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Fourth Hankel Determinant for a subclass of analytic functions related to modified sigmoid functions

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Abstract

The main purpose of this paper is to study the fourth Hankel determinant for a subclass of starlike functions associated with modified sigmoid function and also determine an upper bound of the fourth Hankel determinant for the functions in this class. The authors believe that this article will encourage other researchers to work fourth Hankel determinant due to its novelty in literature.

Keywords: *Analytic functions, Hankel determinant, Modified sigmoid functions, Starlike functions.*

1 Introduction

Special functions play an important role in geometric function theory. Actually, special function theory doesn't have a specific definition. Even so, it is quite important for scientists and engineers who are concerned with mathematical calculations. Nowadays, theory of special function has been overshadowed by other fields such as algebra, real analysis, topology, functional analysis and differential equation.

Three types of classification can be done for special functions. These are ramp function, threshold function and sigmoid function. The most popular

of special functions is sigmoid because of its gradient descendent learning algorithm. There are several ways for evaluating sigmoid, most especially by truncated series expansions (see [10, 20, 22]).
The logistic sigmoid function of the form

$$h(z) = \frac{1}{1+e^{-z}} \tag{1}$$

is differentiable and has the accompanying properties:
• It outputs real numbers between 0 and 1.
• It maps a very large input domain to a small range of outputs.
• It never loses information because it is a one-to-one function.
• It increases monotonically.
The four properties which mentioned above show that sigmoid function has an important role in geometric function theory.

Let $\mathcal{A}$ denote the class of functions f which are analytic in the open disk $\mathcal{U} = \{z : z \in \mathcal{C} : |z| < 1\}$ is the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathcal{U}). \tag{2}$$

A function $f \in \mathcal{A}$ is said to be univalent or one-to-one if it never takes the same value more than once in $\mathcal{U}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ which are normalized $f(0) = f'(0) - 1 = 0$ and univalent in $\mathcal{U}$. The study of univalent function was initiated by Koebe in 1907. Various interesting subclasses of $\mathcal{S}$ defined by natural conditions, namely, class starlike functions, convex function, etc. have been extensively investigated in the literature. The class of starlike functions in $\mathcal{S}$ is indicated by $\mathcal{S}^*$. Let $\mathcal{P}(\gamma)$, $0 \leq \gamma < 1$, denote the class of analytic functions p in $\mathcal{U}$ with $p(0) = 1$ and $Re\{p(z)\} > \gamma$. Especially, we use $\mathcal{P}(0) = \mathcal{P}$ as $\gamma = 0$. Analytically, the functions in $\mathcal{S}^*$ characterized by $\Re(\frac{zf'(z)}{f(z)}) > 0$. Denote by $\mathcal{S}^*(\gamma)$ the subclass of starlike functions, so that $f \in \mathcal{S}^*(\gamma)$ if and only if, for $z \in \mathcal{U}$

$$\Re(\frac{zf'(z)}{f(z)}) > \gamma. \tag{3}$$

Especially, $S^*(0) = S^*$. In [30] Uzoamaka et al studied analytic properties of a sigmoid function. They showed the function h in (1) is starlike in $\mathcal{U}$.

The q^{th} Hankel determinant $H_q(n)$, $(q, n \in \mathbf{N} = 1, 2, \cdots)$ for a function f defined by Noonan and Thomas in [21] of the form (2) was defined by as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix} \qquad (a_1 = 1). \tag{4}$$

Later the q^{th} Hankel determinant is defined for $f \in \mathcal{S}$ of the form (2) was defined by Pommerenke [25, 26], (see also [2, 3]).

For fixed integer q and n, the growth of $H_q(n)$ has been studied for different subfamilies of univalent functions. For $q = 2$ and $n = 2$ we obtain

$$H_2(2) = a_2a_4 - a_3^2 \tag{5}$$

which is called second Hankel determinant. The sharp bounds of $H_2(2)$ for the family $\mathcal{S}^*$ was investigated by Janteng et al [13, 12]. For the family of Bazilevič functions, the exact estimate of $H_2(2)$ was obtained by Krishna et al [15]. For more papers on $H_2(2)$ for subfamilies of $\mathcal{S}$ see the references [7, 11, 14, 17, 21, 19, 23]. The sharp bound of $H_2(2)$ for the class $\mathcal{S}$ still not known. In [24], Thomas conjectured that if $f \in \mathcal{S}$, then $|H_2(2)| \leq 1$. As it was shown by Li and Srivastava in [18], this conjecture is not true for $n \geq 4$. Similarly, Răducanu and Zaprawa in [27] proved that it is also false for $n = 2$ In fact, they showed the $max\{|H_2(2)| : f \in \mathcal{S}\} \geq 1.175$.

For $q = 3$ and $n = 1$ we obtain

$$H_3(1) = a_3(a_2a_4 - a_3^2) + a_4(a_2a_3 - a_4) + a_5(a_3 - a_2^2) \tag{6}$$

which is called third Hankel determinant. The estimation of $H_3(1)$ is much more complex than that of $H_2(2)$. In 2010, the first paper on $H_3(1)$ written by Babalola [5] in which he obtained the upper bound of $H_3(1)$ for the family $\mathcal{S}^*$ and other some subclasses of $\mathcal{S}$. Later on some other authors [1, 8, 9, 28, 29, 16] published their papers concerning $H_3(1)$ for different subfamilies of analytic and univalent functions.

If we take $q = 4$ and $n = 1$ in $H_q(n)$, we obtain the fourth Hankel determinant which is $H_4(1)$. Firstly, the fourth Hankel determinant was studied by Arif et al in [4].They found an upper bound of the fourth Hankel determinant for the class of functions with positive real part.

We can write $H_4(1)$ in the form

$$H_4(1) = a_7H_3(1) - a_6\triangle_1 + a_5\triangle_2 - a_4\triangle_3 \tag{7}$$

where $\triangle_1, \triangle_2$ and $\triangle_3$ are the determinants given by

$$\triangle_1 = (a_3a_6 - a_4a_5) - a_2(a_2a_6 - a_3a_5) + a_4(a_2a_4 - a_3^2) \tag{8}$$

$$\triangle_2 = (a_4a_6 - a_5^2) - a_2(a_3a_6 - a_4a_5) + a_3(a_3a_5 - a_4^2) \tag{9}$$

and

$$\triangle_3 = a_2(a_4a_6 - a_5^2) - a_3(a_3a_6 - a_4a_5) + a_4(a_3a_5 - a_4^2), \tag{10}$$

respectively, such as given in [4]. As one can see, the fourth Hankel determinant $H_4(1)$ in (7) is a polynomial of six successive coefficients from a_2 to a_7 of a function f in a given class.

For our main results, the following lemmas shall be necessary.

Lemma 1.1 *[10] Let h be a sigmoid function and*

$$\Phi(z) = 2h(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m \tag{11}$$

then $\Phi \in \mathcal{P}$, $|z| < 1$ where Φ is a modified sigmoid function.

Lemma 1.2 *[10] Let*

$$\Phi_{n,m}(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m \tag{12}$$

then $|\Phi_{n,m}(z)| < 2$.

Lemma 1.3 *[10] If $\Phi(z) \in \mathcal{P}$ and it is starlike,then f is a normalized univalent function of the form*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathcal{U}) \tag{13}$$

The following remark was obtained by Fadipe-Joseph et al [10] by taking $m = 1$.

Remark 1.4 *Let $\Phi(z) = 1 + \sum_{n=1}^{\infty} C_n z^n$ where $C_n = \frac{-(1)^n}{2n!}$ then $|C_n| \leq 2$, $n = 1, 2, 3, \cdots$ this result is sharp for each n.*

In 2013, Babalola [6] defined a new subclass λ-pseudo starlike functions of order γ $(0 \leq \gamma < 1)$ satisfying the analytic condition

$$\Re\left(\frac{zf'(z)^\lambda}{f(z)}\right) > \gamma, \quad \lambda \geq 1 \tag{14}$$

and denoted by $\mathcal{L}_\lambda(\gamma)$. Especially, for $\lambda = 1$, we can obtain (3). Later, in 2015, Murugusundaramoorthy et.al [20] denoted the class of λ-pseudo starlike functions satisfying the condition (14) and related with sigmoid functions by $\mathcal{L}_\lambda^\gamma(\Phi)$. Also they showed that $\mathcal{L}_1^\gamma(\Phi) = \mathcal{S}^*(\gamma, \Phi)$. In [20], Murugusundaramoorthy et al. obtained the bounds of initial coefficients for the functions in $\mathcal{S}^*(\gamma, \Phi)$ as follows.

Corollary 1.5 *[20] If $f \in \mathcal{A}$ given by (2) belongs to $\mathcal{S}^*(\gamma, \Phi)$ then*

$$|a_2| \leq \frac{1-\gamma}{2} \tag{15}$$

$$|a_3| \leq \frac{(1-\gamma)^2}{8} \tag{16}$$

and

$$|a_4| \leq \frac{1-\gamma}{72} + \frac{(1-\gamma)^3}{48}. \tag{17}$$

In the present paper, we consider $\mathcal{S}^*(\gamma, \Phi)$, the class related to modified sigmoid functions satisfying the condition given by (3), which has been defined in [20]. And we obtain the coefficients a_5, a_6 and a_7 and estimate an upper bound of the fourth Hankel determinant for the functions in this class.

2 Main Results

We obtain the bounds for modulus of the coefficients a_5, a_6 and a_7 of the functions in $\mathcal{S}^*(\gamma, \Phi)$ for using in our main result.

Theorem 2.1 *If $f(z) \in \mathcal{A}$ given by (2) belongs to $\mathcal{S}^*(\gamma, \Phi)$, then*

$$|a_5| \leq \frac{(1-\gamma)^4}{384} + \frac{(1-\gamma)^2}{144} \tag{18}$$

$$|a_6| \leq \frac{(1-\gamma)^5}{3840} + \frac{(1-\gamma)^3}{576} + \frac{(1-\gamma)}{1200} \tag{19}$$

and

$$|a_7| \leq \frac{(1-\gamma)^6}{46080} + \frac{(1-\gamma)^4}{3456} + \frac{133(1-\gamma)^2}{259200}. \tag{20}$$

Proof. Let $f \in \mathcal{S}^*(\gamma, \Phi)$. So, there exist $\Phi(z) \in \mathcal{P}$ such that

$$\frac{zf'(z)}{f(z)} = \gamma + (1-\gamma)\Phi(z) \qquad (z \in \mathcal{U}) \tag{21}$$

where the function $\Phi(z)$ is a modified sigmoid function given by

$$\Phi(z) = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{17}{40320}z^7 + \cdots. \tag{22}$$

Thus we write

$$zf'(z) = f(z)[\gamma + (1-\gamma)\Phi(z)]. \tag{23}$$

Considering (21),(22) and (23), we have

$$z+2a_2z^2+3a_3z^3+\cdots = \tag{24}$$

$$z+(\frac{1-\gamma}{2}+a_2)z^2+(\frac{a_2(1-\gamma)}{2}+a_3)z^3+(\frac{-(1-\gamma)}{24}+\frac{a_3(1-\gamma)}{2}+a_4)z^4 \tag{25}$$

$$-(\frac{a_2(1-\gamma)}{24}-\frac{a_4(1-\gamma)}{2}-a_5)z^5+(\frac{(1-\gamma)}{240}-\frac{a_3(1-\gamma)}{24}+\frac{a_5(1-\gamma)}{2}+a_6)z^6 \tag{26}$$

$$+\frac{a_2(1-\gamma)}{240}-\frac{a_4(1-\gamma)}{24}+\frac{a_6(1-\gamma)}{2}+a_7)z^7+\cdots. \tag{27}$$

Comparing the coefficients of z_5, z_6 and z_7 in (24) we obtain

$$a_5 = \frac{(1-\gamma)^4}{384} - \frac{(1-\gamma)^2}{144} \tag{28}$$

$$a_6 = \frac{(1-\gamma)^5}{3840} - \frac{(1-\gamma)^3}{576} + \frac{(1-\gamma)}{1200} \tag{29}$$

and

$$a_7 = \frac{(1-\gamma)^6}{46080} - \frac{(1-\gamma)^4}{3456} + \frac{133(1-\gamma)^2}{259200}. \tag{30}$$

this is also the required result.

If we take $\gamma = 0$, we obtain the following corollary.

Corollary 2.2 *If $f \in \mathcal{A}$ given by (2) belongs to $\mathcal{S}^*(0,\Phi) = \mathcal{S}^*(\Phi)$, then*

$$|a_5| \le \frac{11}{1152} \tag{31}$$

$$|a_6| \le \frac{163}{57600} \tag{32}$$

and

$$|a_7| \le \frac{1709}{2073600}. \tag{33}$$

Theorem 2.3 *If $f \in \mathcal{S}^*(\gamma,\Phi)$ then*

$$|H_3(1)| \le (1-\gamma)^6\frac{1}{9216} + (1-\gamma)^4\frac{1}{3456} + (1-\gamma)^2\frac{1}{5184}. \tag{34}$$

Proof. Considering

$$H_3(1) = a_3(a_2a_4 - a_3^2) + a_4(a_2a_3 - a_4) + a_5(a_3 - a_2^2) \tag{35}$$

and by few simple calculations we can obtain the required result.

Using Theorem 2.1 and Theorem 2.3, we can obtain an upper bound for the fourth Hankel determinant for functions in the class $\mathcal{S}^*(\gamma,\Phi)$ as follows:

Theorem 2.4 *If $f \in \mathcal{S}^*(\gamma, \Phi)$ then*

$$|H_4(1)| \leq (1-\gamma)^3 10^{-9}[(1-\gamma)^9(16.60099736) + (1-\gamma)^7(33.847911823) \\ +(1-\gamma)^5(48.292288237) + (1-\gamma)^3(30.128884755) \\ +(1-\gamma)(30.932665562)]. \tag{36}$$

Proof. Let $f \in \mathcal{S}^*(\gamma, \Phi)$. Then equations (8)-(10) can be rewritten as follows:

$$\triangle_1 = -(1-\gamma)^7 \frac{1}{30720} - (1-\gamma)^5 \frac{1}{9216} - (1-\gamma)^3 \frac{1}{9600} \tag{37}$$

$$\triangle_2 = -(1-\gamma)^8 \frac{1}{245760} - (1-\gamma)^6 \frac{1}{55296} - (1-\gamma)^4 \frac{1}{28800} - (1-\gamma)^2 \frac{1}{86400} \tag{38}$$

$$\triangle_3 = -(1-\gamma)^9 \frac{1}{4423680} - (1-\gamma)^7 \frac{7}{276480} - (1-\gamma)^5 \frac{1}{230400} - (1-\gamma)^3 \frac{29}{9331200}. \tag{39}$$

Now, using (18),(19),(19),(34), taking modulus of $\triangle_1$, $\triangle_2$, $\triangle_3$ and considering that the modulus of $H_4(1)$ is

$$|H_4(1)| \leq |a_7||H_3(1)| + |a_6||\triangle_1| + |a_5||\triangle_2| + |a_4||\triangle_3| \tag{40}$$

we can obtain the required result in (36).

3 Open Problem

In this paper, the coefficients a_5, a_6, a_7 and the fourth Hankel determinant for a certain class of starlike functions related to modified sigmoid function were presented. The results of this article will encourage other researchers to work fourth Hankel determinant due to its novelty in literature. We hope that this work motivate the researchers to determine an upper bound of the fourth Hankel determinant for the functions in other classes of univalent functions.

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