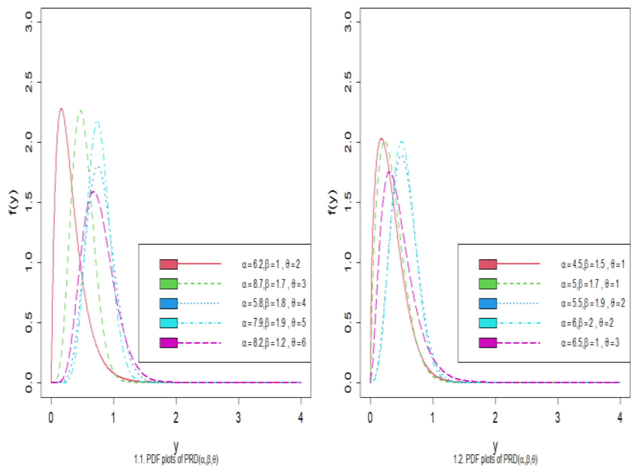


Figure (1.1) and (1.2), illustrates the behaviour of the p.d.f of power Erlang distribution for varying values of the parameters.

The corresponding cumulative distribution function (c.d.f) is given by

$$F(y,\alpha,\beta,\theta)=\frac{\gamma(\theta,\alpha y^{\beta})}{\Gamma(\theta)} \quad ;y>0;\alpha,\beta>0,\theta=1,2,3,\ldots \qquad (2.2)$$

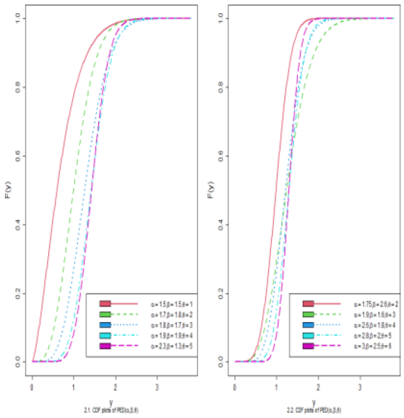

Figure (2.1) and (2.2), illustrates the behaviour of the c.d.f of power Erlang distribution for varying values of the parameters.

## 3 Reliability Analysis

Suppose $Y$ be a continuous random variable with c.d.f $F(y)$, $y\geq 0$.Then its reliability function which is also called survival function is defined as

$$S(y)=p_r(Y>y)=\int_y^{\infty} f(y)dy=1-F(y)$$

Therefore, the survival function for power Erlang distribution is given as

$$S(y,\alpha,\beta,\theta)=1-F(y,\alpha,\beta,\theta)$$

$$=1-\frac{\gamma(\theta,\alpha y^{\beta})}{\Gamma(\theta)} \qquad (3.1)$$

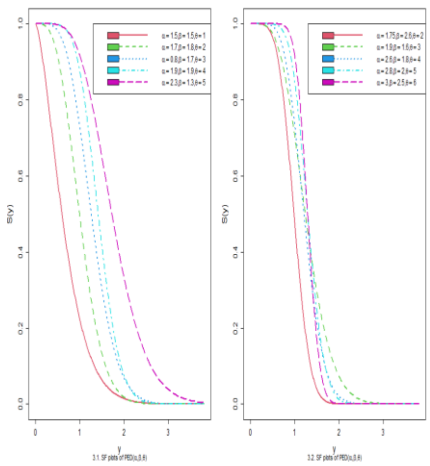

Figure (3.1) and (3.2), illustrates the behaviour of the survival function of power Erlang distribution for varying values of the parameters.

The hazard rate function of a random variable $y$ is given as

$$H(y,\alpha,\beta,\theta)=\frac{f(y,\alpha,\beta,\theta)}{S(y,\alpha,\beta,\theta)} \qquad (3.2)$$

Using equation $(2.1)$ and equation $(3.1)$ in $(3.2)$, we get

$$H(y,\alpha,\beta,\theta)=\frac{\alpha^{\beta}\beta y^{\beta\theta-1}e^{-\alpha y^{\beta}}}{\Gamma(\theta)-\gamma(\theta,\alpha y^{\beta})}$$

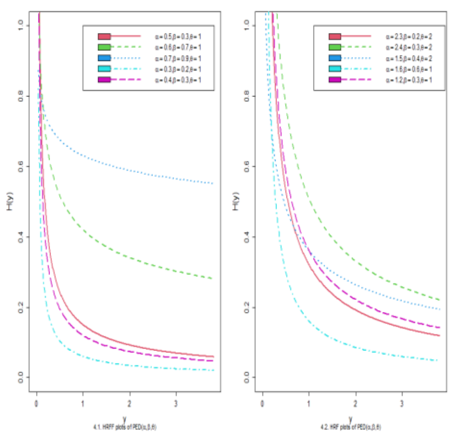

Figure (4.1) and (4.2), illustrates the behaviour of the hazard rate function of power Erlang distribution for varying values of the parameters.

Reverse hazard rate function of random variable $y$ is given by

$$h_r(y,\alpha,\beta,\theta)=\frac{f(y,\alpha,\beta,\theta)}{F(y,\alpha,\beta,\theta)} \tag{3.3}$$

Using equation $(2.1)$ and equation $(2.2)$ in $(3.3)$, we get

$$h_r(y,\alpha.\beta.\theta)=\frac{\alpha^{\theta}\beta y^{\beta\theta-1}e^{-\alpha y^{\beta}}}{\gamma(\theta,\alpha y^{\beta})}$$

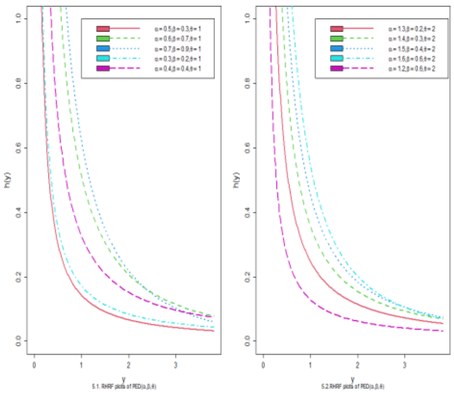


Figure (5.1) and (5.2), illustrates the behaviour of the reverse hazard rate function of power Erlang distribution for varying values of the parameters.

Mean residual function of random variable $y$ can be obtained as

$$m(y,\alpha,\beta,\theta)=\frac{1}{S(y,\alpha,\beta,\theta)}\int_{y}^{\infty} tf(t,\alpha,\beta,\theta)dt-y$$

$$=\frac{\Gamma(\theta)}{\Gamma(\theta)-\gamma(\alpha,\beta,\theta)}\int_{y}^{\infty} t\frac{\alpha^{\theta}\beta}{\Gamma(\theta)}t^{\beta\theta-1}e^{-\alpha y^{\beta}}dt-y$$

$$=\frac{\alpha^{\theta}\beta}{\Gamma(\theta)-\gamma(\theta,\alpha y^{\beta})}\int_{y}^{\infty} t^{\beta\theta}e^{-\alpha y^{\beta}}dt-y$$

After solving the integral, we get

$$m(y,\alpha,\beta,\theta)=\frac{\alpha^{\theta-1}\Gamma\left(\theta+\frac{1}{\beta},y^{\beta}\right)}{\Gamma(\theta)-\gamma\left(\theta,\alpha y^{\beta}\right)}-y$$

# 4 Mathematical Properties Power Erlang Distribution

## 4.1 Moments of power Erlang distribution

Let $Y$ be a random variable follows power Erlang distribution. Then $r^{th}$ moment denoted by $\mu_r^{'}$ is given as

$$\mu_r^{'}=E\left(Y^r\right)=\int_0^{\infty} y^r f\left(y,\alpha,\beta,\theta\right)dy$$

$$=\int_0^{\infty} y^r \frac{\alpha^{\theta}\beta}{\Gamma(\theta)} y^{\beta\theta-1}e^{-\alpha y^{\beta}}dy$$

$$=\frac{\alpha^{\theta}\beta}{\Gamma(\theta)}\int_0^{\infty} y^{r+\beta\theta-1}e^{-\alpha y^{\beta}}dy$$

Making substitution $y^{\beta}=z$ and $0<z<\infty$, we have

$$\mu_r^{'}=\frac{\alpha^{\theta}}{\Gamma(\theta)}\int_0^{\infty} z^{\frac{r}{\beta}+\theta-1}e^{-\alpha z}dz$$

$$\mu_r^{'}=\frac{\Gamma\left(\theta+\frac{r}{\beta}\right)}{\alpha^{\frac{r}{\beta}}\Gamma(\theta)}$$

Now substituting $r=1,2,3,4$ we obtain first four moments about origin of power Erlang distribution

$$\mu_1^{'}=\frac{\Gamma\left(\theta+\frac{1}{\beta}\right)}{\alpha^{\frac{1}{\beta}}\Gamma(\theta)}\quad,\quad \mu_2^{'}=\frac{\Gamma\left(\theta+\frac{2}{\beta}\right)}{\alpha^{\frac{2}{\beta}}\Gamma(\theta)}$$

$$\mu_3' = \frac{\Gamma\left(\theta + \frac{3}{\beta}\right)}{\alpha^{\frac{3}{\beta}}\Gamma(\theta)} \quad , \quad \mu_4' = \frac{\Gamma\left(\theta + \frac{4}{\beta}\right)}{\alpha^{\frac{4}{\beta}}\Gamma(\theta)}$$

The moments about mean of the power Erlang distribution are obtained by using relationship between moments about mean and moments about origin

$$\mu_2 = \frac{\Gamma\left(\theta + \frac{2}{\beta}\right)\Gamma(\theta) - \left\{\Gamma\left(\theta + \frac{1}{\beta}\right)\right\}^2}{\alpha^{\frac{2}{\beta}}(\Gamma(\theta))^2}$$

$$\mu_3 = \frac{\left\{(\Gamma(\theta))^2\Gamma\left(\theta + \frac{3}{\beta}\right) - 3\Gamma(\theta)\Gamma\left(\theta + \frac{2}{\beta}\right) + 2\left(\Gamma\left(\theta + \frac{1}{\beta}\right)\right)^3\right\}}{\alpha^{\frac{3}{\beta}}(\Gamma(\theta))^3}$$

$$\mu_4 = \frac{\left\{\begin{matrix}(\Gamma(\theta))^3\Gamma\left(\theta + \frac{4}{\beta}\right) - 4(\Gamma(\theta))^2\Gamma\left(\theta + \frac{3}{\beta}\right)\Gamma\left(\theta + \frac{1}{\beta}\right) \\ + 6\Gamma(\theta)\Gamma\left(\theta + \frac{2}{\beta}\right)\left(\Gamma\left(\theta + \frac{1}{\beta}\right)\right)^2 - 3\left(\Gamma\left(\theta + \frac{1}{\beta}\right)\right)^4\end{matrix}\right\}}{\alpha^{\frac{4}{\beta}}(\Gamma(\theta))^4}$$

The standard deviation (S.D), coefficient of variation (c.v), coefficient of skewness $\left(\sqrt{\beta_1}\right)$, coefficient of kurtosis $(\beta_2)$, index of dispersion $(\gamma)$ of power Erlang distribution are determined as

$$\sigma = \frac{\sqrt{\Gamma\left(\theta + \frac{2}{\beta}\right)\Gamma(\theta) - \left\{\Gamma\left(\theta + \frac{1}{\beta}\right)\right\}^2}}{\alpha^{\frac{1}{\beta}}\Gamma(\theta)}$$

$$C.V = \frac{\sigma}{\mu_1'} = \frac{\sqrt{\Gamma\left(\theta + \frac{2}{\beta}\right)\Gamma(\theta) - \left\{\Gamma\left(\theta + \frac{1}{\beta}\right)\right\}^2}}{\Gamma\left(\theta + \frac{1}{\beta}\right)}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}} = \frac{\left\{(\Gamma(\theta))^2\Gamma\left(\theta+\frac{3}{\beta}\right) - 3\Gamma(\theta)\Gamma\left(\theta+\frac{2}{\beta}\right) + 2\left(\Gamma\left(\theta+\frac{1}{\beta}\right)\right)^3\right\}}{\Gamma\left(\theta+\frac{2}{\beta}\right)\Gamma(\theta) - \left\{\Gamma\left(\theta+\frac{1}{\beta}\right)\right\}^2}$$

$$\beta_1 = \frac{\mu_4}{(\mu_2)^2} = \frac{\left\{\begin{array}{l}(\Gamma(\theta))^3\Gamma\left(\theta+\frac{4}{\beta}\right) - 4(\Gamma(\theta))^2\Gamma\left(\theta+\frac{3}{\beta}\right)\Gamma\left(\theta+\frac{1}{\beta}\right)\\ +6\Gamma(\theta)\Gamma\left(\theta+\frac{2}{\beta}\right)\left(\Gamma\left(\theta+\frac{1}{\beta}\right)\right)^2 - 3\left(\Gamma\left(\theta+\frac{1}{\beta}\right)\right)^4\end{array}\right\}}{\left[\Gamma\left(\theta+\frac{2}{\beta}\right)\Gamma(\theta) - \left\{\Gamma\left(\theta+\frac{1}{\beta}\right)\right\}^2\right]^2}$$

$$\gamma = \frac{\sigma^2}{\mu_1^{'}} = \frac{\Gamma\left(\theta+\frac{2}{\beta}\right)\Gamma(\theta) - \left\{\Gamma\left(\theta+\frac{1}{\beta}\right)\right\}^2}{\alpha^{\frac{1}{\beta}}\Gamma(\theta)\Gamma\left(\theta+\frac{1}{\beta}\right)}$$

**Table 1**

**Brief description of PErD for different values of parameter combinations**

| $\alpha$ | $\beta$ | $\theta$ | Mean | Var. $\sigma^2$ | C.V | Skew. | Kurt. | I.D $\gamma$ |
|---|---|---|---|---|---|---|---|---|
| 1.5 | 0.9 | 0.3 | 1.6905 | 1.1819 | 2.0342 | 7.3239 | -194.5 | 0.7816 |
| 1.7 | 1.3 | 0.5 | 1.0637 | 0.6708 | 1.1009 | 1.3820 | -147.2 | 0.4104 |
| 1.9 | 1.7 | 0.6 | 0.9406 | 0.4932 | 0.7894 | 0.2328 | -199.7 | 0.2643 |
| 2.0 | 1.8 | 0.7 | 0.7995 | 0.5736 | 0.6928 | -0.270 | -230.7 | 0.2277 |
| 2.2 | 2.3 | 0.8 | 0.7516 | 0.4865 | 0.5203 | -1.016 | -418.7 | 0.1501 |
| 2.4 | 2.4 | 1.5 | 0.5954 | 0.7006 | 0.3564 | -0.761 | -1407.6 | 0.0963 |
| 3.0 | 2.4 | 2.0 | 0.7945 | 1.0931 | 0.3056 | 9.071 | -4562.9 | 0.0742 |

We observed from table 1 that by increasing values of parameters mean, variance, coefficient of variation and index of dispersion decreases. But skewness and kurtosis are in negative values.

4.2 Moment generating function of power Erlang distribution

Let $Y$ be a random variable follows power Erlang distribution. Then the moment generating function of the distribution denoted by $M_Y(t)$ is given

$$M_Y(t) = E(e^{ty}) = \int_0^\infty e^{ty} f(y,\alpha,\beta,\theta)dy$$

Using Taylor's series

$$= \int_0^\infty \left(1 + ty + \frac{(ty)^2}{2!} + \frac{(ty)^3}{3!} + \dots\right) f(y,\alpha,\beta,\theta)dy$$

$$= \sum_{r=0}^\infty \frac{t^r}{r!} \int_0^\infty y^r f(y,\alpha,\beta,\theta)dy$$

$$= \sum_{r=0}^\infty \frac{t^r}{r!} E(Y^r)$$

$$M_Y(t) = \sum_{r=0}^\infty \frac{t^r}{r!} \frac{\Gamma\left(\theta + \frac{r}{\beta}\right)}{\alpha^{\frac{r}{\beta}} \Gamma(\theta)}$$

4.3 Characteristics function of power Erlang distribution

Let $Y$ be a random variable follows power Erlang distribution. Then the characteristics function of the distribution denoted by $\phi_Y(t)$ is given

$$\phi_Y(t) = E(e^{ity}) = \int_0^\infty e^{ity} f(y,\alpha,\beta,\theta)dy$$

Using Taylor's series

$$= \int_0^\infty \left(1 + ity + \frac{(ity)^2}{2!} + \frac{(ity)^3}{3!} + \dots\right) f(y,\alpha,\beta,\theta)dy$$

$$= \sum_{r=0}^\infty \frac{(it)^r}{r!} \int_0^\infty y^r f(y,\alpha,\beta,\theta)dy$$

$$= \sum_{r=0}^\infty \frac{(it)^r}{r!} E(Y^r)$$

$$\phi_Y(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \frac{\Gamma\left(\theta + \frac{r}{\beta}\right)}{\alpha^{\frac{r}{\beta}} \Gamma(\theta)}$$

4.4 Incomplete moments of power Erlang distribution

The $q^{th}$ incomplete moment of power Erlang distribution about origin is given by

$$T_q(s) = \int_0^s y^s f(y, \alpha, \beta, \theta) dy$$

$$= \frac{\alpha^\theta \beta}{\Gamma(\theta)} \int_0^s y^{s+\beta\theta-1} e^{-\alpha y^\beta} dy$$

Making substitution $\alpha y^\beta = z$ and $0 < z < \alpha s^\beta$, we have

$$T_q(s) = \frac{\alpha^{-\frac{s}{\beta}}}{\Gamma(\theta)} \int_0^{\alpha s^\beta} z^{\frac{s}{\beta}+\theta-1} e^{-z} dz$$

$$T_q(s) = \frac{\alpha^{-\frac{s}{\beta}}}{\Gamma(\theta)} \gamma\left(\frac{s}{\beta} + \theta, \alpha s^\beta\right)$$

4.5 Harmonic mean of power Erlang distribution

The harmonic mean of power Erlang distribution is given by

$$\frac{1}{H} = E\left(\frac{1}{Y}\right) = \int_0^\infty \frac{1}{y} f(y, \alpha, \beta, \theta) dy$$

$$= \int_0^\infty \frac{1}{y} \frac{\alpha^\theta \beta}{\Gamma(\theta)} y^{\beta\theta-1} e^{-\alpha y^\beta} dy$$

$$= \frac{\alpha^\theta \beta}{\Gamma(\theta)} \int_0^\infty y^{\beta\theta-2} e^{-\alpha y^\beta} dy$$

Making substitution $y^\beta = z$, we have

$$\frac{1}{H}=\frac{\alpha^{\theta}\beta}{\Gamma(\theta)}\int_0^\infty z^{\theta-\frac{1}{\beta}-1}e^{-\alpha z}dz$$

$$=\frac{\alpha^{\frac{1}{\beta}}}{\Gamma(\theta)}\Gamma\left(\theta-\frac{1}{\beta}\right)$$

4.6 Mode and median of power Erlang distribution

Taking logarithm of equation $(2.1)$, we have

$$\log f(y,\alpha,\beta,\theta)=\theta\log\alpha+\log\beta-\log\Gamma(\theta)+(\beta\theta-1)\log y-\alpha y^{\beta}$$

$(4.1)$ Differentiate equation $(4.1)$, with respect to $y$, we obtain

$$\frac{\partial\log f(y,\alpha,\beta,\theta)}{\partial y}=\frac{\beta\theta-1}{y}-\alpha\beta y^{\beta-1}$$

Substituting $\frac{\partial\log f(y,\alpha,\beta,\theta)}{\partial y}=0$, we get

$$y=\left(\frac{\beta\theta-1}{\alpha\beta}\right)^{\frac{1}{\beta-2}}\Rightarrow M_0=y_0=\left(\frac{\beta\theta-1}{\alpha\beta}\right)^{\frac{1}{\beta-2}}$$

Using the empirical formula for median, we get

$$M_d=\frac{1}{3}M_0+\frac{2}{3}\mu_1'$$

$$M_d=\frac{1}{3}\left\{\left(\frac{\beta\theta-1}{\alpha\beta}\right)^{\frac{1}{\beta-2}}+\frac{2\Gamma\left(\theta+\frac{1}{\beta}\right)}{\alpha^{\frac{1}{\beta}}\Gamma(\theta)}\right\}$$

## 5 Shanon's Entropy Of Power Erlang Distribution

The notion of entropy was introduced by Shanon in 1948. The entropy can be interpreted as the average rate at which information is produced by a random source of data and is given as

$$H(y) = -E[\log f(y)] = -\int_0^\infty [\log f(y)] f(y) dy$$

Provided the integral is convergent

Thus the Shanon's entropy for power Erlang distribution can be calculated as

$$H(y,\alpha,\beta,\theta) = -E[\log f(y,\alpha,\beta,\theta)]$$

$$= -E\left[\log\left(\frac{\alpha^\theta \beta}{\Gamma(\theta)} y^{\beta\theta-1} e^{-\alpha y^\beta}\right)\right]$$

$$= -E\left\{\log\left(\frac{\alpha^\theta \beta}{\Gamma(\theta)}\right)\right\} - (1-\beta\theta)E(\log y) + \alpha E\left(y^\beta\right) \qquad (5.1)$$

Now

$$E(\log y) = \int_0^\infty (\log y) f(y,\alpha,\beta,\theta) dy$$

$$= \frac{\alpha^\theta}{\Gamma(\theta)} \int_0^\infty (\log y) y^{\beta\theta-1} e^{-\alpha y^\beta} dy$$

Making substitution $\alpha^\theta \beta = z$ we have

$$E(\log y) = \frac{\alpha^{\frac{1}{\beta}-1}}{\beta\Gamma(\theta)} \int_0^\infty \left(\log \frac{z}{\alpha}\right) z^{\theta-\frac{1}{\beta}} e^{-z} dz$$

After solving the integral we get

$$E(\log y) = \frac{\alpha^{\frac{1}{\beta}-1}}{\beta\Gamma(\theta)} \left[\Gamma^{'}\left(\theta - \frac{1}{\beta} + 1\right) - \log \alpha \Gamma\left(\theta - \frac{1}{\beta}\right)\right] \qquad (5.2)$$

Also

$$E\left(y^{\beta}\right)=\frac{\alpha^{\theta}\beta}{\Gamma(\theta)}\int_{0}^{\infty} y^{\beta(\theta+1)-1}e^{-\alpha y^{\beta}}\,dy$$

Making substitution $y^{\beta}=z$, we get

$$E\left(y^{\beta}\right)=\frac{\alpha^{\theta}}{\Gamma(\theta)}\int_{0}^{\infty} z^{\theta}e^{-\alpha z}\,dz$$

After solving the integral, we get

$$E\left(y^{\beta}\right)=\frac{\theta+1}{\alpha} \tag{5.3}$$

Using equation $(5.2)$, $(5.3)$ in equation $(5.1)$, we get

$$H(y,\alpha,\beta,\theta)=\log\left(\frac{\Gamma(\theta)}{\alpha^{\theta}\beta}\right)+\frac{(1-\beta\theta)\alpha^{\frac{1-\beta}{\beta}}}{\beta\Gamma\theta}\left[\begin{array}{l}\Gamma'\left(\frac{\beta\theta+\beta-1}{\beta}\right)-\Gamma\left(\frac{\beta\theta+\beta-1}{\beta}\right)\log\alpha\\ +\theta+1\end{array}\right]$$

# 6 Renyi Entropy Of Power Erlang Distribution

If $Y$ is a continuous random variable having probability density function $f(y,\alpha,\beta,\theta)$. Then

Renyi entropy is defined as

$$T_R(\gamma)=\frac{1}{1-\gamma}\log\left\{\int_{0}^{\infty} f^{\gamma}(y)\,dy\right\},\ \text{where } \gamma>0 \text{ and } \gamma\neq 1$$

Thus, the Renyi entropy for power Erlang distribution $(2.1)$, is given as

$$T_R(\gamma)=\frac{1}{1-\gamma}\log\left\{\int_{0}^{\infty}\left[\frac{\alpha^{\theta}\beta}{\Gamma(\theta)}y^{\beta\theta-1}e^{-\alpha y^{\beta}}\right]^{\gamma}dy\right\}$$

$$=\frac{1}{1-\gamma}\log\left\{\left(\frac{\alpha^{\theta}\beta}{\Gamma(\theta)}\right)^{\gamma}\int_{0}^{\infty} y^{(\beta\theta-1)\gamma}e^{-\alpha\gamma y^{\beta}}\,dy\right\}$$

Making substitution $y^{\beta}=z$, we have

$$T_R(\gamma)=\frac{1}{1-\gamma}\log\left\{\frac{\alpha^{\theta\gamma}\beta^{\gamma-1}}{(\Gamma\theta)^{\gamma}}\int_0^{\infty} z^{\left[\frac{(\beta\theta-1)\gamma+1}{\beta}-1\right]}e^{-\alpha\gamma z}dz\right\}$$

After solving the integral, we get

$$T_R(\gamma)=\frac{1}{1-\gamma}\log\left\{\frac{\alpha^{\theta\gamma}\beta^{\gamma-1}}{(\Gamma\theta)^{\gamma}}\frac{\Gamma((\beta\theta-1)\gamma+1)}{(\alpha\gamma)^{(\beta\theta-1)\gamma+1}}\right\}$$

$$T_R(\gamma)=\frac{1}{1-\gamma}\log\left\{\frac{\alpha^{(\theta-\beta\theta+1)\gamma-1}}{(\Gamma(\theta))^{\gamma}}\Gamma((\beta\theta-1)\gamma+1)\right\}$$

## 7 Mathai and Haubold Entropy of Erlang Distribution

The measure of entropy given by Mathai and Haubold is defined as

$$T_{MH}(\gamma)=\frac{1}{1-\gamma}\log\left\{\int_0^{\infty} f^{\gamma-2}(y)dy\right\},\quad \text{where } \gamma>0 \text{ and } \gamma\neq 1$$

Thus, the Mathai and Haubold entropy for power Erlang distribution $(2.1)$, is given as

$$T_{MH}(\gamma)=\frac{1}{1-\gamma}\log\left\{\int_0^{\infty}\left[\frac{\alpha^{\theta}\beta}{\Gamma(\theta)}y^{\beta\theta-1}e^{-\alpha y^{\beta}}\right]^{\gamma-2}dy\right\}$$

$$=\frac{1}{1-\gamma}\log\left\{\left(\frac{\alpha^{\theta}\beta}{\Gamma(\theta)}\right)^{\gamma-2}\int_0^{\infty} y^{(\beta\theta-1)(\gamma-2)}e^{-\alpha(\gamma-2)y^{\beta}}dy\right\}$$

After solving the integral, we get

$$T_{MH}(\gamma)=\frac{1}{1-\gamma}\log\left\{\frac{\alpha^{(\theta-\beta\theta+1)(\gamma-2)-1}}{(\Gamma(\theta))^{\gamma-2}}\Gamma((\beta\theta-1)(\gamma-2)+1)\right\}$$

## 8 Tsallis Entropy of Power Erlang Distribution

Tsallis entropy of order $\gamma$ for power Erlang distribution $(2.1)$, is given as

$$S_\gamma = \frac{1}{1-\gamma}\left\{1-\int_0^\infty f^\gamma(y)dy\right\}, \quad \text{where } \gamma > 0 \text{ and } \gamma \neq 1$$

$$S_\gamma = \frac{1}{1-\gamma}\left\{1-\int_0^\infty\left[\frac{\alpha^\theta\beta}{\Gamma(\theta)}y^{\beta\theta-1}e^{-\alpha y^\beta}\right]^\gamma dy\right\}$$

After solving the integral, we get

$$S_\gamma = \frac{1}{1-\gamma}\left(1-\frac{\alpha^{(\theta-\beta\theta+1)-1}}{(\Gamma(\theta))^\gamma}\Gamma((\beta\theta-1)+1)\right)$$

## 9 Mean Deviation From Mean of Power Erlang Distribution

The quantity of scattering in a population is evidently measured to some extent by the totality of the deviations. Let $Y$ be a random variable from power Erlang distribution with mean $\mu$ .Then the mean deviation from mean is defined as.

$$D(\mu) = E(|Y-\mu|)$$

$$= \int_0^\infty |Y-\mu| f(y)dy$$

$$= \int_0^\mu (\mu-y)f(y)dy + \int_\mu^\infty (y-\mu)f(y)dy$$

$$= \mu\int_0^\mu f(y)dy - \int_0^\mu yf(y)dy + \int_\mu^\infty yf(y)dy - \int_\mu^\infty \mu f(y)dy$$

$$= \mu F(\mu) - \int_0^\mu yf(y)dy - \mu[1-F(\mu)] + \int_\mu^\infty yf(y)dy$$

$$= 2\mu F(\mu) - 2\int_0^\mu yf(y)dy \qquad (9.1)$$

Now

$$\int_0^{\mu} yf(y)dy = \int_0^{\mu} y \frac{\alpha^{\theta}\beta}{\Gamma(\theta)} y^{\beta\theta-1} e^{-\alpha y^{\beta}} dy$$

$$= \frac{\alpha^{\theta}\beta}{\Gamma(\theta)} \int_0^{\mu} y^{\beta\theta} e^{-\alpha y^{\beta}} dy$$

Making the substation $\alpha y^{\beta} = z$ , $0 < z < \mu^{\frac{1}{\beta}}$ , we have

$$= \frac{\alpha^{(1-\beta)\theta-1}}{\Gamma(\theta)} \int_0^{\mu^{\frac{1}{\beta}}} z^{\theta+\frac{1}{\beta}-1} e^{-z} dz$$

After solving the integral, we get

$$\int_0^{\mu} yf(y)dy = \frac{\alpha^{(1-\beta)\theta-1}}{\Gamma(\theta)} \gamma\left(\theta + \frac{1}{\beta}, \mu^{\frac{1}{\beta}}\right) \tag{9.2}$$

Using equation $(9.2)$ in equation $(9.1)$, we have

$$D(\mu) = \frac{2}{\Gamma(\theta)}\left[\mu\gamma(\theta, \alpha\mu^{\beta}) - \alpha^{(1-\beta)\theta-1}\gamma\left(\theta + \frac{1}{\beta}, \mu^{\frac{1}{\beta}}\right)\right]$$

# 10 Mean Deviation From Median of Power Erlang Distribution

Let $Y$ be a random variable from power Erlang distribution with median $M$ .Then the mean deviation from median is defined as.

$$D(M) = E(|Y - M|)$$

$$= \int_0^{\infty} |Y - M| f(y) dy$$

$$= \int_0^{M} (M - y) f(y) dy + \int_M^{\infty} (y - M) f(y) dy$$

$$= MF(M) - \int_0^{M} yf(y)dy - M[1 - F(M)] + \int_M^{\infty} yf(y)dy$$

$$= \mu - 2\int_0^M y f(y) dy \qquad (10.1)$$

Now

$$\int_0^M y f(y) dy = \int_0^M y \frac{\alpha^\theta \beta}{\Gamma(\theta)} y^{\beta\theta - 1} e^{-\alpha y^\beta} dy$$

$$= \frac{\alpha^\theta \beta}{\Gamma(\theta)} \int_0^M y^{\beta\theta} e^{-\alpha y^\beta} dy$$

Making the substation $\alpha y^\beta = z$ , $0 < z < M^{\frac{1}{\beta}}$, we have

$$= \frac{\alpha^{(1-\beta)\theta - 1}}{\Gamma(\theta)} \int_0^{M^{\frac{1}{\beta}}} z^{\theta + \frac{1}{\beta} - 1} e^{-z} dz$$

After solving the integral, we get

$$\int_0^M y f(y) dy = \frac{\alpha^{(1-\beta)\theta - 1}}{\Gamma(\theta)} \gamma\left(\theta + \frac{1}{\beta}, M^{\frac{1}{\beta}}\right) \qquad (10.2)$$

Using equation $(10.2)$ in equation $(10.1)$, we have

$$D(M) = \mu - 2\frac{\alpha^{(1-\beta)\theta - 1}}{\Gamma(\theta)} \gamma\left(\theta + \frac{1}{\beta}, M^{\frac{1}{\beta}}\right)$$

# 11 Bonferroni and Lorenz Curves

In economics the relation between poverty and economy is well studied by using Bonferroni and Lorenz curves. Besides that these curves have been used in different fields such as reliability, insurance and biomedicine.

The Bonferroni curves, $B(s)$ is given as

$$B(s) = \frac{1}{s\mu} \int_0^t y f(y) dy \qquad (11.1)$$

Or

$$B(s)=\frac{1}{s\mu}\int_0^s F^{-1}(y)dy$$

And Lorenz curves, $L(s)$ is given as

$$L(s)=\frac{1}{\mu}\int_0^t yf(y)dy \qquad (11.2)$$

Or

$$L(s)=\frac{1}{\mu}\int_0^s F^{-1}(y)dy$$

Where $E(X)=\mu$ and $t=F^{-1}(s)$

Now

$$\int_0^t yf(y)dy=\int_0^t y\frac{\alpha^\theta\beta}{\Gamma(\theta)}y^{\beta\theta-1}e^{-\alpha y^\beta}dy$$

$$=\frac{\alpha^\theta\beta}{\Gamma(\theta)}\int_0^t y^{\beta\theta}e^{-\alpha y^\beta}dy$$

After solving the integral, we get

$$\int_0^t yf(y)dy=\frac{\alpha^{(1-\beta)\theta-1}}{\Gamma(\theta)}\gamma\left(\theta+\frac{1}{\beta},t^{\frac{1}{\beta}}\right) \qquad (11.3)$$

Substituting equation $(11.3)$ in equations $(11.1)$ and $(11.2)$, we get

And $B(s)=\dfrac{\alpha^{(1-\beta)\theta-1}}{s\mu\Gamma(\theta)}\gamma\left(\theta+\dfrac{1}{\beta},t^{\frac{1}{\beta}}\right)$

$$L(s)=\frac{\alpha^{(1-\beta)\theta-1}}{\mu\Gamma(\theta)}\gamma\left(\theta+\frac{1}{\beta},t^{\frac{1}{\beta}}\right)$$

## 12 Order Statistics of Power Erlang Distribution

Let $Y_{(1)}, Y_{(2)}, Y_{(3)}, \ldots, Y_{(n)}$ denotes the order statistics of a random sample drawn from power a continuous distribution with cdf $F(y)$ and pdf $f(y)$, then the pdf of $Y_{(k)}$ is given by

$$f_{Y(k)}(Y,\theta)=\frac{n!}{(k-1)!(n-k)!} f_Y(y)[F_Y(y)]^{k-1}[1-F_Y(y)]^{n-k} \quad k=1,2,3\ldots,n \quad (12.1)$$

Substitute the equation $(2.1)$ and $(2.2)$ in equation $(12.1)$, we obtain the probability function of $k^{th}$ order statistics of power Erlang distribution is given by

$$f_{Y(k)}(Y,\theta)=\frac{n!}{(k-1)!(n-k)!}\frac{\alpha^{\theta}\beta}{\Gamma(\theta)} y^{\beta\theta-1}\left[\frac{\gamma(\theta,\alpha y^{\beta})}{\Gamma(\theta)}\right]^{k-1}\left[1-\frac{\gamma(\theta,\alpha y^{\beta})}{\Gamma(\theta)}\right]^{n-k} \quad (12.1)$$

Then, the pdf of first order $Y_{(1)}$ power Erlang distribution given by

$$f_{Y(1)}(Y,\theta)=\frac{n\alpha^{\theta}\beta}{\Gamma(\theta)} y^{\beta\theta-1}\left[1-\frac{\gamma(\theta,\alpha y^{\beta})}{\Gamma(\theta)}\right]^{1-k}$$

And the pdf of nth order $Y_{(n)}$ power Erlang distribution is given by

$$f_{Y(n)}(Y,\theta)=\frac{n\alpha^{\theta}\beta}{\Gamma(\theta)} y^{\beta\theta-1}\left[\frac{\gamma(\theta,\alpha y^{\beta})}{\Gamma(\theta)}\right]^{n-1}$$

## 13 Maximum Likelihood Estimator Of Power Erlang Distribution

Let $Y_1, Y_2, .., Y_n$ be a random sample of size n from power Erlang distribution then its likelihood function is given by

$$l=\prod_{i=1}^{n} f(y_i,\alpha,\beta,\theta)$$

$$=\prod_{i=1}^{n}\frac{\alpha^{\theta}\beta}{\Gamma(\theta)} y^{\beta\theta-1} e^{-\alpha\sum_{i=1}^{n} y_i^{\beta}}$$

$$=\left(\frac{\alpha^{\theta}\beta}{\Gamma(\theta)}\right)^{n}\prod_{i==1}^{n} y_i^{\beta\theta-1} e^{-\alpha\sum_{i=1}^{n} y_i^{\beta}}$$

The log likelihood function becomes

$$\log l = n\theta\log\alpha + n\log\beta - n\log\Gamma(\theta) + (\beta\theta-1)\sum_{i=1}^{n}\log y_i - \alpha\sum_{i=1}^{n} y_i^{\beta} \qquad (13.1)$$

Differentiating $(13.1)$ w.r.t the parameters, we get

$$\frac{\partial\log l}{\partial\alpha} = \frac{n\theta}{\alpha} - \sum_{i=1}^{n} y_i^{\beta}$$

$$\frac{\partial\log l}{\partial\beta} = \frac{n}{\beta} + \theta\sum_{i=1}^{n} y_i - \alpha\sum_{i=1}^{n} y_i^{\beta}\log(y_i)$$

$$\frac{\partial\log l}{\partial\theta} = n\log\alpha - n\psi(\theta) + \beta\sum_{i=1}^{n}\log y_i$$

Where $\psi(\theta) = \frac{\Gamma'(\theta)}{\Gamma(\theta)}$

The above equations are non-linear equations which cannot be expressed in compact form and it is difficult to solve these equations explicitly for $\alpha, \beta$ and $\theta$ .Applying the iterative methods such as Newton–Raphson method, secant method, Regula-falsi method etc. The MLE of the parameters denoted as $\hat{\xi}(\hat{\alpha}, \hat{\beta}, \hat{\theta})$ of $\xi(\alpha, \beta, \theta)$ can be obtained by using the above methods.

Since the MLE of $\hat{\xi}$ follows asymptotically normal distribution which is given as

$$\sqrt{n}(\hat{\xi} - \xi) \rightarrow N(0, I(\xi))$$

Where $I^{-1}(\xi)$ is the limiting variance-covariance matrix $\hat{\xi}$ and $I(\xi)$ is a $3\times3$ Fisher information matrix

i,e

$$I(\xi) = -\frac{1}{n}\begin{bmatrix} E\left(\frac{\partial^2 \log l}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 \log l}{\partial \alpha \partial \beta}\right) & E\left(\frac{\partial^2 \log l}{\partial \alpha \partial \theta}\right) \\ E\left(\frac{\partial^2 \log l}{\partial \beta \partial \alpha}\right) & E\left(\frac{\partial^2 \log l}{\partial \beta^2}\right) & E\left(\frac{\partial^2 \log l}{\partial \beta \partial \theta}\right) \\ E\left(\frac{\partial^2 \log l}{\partial \theta \partial \alpha}\right) & E\left(\frac{\partial^2 \log l}{\partial \theta \partial \beta}\right) & E\left(\frac{\partial^2 \log l}{\partial \theta^2}\right) \end{bmatrix}$$

Where

$$\frac{\partial^2 \log l}{\partial \alpha^2} = \frac{-n\theta}{\alpha^2}\,;\, \frac{\partial^2 \log l}{\partial \beta^2} = \frac{-n}{\beta^2} - \alpha \sum_{i=1}^{n} y_i^{\beta} (\log(y_i))^2\,;$$

$$\frac{\partial^2 \log l}{\partial \theta^2} = -n\psi^{'}(\theta)$$

$$\frac{\partial^2 \log l}{\partial \alpha \partial \beta} = \frac{\partial^2 \log l}{\partial \beta \partial \alpha} = -\beta \sum_{i=1}^{n} y_i^{\beta} \log(y_i)$$

$$\frac{\partial^2 \log l}{\partial \alpha \partial \theta} = \frac{\partial^2 \log l}{\partial \theta \partial \alpha} = \frac{n}{\alpha}$$

$$\frac{\partial^2 \log l}{\partial \beta \partial \theta} = \frac{\partial^2 \log l}{\partial \theta \partial \beta} = \sum_{i=1}^{n} \log y_i$$

Hence the approximate $100(1-\psi)\%$ confidence interval for $\alpha, \beta$ and $\theta$ are respectively given by

$$\hat{\alpha} \pm Z_{\frac{\psi}{2}} \sqrt{I_{\alpha\alpha}^{-1}(\hat{\xi})}\ ,\ \hat{\beta} \pm Z_{\frac{\psi}{2}} \sqrt{I_{\beta\beta}^{-1}(\hat{\xi})} \text{ and } \hat{\theta} \pm Z_{\frac{\psi}{2}} \sqrt{I_{\theta\theta}^{-1}(\hat{\xi})}$$

Where $Z_{\frac{\psi}{2}}$ is the $\psi^{th}$ percentile of the standard distribution.

# 14 Applications

In this section the versatility of the formulated distribution is examined through two data sets which are related different fields of science. To examine the significance and potentiality of the formulated distribution, we compare the formulated distribution with its sub models having following densities.

1) Area-biased Erlang distribution (ABED) with p.d.f

$$f(y,\alpha,\theta)=\frac{\alpha^{\theta+2}}{\Gamma(\theta+2)}y^{\theta+1}e^{-\alpha y}; y>0, \alpha>0, \theta=1,2,3,...$$

Corresponding c.d.f

$$F(y,\alpha,\theta)=\frac{\gamma(\theta+2,\alpha y)}{\Gamma(\theta+2)}; y>0, \alpha>0, \theta=1,2,3,...$$

2) Length-biased Erlang distribution (LBED) with p.d.f

$$f(y,\alpha,\theta)=\frac{\alpha^{\theta+1}}{\Gamma(\theta+1)}y^{\theta}e^{-\alpha y}; y>0, \alpha>0, \theta=1,2,3,...$$

Corresponding c.d.f

$$F(y,\alpha,\theta)=\frac{\gamma(\theta+1,\alpha y)}{\Gamma(\theta+1)}; y>0, \alpha>0, \theta=1,2,3,...$$

3) Erlang distribution (ED) with p.d.f

$$f(y,\alpha,\theta)=\frac{\alpha^{\theta}}{\Gamma(\theta)}y^{\theta-1}e^{-\alpha y}; y>0, \alpha>0, \theta=1,2,3,...$$

Corresponding cdf

$$F(y,\alpha,\theta)=\frac{\gamma(\theta,\alpha y)}{\Gamma(\theta)}; y>0, \alpha>0, \theta=1,2,3,...$$

4) Exponential distribution (EXD) with p.d.f

$$f(y,\alpha,\theta)=\alpha e^{-\alpha y}; y>0, \alpha>0$$

Corresponding c.d.f

$$F(y,\alpha,\theta)=1-e^{-\alpha y}; y>0, \alpha>0$$

The analytical measures of the goodness of fit including the AIC (Akaike information criterion), CAIC (Consistent Akaike information criterion), BIC (Bayesian information criterion), HQIC (Hannan-Quinn information criteria) and KS (Kolmogorov-smirov) are used to compare the fitted models. Also the p-value of each model is indicated. A distribution is considered better having lesser AIC, CAIC, BIC, HQIC and KS values with large p-value.

$$AIC=2k-2\ln l \qquad CAIC=\frac{2kn}{n-k-1}-2\ln l$$

$$BIC=k\ln n-2\ln l \qquad HQIC=2k\ln(\ln(n))-2\ln l$$

The descriptive statistics of the data set 1 and data set 2 are presented in Table 1 and Table 4.The estimates of the parameters are shown in Table 2 and Table5 for data set 1 and data set 2 respectively. Log-likelihood, Akaike information criteria (AIC) etc for the data set 1and data set 2 are generated and presented in Table 3 and Table 6 respectively.

**Data Set 1:-** The following data represent 40 patients suffering from blood cancer (leukaemia) from one ministry of health hospitals in Saudi Arabia (see Abouammah et al.). The ordered lifetimes (in years) are given.

0.315, 0.496, 0.616, 1.145, 1.208, 1.263, 1.414, 2.025, 2.036, 2.162, 2.211, 2.37, 2.532, 2.693, 2.805, 2.91, 2.912, 3.192, 3.263, 3.348, 3.348, 3.427, 3.499, 3.534, 3.767, 3.751, 3.858, 3.986, 4.049, 4.244, 4.323, 4.381, 4.392, 4.397, 4.647, 4.753, 4.929, 4.973, 5.074, 5.381

**Table 2**

**The descriptive statistics of data set 1 of 40 patients suffering from leukaemia**

| Min | $Q_1$ | $Q_3$ | Mean | Median | Skew. | Kurt. | Max |
|---|---|---|---|---|---|---|---|
| 0.315 | 2.199 | 4.264 | 3.141 | 3.348 | -0.41 | 2.273 | 5.381 |

**Table 3**

**The ML Estimates of the unknown parameters for Data set 1st**

**Table 4**

| Model | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | S.E | | |
|---|---|---|---|---|---|---|
| | | | | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ |
| PED | 0.0036 | 3.803 | 0.5265 | 0.0019 | 0.336 | 0.099 |
| ABED | 1.1031 | …… | 1.4647 | 0.2537 | …… | 0.7404 |
| LBED | 1.4647 | …… | 2.4647 | 0.2537 | …… | 0.2537 |
| ED | 1.1031 | ……. | 3.4647 | 0.2537 | …… | 0.7404 |
| EXD | 0.3183 | ……. | …… | 0.0503 | ……. | ……. |

**Performance of distributions for Data set 1st**

| Model | $-2\log l$ | AIC | CAIC | BIC | HQIC | K-S | P-value |
|---|---|---|---|---|---|---|---|
| PED | 135.171 | 141.171 | 141.838 | 146.238 | 143.003 | 0.115 | 0.659 |
| ABED | 147.096 | 151.097 | 151.421 | 154.475 | 152.318 | 0.158 | 0.269 |
| LBED | 147.096 | 151.097 | 151.421 | 154.475 | 152.318 | 0.158 | 0.269 |
| ED | 147.096 | 151.097 | 151.497 | 154.475 | 152.104 | 0.637 | 4.6e-12 |
| EXD | 171.556 | 173.556 | 173.685 | 175.245 | 174.059 | 0.599 | 9.6e-11 |

**Data Set 2:** The data represents the breaking stress of carbon fibres of 50mm length (GPa) and has been already used by Al-Aqtash et al. [3] demonstrate the appropriateness of Gumbell-Weibull distribution. The data set is follows

0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79,2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65,3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90

**Table 5**

**The descriptive statistics of data set 2 of breaking stress of carbon fibres**

| Min | $Q_1$ | $Q_3$ | Mean | Median | Skew. | Kurt. | Max |
|---|---|---|---|---|---|---|---|
| 0.390 | 2.178 | 3.277 | 2.760 | 2.835 | -0.13 | 3.223 | 4.90 |

**Table 6**

**The ML Estimates of the unknown parameters for Data set 2nd**

**Table 7**

| Model | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | S.E | | |
|---|---|---|---|---|---|---|
| | | | | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ |
| PED | 0.0122 | 3.7617 | 0.8567 | 0.0133 | 0.6665 | 0.2515 |
| ABED | 2.7135 | …… | 5.4880 | 0.4780 | …… | 1.2755 |
| LBED | 2.7135 | …… | 6.4880 | 0.4780 | …… | 1.2755 |
| ED | 2.7135 | ……. | 7.4880 | 0.4780 | …… | 1.2755 |
| EXD | 0.3623 | ……. | …… | 0.044 | ……. | ……. |

**Performance of distributions for Data set 2nd**

| Model | $-2\log l$ | AIC | CAIC | BIC | HQIC | K-S | P-value |
|---|---|---|---|---|---|---|---|
| PED | 171.8988 | 177.898 | 178.286 | 184.467 | 180.494 | 0.083 | 0.751 |
| ABED | 182.334 | 186.335 | 186.525 | 190.714 | 188.065 | 0.132 | 0.194 |
| LBED | 182.334 | 186.335 | 186.525 | 190.714 | 188.065 | 0.132 | 0.194 |
| ED | 182.334 | 186.335 | 186.525 | 190.714 | 188.065 | 0.703 | 1.3e-14 |
| EXD | 265.988 | 267.988 | 268.117 | 270.178 | 268.492 | 0.603 | 7.5e-11 |

Since it has been observed from table 4 and table 7 that the power Erlang distribution has smaller values for the AIC, AICC, CAIC, BIC, HQIC and K-S statistics as compared with its sub models. Accordingly we arrive the conclusion that power Erlang distribution provides an adequate fit than compared ones.

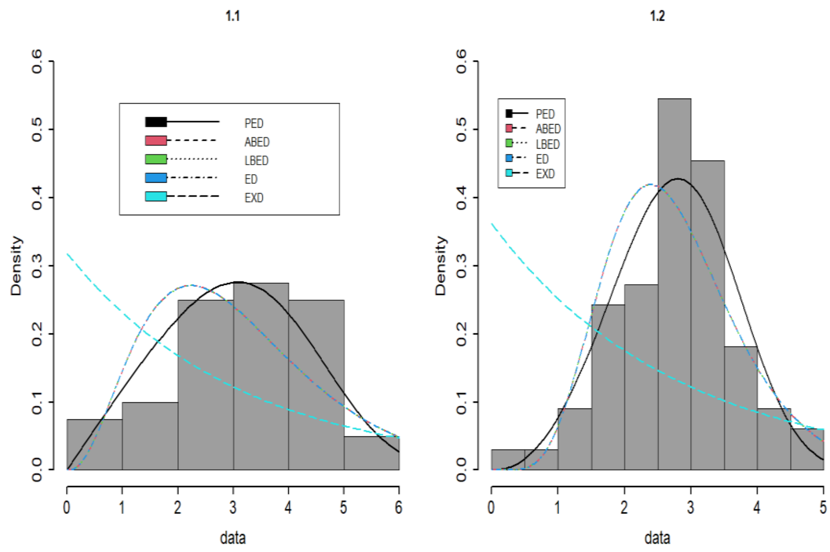


Figure (1.1) and (1.2) represents the estimated densities of the fitted distributions to data set 1st and 2nd.

## 15 Conclusion

This paper introduces an extension of Erlang distribution which is obtained by applying power transformation method. For this distribution several mathematical quantities are derived including moments, moment generating function, skewness, kurtosis, incomplete moments, mode, median, order statistics, different measure of entropies, mean deviations, Bonferroni and Lorenz curves. An account on reliability analysis has been discussed. Different plots have been drawn to show the behaviour of p.d.f, c.d.f and other related measures. The method of maximum likelihood estimation has been applied for estimating the parameters of the distribution. Lastly it has been carried out through two real life data sets that the formulated distribution leads an improved fit than compared ones.

## 16 Open Questions

The following probability density function represents power size-biased and power area-biased Erlang distribution.

1. $f(y,\alpha,\beta,\theta)=\frac{\alpha^{\theta+1}\beta}{\Gamma(\theta+1)}y^{\beta(\theta+1)-1}e^{-\alpha y^{\beta}}$ ; $y>0,\alpha,\beta>0$, $\theta=1,2,3...$

2. $f(y,\alpha,\beta,\theta)=\frac{\alpha^{\theta+2}\beta}{\Gamma(\theta+2)}y^{\beta(\theta+2)-1}e^{-\alpha y^{\beta}}$ ; $y>0,\alpha,\beta>0$, $\theta=1,2,3...$

Show that power size-biased and power area-biased Erlang distribution are more efficient than proposed distribution for analysing data ?

# References

[1] A.A Bhat, and S.P. Ahmad. A new generalization of Raylieh distribution properties and applications. Pakistan Journal of Statistics. 36(3), (2020), 225-250.

[2] Akaike H. A new look at the statistical model identification. Selected Papers of Hirotugu. Springer, (1974), 215-222.

[3] Al-Aqtash R, Lee C and Famoye F. Gumbel-Weibull distribution: properties and applications. Journal of Modern Applied Statistical Methods. 16(2), (2014), 201- 225.

[4] Abouammoh A.M, Ahmad R and Khalique A. On new renewal better than used classes of life distribution. Statistics and Probability Letters. 48, (2000),189-194.

[5] Bhattacharya Sk, Singh NK. Intensity in M/KK/1queue, Far. East. Journal of Math and Science. 2, (1994),57-62.

[6] Erlang A.K. The theory of probabilities and telephone conversations. Nyt Tidsskrift for Matematik, B. 20(6), (1909), 87-98.

[7] Ghitnay M.E, Al-Mutairi, D.K. Balakrishanan N. and Al-Enezi L.J. Power Lindley distribution and associated inference. Computational Statistics and Data Analysis. 64, (2013), 20-30.

[8] Haq A, Day S.Bayesian estimation of Erlang distribution under different prior distributions.Journal of Reliability and Statistical Studies. 4, (2011), 1-30.

[9] Hesham M. Reyad, Soha A. Othman and Alaaedin A. Moussa. The Length-biased weighted Erlang distribution. Asian Research Journal of Mathematics. 3, (2017),1-15

[10] Hannan E.J and Quinn B.G. The determination of the order of an auto regression.Journal of the Royal Statistical Society, Series B.41, (1979),190-195.

[11] Khan AH, Jan TR. Bayesian estimation of Erlang distribution under differ generalized truncated distributions as priors. Journal of Modern Applie Statistical Methods.11, (2012), 416-442.

[12] Krishnarani S.D. On a power transformation of Half-Logistic distribution. Journal of Probability and Statistics. 5, (2016), 1-10.

[13] Mathai A.M and Haubold H.J (2007). Pathway model, Super statistics, Tsallis Statistics and generalized measures of entropy. Physica A: Statistical Mechanics and its Applications. 375(1), (2016),110-1212.

[14] Sofi Mudasir and S.P .Ahmad. Characterization and information measures of weighted Erlang distribution. Journal of Statistics Applications and Probability Letters. 4(3), (2017), 109-122.

[16] Sofi Mudasir and S.P .Ahmad. Parameter estimation of weighted Erlang distribution using R software. Mathematical theory and Modelling. 7(6), (2017), 1- 21.

[17] Shukla K.K and Shanker R (2018). Power Ishita distribution and its application to model lifetime data. Statistics in Transition. 19(1), (2017), 135-148.

[18] Shanon E. A mathematical theory of communication. Bell System Technical Journal. 27(3), (1948),379-423.

[19] Rady E.H.A, Hassanien W.A and Elhaddad T.A. The power Lomax distribution with an application to bladder cancer data. Springer Plus. 5, (2016).

[21] Renyi A. On measures of entropy and information. Berkeley Symposium on Mathematical Statistics and Probability. 1(1), (1960),547-561.

[22] Zaka A and Akhtar A.S. Methods for estimating the parameters of power Function distribution. Pakistan Journal of Statistics and Operation Research.9, (2013), 213-224.