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A characterization of the quaternion group by the sum of subgroup orders

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Abstract

Let G be a finite group and $\sigma_1(G) = \frac{1}{|G|} \sum_{H \leq G} |H|$. In this note, we give a characterization of the well-known quaternion group Q_8 by using the function σ_1. A related open problem is also formulated.

Keywords: *subgroup orders, quaternion group.*

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1 Introduction

Given a finite group G, we consider the function

$$\sigma_1(G) = \frac{1}{|G|} \sum_{H \leq G} |H|$$

studied in our previous papers [6, 10, 11]. Recall some basic properties of σ_1:

- if G is cyclic of order n and $\sigma(n)$ denotes the sum of all divisors of n, then $\sigma_1(G) = \frac{\sigma(n)}{n}$;

- σ_1 is multiplicative, i.e. if G_i, $i = 1, 2, \ldots, m$, are finite groups of coprime orders, then $\sigma_1(\prod_{i=1}^{m} G_i) = \prod_{i=1}^{m} \sigma_1(G_i)$;

- $\sigma_1(G) \geq \sigma_1(G/H) + \frac{1}{(G:H)}(\sigma_1(H) - 1) \geq \sigma_1(G/H)$, for all $H \trianglelefteq G$.

The function σ_1 was used to provide some criteria for a group to belong to one of the well-known classes of groups. For instance, if $\sigma_1(G) \leq 2$ then G is cyclic of deficient or perfect order by [6], while if $\sigma_1(G) \leq \frac{117}{20}$ then G is solvable by [4, 11]. Also, we note that there is no constant $c \in (2, \infty)$ such that $\sigma_1(G) < c$ implies the nilpotency of G, but this can be obtained from the condition $\sigma_1(G) < 2 + \frac{4}{|G|}$ (see [10]).

We observe that there are pairs (G_1, G_2) of non-isomorphic finite groups such that $\sigma_1(G_1) = \sigma_1(G_2)$, e.g. $G_1 = M_4(2) = SmallGroup(16, 6)$ and $G_2 = \mathbb{Z}_2 \times \mathbb{Z}_8 = SmallGroup(16, 5)$. More generally, for any prime p and any integer $n \geq 3$ (where $n \geq 4$ for $p = 2$) the modular group

$$M_n(p) = \langle x, y \mid x^{p^{n-1}} = y^p = 1,\ y^{-1}xy = x^{p^{n-2}+1} \rangle$$

and the direct product $\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}$ satisfy this property.

In the current note, we will prove that Q_8 is completely determined by the function σ_1 among finite p-groups. Our main result can be stated as follows.

Theorem 1.1. *Let G be a finite p-group such that $\sigma_1(G) = \frac{23}{8} = \sigma_1(Q_8)$. Then $G \cong Q_8$.*

For the proof of Theorem 1.1, we need the classification of finite p-groups having a maximal subgroup which is cyclic (see e.g. Theorem 4.1, [8], II).

Theorem A. *Let p be a prime, $n \geq 3$ be an integer and G be a group of order p^n possessing a cyclic maximal subgroup. Then either G is abelian of type $\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}$ or one of the following non-abelian groups:*

a) *$M_n(p)$, when p is odd;*

b) – *$M_n(2)$ $(n \geq 4)$,*

– *the dihedral group $D_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2 = 1,\ yxy = x^{-1} \rangle$,*

– *the generalized quaternion group*

$$Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^4 = 1,\ yxy^{-1} = x^{2^{n-1}-1} \rangle,$$

– *the quasi-dihedral group*

$$S_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2 = 1,\ y^{-1}xy = x^{2^{n-2}-1} \rangle \ (n \geq 4),$$

when $p = 2$.

We also need the following auxiliary result, taken from [6].

Lemma B. *Let G be a finite group. Then*

$$\sum_{H\leq G,\, H=\text{cyclic}} |H| = \sum_{a\in G} \frac{o(a)}{\phi(o(a))} \geq \sum_{a\in G} 1 = |G|.$$

Note that similar problems for some other functions related to the structure of a finite group G, for example for the function $\psi(G) = \sum_{x\in G} o(x)$ (where $o(x)$ denotes the order of the element x), have been recently investigated by many authors (see e.g. [1, 2, 3]).

Most of our notation is standard and will not be repeated here. Basic definitions and results on groups can be found in [5, 8]. For subgroup lattice concepts we refer the reader to [7].

2 Proof of the main result

The computation of subgroups of groups in Theorem A has been made in [9]. It easily leads to the following lemma.

Lemma 2.1. *We have:*

a) $\sigma_1(M_n(p)) = \sigma_1(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = \frac{p^{n+1}+2p^n-p^2-1}{p^n(p-1)}$ *, for any prime p and any integer $n \geq 3$ (where $n \geq 4$ for $p = 2$);*

b) – $\sigma_1(D_{2^n}) = \frac{2^n(n+1)-1}{2^n}$ *, for any integer $n \geq 3$;*
– $\sigma_1(Q_{2^n}) = \frac{n2^n-1}{2^n}$ *, for any integer $n \geq 3$;*
– $\sigma_1(S_{2^n}) = \frac{(2n+1)2^{n-1}-1}{2^n}$ *, for any integer $n \geq 4$.*

We are now able to prove our main result.

Proof of Theorem 1.1. Let $|G| = p^n$, where p is a prime and n is a positive integer.

Assume first that G is cyclic. Then

$$\sigma_1(G) = \frac{1+p+...+p^n}{p^n} = 1 + \frac{1}{p-1} - \frac{1}{p^n(p-1)}$$

$$< 1 + \frac{1}{p-1} \leq 2 < \frac{23}{8} = \sigma_1(Q_8),$$

a contradiction.

Assume now that G is not cyclic. Then it has at least $p+1$ subgroups of order p^{n-1}. If G has no cyclic maximal subgroup, then, by using Lemma B, we infer that

$$\sigma_1(G) \geq \frac{1}{p^n}\left[p^n + (p+1)p^{n-1} + \sum_{H \leq G,\, H=\text{cyclic}} |H|\right]$$

$$\geq \frac{1}{p^n}\left[p^n + (p+1)p^{n-1} + p^n\right] > 3 > \frac{23}{8} = \sigma_1(Q_8),$$

contradicting again our hypothesis. Thus G has a cyclic maximal subgroup, i.e. it is one of the groups described in Theorem A. According to Lemma 2.1, one obtains:

a) For $p = \text{odd}$:

$$\sigma_1(M_n(p)) = \frac{23}{8} \Rightarrow 8 \mid p-1 \Rightarrow p \geq 17.$$

Then $\sigma_1(M_n(p)) < 2$. Indeed, we have

$$\sigma_1(M_n(p)) < 2 \Leftrightarrow \frac{p^{n+1} + 2p^n - p^2 - 1}{p^{n+1} - p^n} < 2 \Leftrightarrow (p-4)p^n + p^2 + 1 > 0,$$

which is obviously true. Consequently, in this case we cannot have $\sigma_1(M_n(p)) = \frac{23}{8}$.

b) For $p = 2$:

– $\sigma_1(M_n(2)) = \sigma_1(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}) = \frac{2^{n+2}-5}{2^n} > 3 > \frac{23}{8}$;
– $\sigma_1(D_{2^n}) = n + 1 - \frac{1}{2^n} > 3 > \frac{23}{8}$;
– $\sigma_1(Q_{2^n}) = n - \frac{1}{2^n} = \frac{23}{8} \Leftrightarrow n = 3$;
– $\sigma_1(S_{2^n}) = \frac{2n+1}{2} - \frac{1}{2^n} > n \geq 4 > \frac{23}{8}$.

Hence the unique finite group G with $\sigma_1(G) = \frac{23}{8}$ is Q_8, as desired. □

Finally, we formulate a natural problem concerning the above study.

3 Open Problem

Find other finite groups G which are completely determined by the function σ_1.

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