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New uncertainty principles for the Dunkl Gabor transform

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Abstract

After reviewing Dunkl Pitt's and Dunkl Beckner's inequalities we connect both the inequalities to show a generalization of uncertainty principles for the Dunkl Gabor transform. Next we present two concentration uncertainty principles such as Benedick-Amrein-Berthier's uncertainty principle and local uncertainty principle. Finally, we study the Dunkl logarithmic Sobolev inequalities. Obtaining best possible constants of inequalities, we connect the inequalities to show a generalization of the uncertainty principles of Heisenberg type.

Keywords: *Dunkl transform, Dunkl Gabor transform, Dunkl Pitt's inequality, Dunkl Beckner's inequality, Dunkl logarithm Sobolev inequality, Dunkl Benedick-Amrein-Berthier's uncertainty principle, Local uncertainty principle.*
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1 Introduction

In quantum mechanics, the Heisenberg uncertainty principle states that the position and momentum of a particle described by a wave function in $L^2(\mathbb{R})$ cannot be simultaneously and arbitrary small. Motivated by this principle in 1946, D. Gabor, who won the 1971 Nobel Prize in physics, first recognized the great importance of localized time and frequency concentrations in signal processing [15]. In order to incorporate both time and frequency localization properties in one single transform function, Gabor introduced the windowed Fourier transform (or Gabor transform) by using a Gaussian distribution function as window function. Subsequently, various other functions have been used as window functions instead of the Gaussian function that was originally introduced by Gabor. The Gabor transformation has been found to be very useful in many physical and engineering applications, including wave propagation, signal processing and quantum optics [6]. For more details on the Gabor transform and its basic properties, we refer the reader to [8]. We may also refer to [18] where the author extends Gabor theory to the setup of locally compact abelian groups, and to [48] for the Gabor transform on Gelfand pairs.

We consider the differential-difference operators $T_j,\ j = 1, 2, ..., d$, associated with a root system $\mathscr{R}$ and a multiplicity function k, introduced by Dunkl in [10], and called the Dunkl operators in the literature.

The Dunkl theory is based on the Dunkl kernel $K(\lambda, .)$, $\lambda \in \mathbb{C}^d$, which is the unique analytic solution of the system

$$T_j u(x) = \lambda_j u(x), \quad j = 1, 2, ..., d,$$

satisfying the normalizing condition $u(0) = 1$.

With the kernel $K(\lambda, .)$, Dunkl have defined in [11] the Dunkl transform $\mathscr{F}_D$. For a family of weighted functions, ω_k, invariant under a finite reflection group W, Dunkl transform is an extension of the Fourier transform that defines an isometry of $L^2(\mathbb{R}^d, \omega_k(x)dx)$ onto itself. The basic properties of the Dunkl transforms have been studied by several authors, see [9, 10, 11, 49] and the references therein.

Very recently, many authors have been investigating the behavior of the Dunkl transform to several problems already studied for the Fourier transform; for instance, uncertainty principles [5, 21], real Paley-Wiener theorems [24], heat equation [39], wavelet transform [50], Gabor transform [25, 28] and so on.

This paper is a continuation of the papers [25, 29, 30] in the study of the quantitative uncertainty principles for the Dunkl Gabor transform on $\mathbb{R}^d$. In the classical setting, the notion of the quantitative uncertainty principles for the Gabor transform was first introduced by Wilczok [52]. Next, this subject has been extended for the generalized Gabor transforms (see [25, 27] and others) and for several classes of groups of the form $K \ltimes \mathbb{R}^d$ (see [2]).

We recall that the classical quantitative uncertainty principles is just another name for some special inequalities. These inequalities give us information about how a function and its Fourier transform relate. They are called uncertainty principles since they are similar to the classical Heisenberg uncertainty principle, which has had a big part to play in the development and understanding of quantum physics.

The quantitative uncertainty principles have been studied by many authors for various Fourier transforms, we refer the reader to the survey [14], the book [19] and the references [1, 4, 5, 13, 21, 22, 38, 43, 44, 51] for numerous versions of uncertainty principles for the Fourier transform in different settings.

To date, several generalizations, modifications and variations of the harmonic based uncertainty principles have appeared in the open literature, for instance, Benedick's uncertainty principle, Amrein and Berthier's uncertainty principles, Slepian and Pollak's uncertainty principles, Donoho and Stark's uncertainty principles and much more [3, 12, 45].

The aim of this article is to formulate some novel uncertainty principles for the Dunkl Gabor transform. Firstly, we derive an analogue of the Pitt inequality for the Dunkl Gabor transform, then we formulate Beckner's uncertainty principle for this transform via two approaches: one based on a sharp estimate from Dunkl Pitt's inequality and the other from the Dunkl Beckner inequality. Secondly, we consider the logarithmic Sobolev inequalities for the Dunkl Gabor transforms which has a dual relation with Beckner's inequality. Thirdly, we derive Benedick-Amrein-Berthier's uncertainty principle for the Dunkl Gabor transforms which shows that it is impossible for a non-trivial function and its Dunkl Gabor transform to be both supported on sets of finite measure. Towards the culmination, we formulate local uncertainty principles for the continuous Dunkl Gabor transforms in arbitrary space dimensions.

The remaining part of the paper is organized as follows. In §2 we recall the main results about the harmonic analysis associated with the Dunkl operators. The §3 is devoted to proving an analogue of the Pitt inequality for the Dunkl Gabor transform. In §4 we derive the Beckner uncertainty principle for this transform. In §5 we present two concentration uncertainty principles for the Dunkl Gabor transform such as Benedick-Amrein-Berthier's uncertainty principle and local uncertainty principle. The last Section is devoted to proving the Dunkl logarithm Sobolev uncertainty principles for the Dunkl Gabor transform.

2 Preliminaries

This section gives an introduction to the Dunkl theory. Main references are [9, 10, 11, 40, 42, 47, 49].

2.1 The Dunkl operators

We consider $\mathbb{R}^d$ with the Euclidean scalar product $\langle,\rangle$ for which the basis $\{e_i,\ i = 1,...,d\}$ is orthogonal and $\|x\| = \sqrt{\langle x,x\rangle}$. For α in $\mathbb{R}^d\backslash\{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α, i.e.

$$\sigma_\alpha(x) = x - 2\frac{\langle\alpha, x\rangle}{\|\alpha\|^2}\alpha. \tag{2.1}$$

A finite set $R \subset \mathbb{R}^d\backslash\{0\}$ is called a root system if $R\bigcap\mathbb{R}\alpha = \{\pm\alpha\}$ and $\sigma_\alpha(R) = R$ for all $\alpha \in R$. For a given root system R the reflections $\sigma_\alpha, \alpha \in R$, generate a finite group $W \subset O(d)$, called the reflection group associated with R.

We fix a positive root system $R_+ = \Big\{\alpha \in R\ \ : \langle\alpha,\beta\rangle > 0\Big\}$ for some $\beta \in \mathbb{R}^d\backslash\bigcup\limits_{\alpha\in R} H_\alpha$. We will assume that $\langle\alpha,\alpha\rangle = 2$ for all $\alpha \in R_+$. A function $k : \mathscr{R} \longrightarrow [0,\infty)$ is called a multiplicity function if it is invariant under the action of the associated reflection group W. For abbreviation, we introduce the index

$$\gamma = \gamma(k) = \sum_{\alpha\in R_+} k(\alpha). \tag{2.2}$$

Moreover, let ω_k denotes the weight function

$$\omega_k(x) = \prod_{\alpha\in R_+} |\langle\alpha, x\rangle|^{2k(\alpha)}, \tag{2.3}$$

which is W-invariant and homogeneous of degree 2γ. We introduce the Mehta-type constant

$$c_k = \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}}\omega_k(x)\, dx. \tag{2.4}$$

In the following we denote by

$C^p(\mathbb{R}^d)$ the space of functions of class C^p on $\mathbb{R}^d$.

$\mathscr{E}(\mathbb{R}^d)$ the space of C^∞-functions on $\mathbb{R}^d$.

$\mathscr{S}(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$.

$D(\mathbb{R}^d)$ the space of C^∞-functions on $\mathbb{R}^d$ which are of compact support.

$\mathscr{S}'(\mathbb{R}^d)$ the topological dual of the Schwartz space $\mathscr{S}(\mathbb{R}^d)$.

The Dunkl operators T_j, $j = 1, ..., d$, on $\mathbb{R}^d$ associated with the finite reflection group W and multiplicity function k are given by

$$T_j f(x) := \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle}, \quad f \in C^1(\mathbb{R}^d), \tag{2.5}$$

where $\alpha_j = \langle \alpha, e_j\rangle$.

We define the Dunkl-Laplacian operator $\triangle_k$ on $\mathbb{R}^d$ by

$$\triangle_k f(x) := \sum_{j=1}^{d} T_j^2 f(x) = \triangle f(x) + 2\sum_{\alpha\in R^+} k(\alpha)\Big(\frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle^2}\Big),$$

where $\triangle$ and ∇ are the usual Euclidean Laplacian and the gardient operators on $\mathbb{R}^d$ respectively.

For $y \in \mathbb{R}^d$, the system

$$\begin{cases} T_j u(x,y) &= y_j u(x,y), \quad j = 1, ..., d, \\ u(0,y) &= 1, \end{cases} \tag{2.6}$$

admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $K(x,y)$ and called Dunkl kernel. This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$.

The function $K(x,z)$ admits for all $x \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$ the following Laplace type integral representation

$$K(x,z) = \int_{\mathbb{R}^d} e^{\langle y,z\rangle} d\mu_x(y), \tag{2.7}$$

where μ_x is the positive probability measure on $\mathbb{R}^d$, with support in the closed ball $\overline{B}_d(0, \|x\|)$ of center 0 and radius $\|x\|$.

2.2 The Dunkl transform

Notation. We denote by $L^p_k(\mathbb{R}^d)$ the space of measurable functions on $\mathbb{R}^d$ such that

$$\begin{aligned} \|f\|_{L^p_k(\mathbb{R}^d)} &:= \Big(\int_{\mathbb{R}^d} |f(x)|^p \, d\gamma_k(x)\Big)^{\frac{1}{p}} < \infty, \quad \text{if} \quad 1 \le p < \infty, \\ \|f\|_{L^\infty_k(\mathbb{R}^d)} &:= \operatorname{ess}\sup_{x\in\mathbb{R}^d} |f(x)| < \infty, \end{aligned}$$

where

$$d\gamma_k(x) := \omega_k(x)dx.$$

For $p = 2$, we provide this space with the scalar product

$$\langle f, g\rangle_{L^2_k(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f(x)\overline{g(x)} d\gamma_k(x).$$

If $\mathscr{F}$ is a space of a $\mathbb{C}$-valued functions on $\mathbb{R}^d$, denote by

$$\mathscr{F}_{rad} := \Big\{ f \in \mathscr{F} : f \circ A = f \text{ for all } A \in O(d, \mathbb{R})\Big\}$$

the subspace of those $f \in \mathscr{F}$ which are radial. For $f \in \mathscr{F}_{rad}$ there exists a unique function $F : \mathbb{R}_+ \to \mathbb{C}$ such that $f(x) = F(\|x\|)$ for all $x \in \mathbb{R}^d$.

Remark 2.1. *By using the homogeneity of ω_k it is shown in [40] that for a radial function $f \in L^1_k(\mathbb{R}^d)$ the function F defined on $[0,\infty)$ by $f(x) = F(\|x\|)$, for all $x \in \mathbb{R}^d$ is integrable with respect to the measure $r^{2\gamma+d-1}dr$. More precisely,*

$$\int_{\mathbb{R}^d} f(x)d\gamma_k(x) = d_k \int_0^\infty F(r)r^{2\gamma+d-1}dr, \tag{2.8}$$

where

$$d_k := \frac{c_k}{2^{\gamma+\frac{d}{2}-1}\Gamma(\gamma+\frac{d}{2})}. \tag{2.9}$$

The Dunkl transform of a function f in $L^1_k(\mathbb{R}^d)$ is given by

$$\mathscr{F}_D(f)(y) = \frac{1}{c_k}\int_{\mathbb{R}^d} f(x)K(-ix,y)d\gamma_k(x), \quad \text{for all } y \in \mathbb{R}^d. \tag{2.10}$$

In the following we give some properties of this transform (cf. [9, 11]).

i) For f in $L^1_k(\mathbb{R}^d)$ we have

$$\|\mathscr{F}_D(f)\|_{L^\infty_k(\mathbb{R}^d)} \leq \frac{1}{c_k}\|f\|_{L^1_k(\mathbb{R}^d)}. \tag{2.11}$$

ii) Inversion formula: Let f be a function in $L^1_k(\mathbb{R}^d)$, such that $\mathscr{F}_D(f) \in L^1_k(\mathbb{R}^d)$. Then

$$\mathscr{F}_D^{-1}(f)(x) = \mathscr{F}_D(f)(-x), \quad a.e.\ x \in \mathbb{R}^d. \tag{2.12}$$

Proposition 2.1. *The Dunkl transform $\mathscr{F}_D$ is a topological isomorphism from $\mathscr{S}(\mathbb{R}^d)$ onto itself. If we put for f in $\mathscr{S}(\mathbb{R}^d)$*

$$\overline{\mathscr{F}_D}(f)(y) = \mathscr{F}_D(f)(-y), \quad y \in \mathbb{R}^d, \tag{2.13}$$

we have

$$\mathscr{F}_D\overline{\mathscr{F}_D} = \overline{\mathscr{F}_D}\mathscr{F}_D = Id.$$

Proposition 2.2. i) *Plancherel's formula for $\mathscr{F}_D$.*
For all f in $\mathscr{S}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} |f(x)|^2 d\gamma_k(x) = \int_{\mathbb{R}^d} |\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi). \tag{2.14}$$

ii) *Plancherel's theorem for $\mathscr{F}_D$.*
The Dunkl transform $f \mapsto \mathscr{F}_D(f)$ can be uniquely extended to an isometric isomorphism on $L^2_k(\mathbb{R}^d)$.

iii) *Parseval's formula for $\mathscr{F}_D$.*
For all f, g in $\mathscr{S}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} f(x)\overline{g(x)}d\gamma_k(x) = \int_{\mathbb{R}^d} \mathscr{F}_D(f)(\xi)\overline{\mathscr{F}_D(g)(\xi)}d\gamma_k(\xi). \tag{2.15}$$

Definition 2.1. *([40]) Let $x \in \mathbb{R}^d$. The Dunkl translation operator $f \mapsto \tau_x f$ is defined on $L^2_k(\mathbb{R}^d)$ by*

$$\mathscr{F}_D(\tau_x f) = K(ix,.)\mathscr{F}_D(f). \tag{2.16}$$

It is useful to have a class of functions in which (2.16) holds pointwise. One such class is given by the generalized Wigner space $\mathscr{W}_k(\mathbb{R}^d)$ given by

$$\mathscr{W}_k(\mathbb{R}^d) := \Big\{f \in L^1_k(\mathbb{R}^d) : \mathscr{F}_D(f) \in L^1_k(\mathbb{R}^d)\Big\}.$$

Proposition 2.3. *([40, 47]) i) Let $x \in \mathbb{R}^d$. For all f in $L^2_k(\mathbb{R}^d)$, we have*

$$\|\tau_x f\|_{L^2_k(\mathbb{R}^d)} \leq \|f\|_{L^2_k(\mathbb{R}^d)}.$$

ii) For all f in $\mathscr{W}_k(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, we have for every $y \in \mathbb{R}^d$

$$\tau_x f(y) = \frac{1}{c_k}\int_{\mathbb{R}^d} K(ix,\xi)K(iy,\xi)\mathscr{F}_D(f)(\xi)d\gamma_k(\xi).$$

iii) For all f in $L^2_k(\mathbb{R}^d)$, we have for $x, y \in \mathbb{R}^d$

$$\tau_x f(y) = \tau_y(f)(x). \tag{2.17}$$

Several essential properties of $\tau_y f$ is established for f being radial functions. This is collected in the following proposition ([42, 47]). Let $L^p_{k,rad}(\mathbb{R}^d)$ stands for the subspace of radial functions in $L^p_k(\mathbb{R}^d)$.

Proposition 2.4. *(i) Let f be in $L^1_{k,rad}(\mathbb{R}^d)$ and nonnegative. Then we have*

$$\forall\, y \in \mathbb{R}^d, \quad \tau_y f \geq 0, \quad \tau_y f \in L^1_k(\mathbb{R}^d)$$

and

$$\int_{\mathbb{R}^d} \tau_y f(x)d\gamma_k(x) = \int_{\mathbb{R}^d} f(x)d\gamma_k(x). \tag{2.18}$$

(ii) Let f be in $L^p_{k,rad}(\mathbb{R}^d)$, $1 \leq p \leq \infty$, we have

$$\forall\, y \in \mathbb{R}^d, \quad \|\tau_y f\|_{L^p_k(\mathbb{R}^d)} \leq \|f\|_{L^p_k(\mathbb{R}^d)}. \tag{2.19}$$

Using the Dunkl translation operator, we define the Dunkl convolution product of functions as follows (see [47, 49]).

Definition 2.2. *For f, g in $D(\mathbb{R}^d)$, we define the Dunkl convolution product by*

$$\forall\, x \in \mathbb{R}^d, \quad f *_D g(x) = \frac{1}{c_k}\int_{\mathbb{R}^d} \tau_x f(-y)g(y)d\gamma_k(y). \tag{2.20}$$

This convolution is commutative, associative and satisfies the following properties (see [47, 49]).

Proposition 2.5. *i) Let $1 \leq p, q, r \leq \infty$, such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. If f is a radial function in $L^p_k(\mathbb{R}^d)$ and g an element of $L^q_k(\mathbb{R}^d)$, then $f *_D g$ belongs to $L^r_k(\mathbb{R}^d)$ and we have*

$$\|f *_D g\|_{L^r_k(\mathbb{R}^d)} \leq \frac{1}{c_k}\,\|f\|_{L^p_k(\mathbb{R}^d)}\,\|g\|_{L^q_k(\mathbb{R}^d)}\,. \tag{2.21}$$

ii) Let $W = \mathbb{Z}_2^d$. For all f in $L^p_k(\mathbb{R}^d)$ and g an element of $L^q_k(\mathbb{R}^d)$, the function $f *_D g$ belongs to $L^r_k(\mathbb{R}^d)$ with $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$ and we have

$$\|f *_D g\|_{L^r_k(\mathbb{R}^d)} \leq \frac{2^{d|\frac{1}{p}-\frac{1}{2}|}}{c_k}\,\|f\|_{L^p_k(\mathbb{R}^d)}\,\|g\|_{L^q_k(\mathbb{R}^d)}\,. \tag{2.22}$$

2.3 Basic Dunkl Gabor transform

In this subsection we recall some results on the Dunkl Gabor transform. Fore more details we refer the reader to [25, 28].

Notation. We denote by:

$\mathbb{R}^{2d} := \mathbb{R}^d \times \mathbb{R}^d$.

For $p \in [1, \infty]$, p' denotes as in all that follows, the conjugate exponent of p.

$L^p_{\mu_k}(\mathbb{R}^{2d})$, $p \in [1, \infty]$, the space of measurable functions f on $\mathbb{R}^{2d}$ with respect to the measure $d\mu_k(x, y) := d\gamma_k(x)d\gamma_k(y)$ such that

$$\begin{aligned} \|f\|_{L^p_{\mu_k}(\mathbb{R}^{2d})} &:= \left(\int_{\mathbb{R}^{2d}} |f(x, y)|^p d\mu_k(x, y)\right)^{\frac{1}{p}} < \infty, \quad 1 \le p < \infty, \\ \|f\|_{L^\infty_{\mu_k}(\mathbb{R}^{2d})} &:= \operatorname*{ess\,sup}_{(x,y)\in\mathbb{R}^{2d}} |f(x, y)| < \infty. \end{aligned}$$

Definition 2.3. *For any function h in $L^2_{k,rad}(\mathbb{R}^d)$ and any $\nu \in \mathbb{R}^d$, we define the modulation of h by ν as :*

$$h_\nu := \mathscr{F}_D(\sqrt{\tau_\nu(|h|^2)}), \tag{2.23}$$

where τ_ν, $\nu \in \mathbb{R}^d$, are the Dunkl translation operators given in Proposition 2.4.

We consider the family $h_{y,\nu}$, $\nu, y \in \mathbb{R}^d$ defined by

$$h_{y,\nu}(x) = \tau_{-y}h_\nu(x), \ x \in \mathbb{R}^d.$$

We note that we have

$$\forall\, y, \nu \in \mathbb{R}^d, \quad \|h_{y,\nu}\|_{L^2_k(\mathbb{R}^d)} \le \|h\|_{L^2_k(\mathbb{R}^d)}. \tag{2.24}$$

Definition 2.4. *Let h be in $L^2_{k,rad}(\mathbb{R}^d)$. For a function f in $L^2_k(\mathbb{R}^d)$ we define its Dunkl Gabor transform by*

$$\mathscr{G}^D_h(f)(y, \nu) := \frac{1}{c_k}\int_{\mathbb{R}^d} f(x)\overline{h_{y,\nu}(x)}d\gamma_k(x), \tag{2.25}$$

which can also be written in the form

$$\mathscr{G}^D_h(f)(y, \nu) := f *_D \overline{\check{h}_\nu}(y), \tag{2.26}$$

where $\check{g}(t) := g(-t)$.

Remark 2.2. *(i) For h in $L^2_{k,rad}(\mathbb{R}^d)$, we have*

$$\|h_\nu\|_{L^2_k(\mathbb{R}^d)} = \|h\|_{L^2_k(\mathbb{R}^d)}. \tag{2.27}$$

(ii) For every $f \in L^2_k(\mathbb{R}^d)$ and h in $L^2_{k,rad}(\mathbb{R}^d)$, for all $\lambda > 0$ and for all $(y, \nu) \in \mathbb{R}^{2d}$, we have

$$\mathscr{G}^D_{h_{\frac{1}{\lambda}}}(f_\lambda)(y, \nu) = \mathscr{G}^D_h(f)(\frac{y}{\lambda}, \lambda\nu), \tag{2.28}$$

where

$$\forall\, t > 0, \ \forall\, x \in \mathbb{R}^d, \ g_t(x) := \frac{1}{t^{\frac{2\gamma+d}{2}}}g(\frac{x}{t}).$$

Proposition 2.6. *For f in $L^2_k(\mathbb{R}^d)$ and h in $L^2_{k,rad}(\mathbb{R}^d)$ we have*

$$\|\mathscr{G}_h^D(f)\|_{L^\infty_{\mu_k}(\mathbb{R}^{2d})} \leq \frac{1}{c_k}\|f\|_{L^2_k(\mathbb{R}^d)}\|h\|_{L^2_k(\mathbb{R}^d)}. \tag{2.29}$$

Proposition 2.7. *(Plancherel's formula)*
Let h be in $L^2_{k,rad}(\mathbb{R}^d)$. Then, for all f in $L^2_k(\mathbb{R}^d)$, we have

$$\|\mathscr{G}_h^D(f)\|_{L^2_{\mu_k}(\mathbb{R}^{2d})} = \|h\|_{L^2_k(\mathbb{R}^d)}\|f\|_{L^2_k(\mathbb{R}^d)}. \tag{2.30}$$

As in the classical case, the Dunkl Gabor transform preserves the orthogonality relation which is shown as follows.

Corollary 2.1. *Let h be in $L^2_{k,rad}(\mathbb{R}^d)$. Then, for all f_1, f_2 in $L^2_k(\mathbb{R}^d)$, we have*

$$\int_{\mathbb{R}^{2d}} \mathscr{G}_h^D(f_1)(y,\nu)\overline{\mathscr{G}_h^D(f_2)(y,\nu)}d\mu_k(y,\nu) = \|h\|^2_{L^2_k(\mathbb{R}^d)}\int_{\mathbb{R}^d} f_1(x)\overline{f_2(x)}d\gamma_k(x). \tag{2.31}$$

By Riesz-Thorin's interpolation theorem we obtain the following.

Proposition 2.8. *We assume that $h \in L^2_{k,rad}(\mathbb{R}^d)$, $f \in L^2_k(\mathbb{R}^d)$ and p belongs in $[2,\infty]$. We have*

$$\|\mathscr{G}_h^D(f)\|_{L^p_{\mu_k}(\mathbb{R}^{2d})} \leq \frac{1}{c_k^{\frac{p-2}{p}}}\|h\|_{L^2_k(\mathbb{R}^d)}\|f\|_{L^2_k(\mathbb{R}^d)}. \tag{2.32}$$

3 Pitt's inequality for the Dunkl Gabor transform

The Pitt inequality in the Dunkl setting expresses a fundamental relationship between a sufficiently smooth function and the corresponding Dunkl transform. This subject was firstly studied by Soltani [46]. Next Gorbachev et all in [17] have improved the result of Soltani and have given the Sharp Pitt inequality and logarithmic uncertainty principle for Dunkl transform on $\mathbb{R}^d$. More precisely they proved that, for every $f \in S(\mathbb{R}^d) \subseteq L^2_k(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \|\xi\|^{-2\lambda}|\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi) \leq C_k(\lambda)\int_{\mathbb{R}^d}\|x\|^{2\lambda}|f(x)|^2 d\gamma_k(x),\ 0 \leq \lambda < \frac{2\gamma+d}{2}, \tag{3.1}$$

where

$$C_k(\lambda) := 2^{-2\lambda}\Big[\frac{\Gamma(\frac{2\gamma+d-2\lambda}{4})}{\Gamma(\frac{2\gamma+d+2\lambda}{4})}\Big]^2 \tag{3.2}$$

and Γ denotes the well known Euler's Gamma function.

The main objective of this section is to formulate an analogue of Pitt's inequality (3.1) for the Dunkl Gabor transform. Formally, we start our investigation with the following lemma.

Lemma 3.1. *Let $h \in L^2_{k,rad}(\mathbb{R}^d)\bigcap L^\infty_k(\mathbb{R}^d)$, then for any $f \in L^2_k(\mathbb{R}^d)$, we have*

$$\mathscr{F}_D\Big(\mathscr{G}_h^D(f)(.,\nu)\Big)(\xi) = \mathscr{F}_D(f)(\xi)\sqrt{\tau_\nu|h|^2(-\xi)}. \tag{3.3}$$

We are now in a position to establish the Pitt inequality for the Dunkl Gabor transforms.

Theorem 3.1. *Let $h \in L^2_{k,rad}(\mathbb{R}^d) \bigcap L^\infty_k(\mathbb{R}^d)$. For any arbitrary $f \in S(\mathbb{R}^d) \subseteq L^2_k(\mathbb{R}^d)$, the Pitt inequality for the Dunkl Gabor transform is given by:*

$$\|h\|^2_{L^2_k(\mathbb{R}^d)} \int_{\mathbb{R}^d} \|\xi\|^{-2\lambda} \left|\mathscr{F}_D(f)(\xi)\right|^2 d\gamma_k(\xi) \leq C_k(\lambda) \int_{\mathbb{R}^d} \|t\|^{2\lambda} \left|\mathscr{G}^D_h(f)(t,\nu)\right|^2 d\mu_k(t,\nu),\ 0 \leq \lambda < \frac{2\gamma+d}{2}, \tag{3.4}$$

where $C_k(\lambda)$ is given by (3.2).

Proof. As a consequence of the inequality (3.1), we can write

$$\int_{\mathbb{R}^d} \|\xi\|^{-2\lambda} \left|\mathscr{F}_D[\mathscr{G}^D_h(f)(.,\nu)](\xi)\right|^2 d\gamma_k(\xi) \leq C_k(\lambda) \int_{\mathbb{R}^{2d}} \|t\|^{2\lambda} \left|\mathscr{G}^D_h(f)(t,\nu)\right|^2 d\gamma_k(t),\ \text{for all } \nu \in \mathbb{R}^d \tag{3.5}$$

which upon integration with respect to the Haar measure $d\gamma_k(\nu)$ yields

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \|\xi\|^{-2\lambda} \left|\mathscr{F}_D[\mathscr{G}^D_h(f)(.,\nu)](\xi)\right|^2 d\mu_k(\xi,\nu) \leq C_k(\lambda) \int_{\mathbb{R}^{2d}} \|t\|^{2\lambda} \left|\mathscr{G}^D_h(f)(t,\nu)\right|^2 d\mu_k(t,\nu). \tag{3.6}$$

Invoking Lemma 3.1, we can express the inequality (3.6) in the following manner:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \|\xi\|^{-2\lambda} |\mathscr{F}_D(f)(\xi)|^2 \tau_\nu |h|^2(-\xi) d\mu_k(\xi,\nu) \leq C_k(\lambda) \int_{\mathbb{R}^{2d}} \|t\|^{2\lambda} \left|\mathscr{G}^D_h(f)(t,\nu)\right|^2 d\mu_k(t,\nu).$$

Equivalently, we have

$$\int_{\mathbb{R}^d} \|\xi\|^{-2\lambda} \left|\mathscr{F}_D(f)(\xi)\right|^2 \Big\{ \int_{\mathbb{R}^d} \tau_\nu |h|^2(-\xi) d\gamma_k(\nu) \Big\} d\gamma_k(\xi) \leq C_k(\lambda) \int_{\mathbb{R}^{2d}} \|t\|^{2\lambda} \left|\mathscr{G}^D_h(f)(t,\nu)\right|^2 d\mu_k(t,\nu).$$

Using the hypothesis on h, the relation (2.18) becomes

$$\|h\|^2_{L^2_k(\mathbb{R}^d)} \int_{\mathbb{R}^d} \|\xi\|^{-2\lambda} \left|\mathscr{F}_D(f)(\xi)\right|^2 d\gamma_k(\xi) \leq C_k(\lambda) \int_{\mathbb{R}^{2d}} \|t\|^{2\lambda} \left|\mathscr{G}^D_h(f)(t,\nu)\right|^2 d\mu_k(t,\nu) \tag{3.7}$$

which establishes the Pitt inequality for the Dunkl Gabor transform in arbitrary space dimensions. □

Remark 3.1. *For $\lambda = 0$, equality holds in (3.4), which is in consonance with Plancherel's formula (2.30).*

Theorem 3.2. *Let $h \in L^2_{k,rad}(\mathbb{R}^d) \bigcap L^\infty_k(\mathbb{R}^d)$. For any function $f \in S(\mathbb{R}^d)$, the following inequality holds:*

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} \log\|t\|\, |\mathscr{G}^D_h(f)(t,\nu)|^2 d\mu_k(t,\nu) + \|h\|^2_{L^2_k(\mathbb{R}^d)} \int_{\mathbb{R}^d} \log\|\xi\|\, |\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi) \geq \\ &\Big[\frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})} + \log 2\Big] \|h\|^2_{L^2_k(\mathbb{R}^d)} \|f\|^2_{L^2_k(\mathbb{R}^d)}. \end{aligned} \tag{3.8}$$

Proof. For every $0 \leq \lambda < \frac{2\gamma+d}{2}$, we define

$$P(\lambda) = \|h\|^2_{L^2_k(\mathbb{R}^d)} \int_{\mathbb{R}^d} \|\xi\|^{-2\lambda} \left|\mathscr{F}_D(f)(\xi)\right|^2 d\gamma_k(\xi) - C_k(\lambda) \int_{\mathbb{R}^{2d}} \|t\|^{2\lambda} \left|\mathscr{G}^D_h(f)(t,\nu)\right|^2 d\mu_k(t,\nu). \tag{3.9}$$

On differentiating (3.9) with respect to λ, we obtain

$$
\begin{aligned}
&P'(\lambda) = -2\|h\|^2_{L^2_k(\mathbb{R}^d)} \int_{\mathbb{R}^d} \|\xi\|^{-2\lambda} \log \|\xi\| \left|\mathscr{F}_D(f)(\xi)\right|^2 d\gamma_k(\xi) \\
&-2C_k(\lambda) \int_{\mathbb{R}^{2d}} \|t\|^{2\lambda} \log \|t\| \, |\mathscr{G}^D_h(f)(t,\nu)|^2 d\mu_k(t,\nu) - C'_k(\lambda) \int_{\mathbb{R}^{2d}} \|t\|^{2\lambda} \left|\mathscr{G}^D_h(f)(t,\nu)\right|^2 d\mu_k(t,\nu),
\end{aligned} \tag{3.10}
$$

where

$$
C'_k(\lambda) \;=\; -C_k(\lambda)\Big(2\log 2 + \frac{\Gamma'(\frac{2\gamma+d-2\lambda}{4})}{\Gamma(\frac{2\gamma+d-2\lambda}{4})} + \frac{\Gamma'(\frac{2\gamma+d+2\lambda}{4})}{\Gamma(\frac{2\gamma+d+2\lambda}{4})}\Big). \tag{3.11}
$$

For $\lambda = 0$, equation (3.11) yields

$$
C'_k(0) = -2\Big[\log 2 + \frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})}\Big]. \tag{3.12}
$$

By virtue of Dunkl Pitt's inequality (3.4), it follows that $P(\lambda) \leq 0$, for all $\lambda \in [0, \frac{2\gamma+d}{2})$ and

$$
P(0) \;=\; \|h\|^2_{L^2_k(\mathbb{R}^d)} \int_{\mathbb{R}^d} \left|\mathscr{F}_D(f)(\xi)\right|^2 d\gamma_k(\xi) - C_k(0) \int_{\mathbb{R}^{2d}} \left|\mathscr{G}^D_h(f)(t,\nu)\right|^2 d\mu_k(t,\nu) \tag{3.13}
$$

$$
\;=\; \|h\|^2_{L^2_k(\mathbb{R}^d)} \|f\|^2_{L^2_k(\mathbb{R}^d)} - \|h\|^2_{L^2_k(\mathbb{R}^d)} \|f\|^2_{L^2_k(\mathbb{R}^d)} = 0. \tag{3.14}
$$

Therefore,

$$
\begin{aligned}
&-2\|h\|^2_{L^2_k(\mathbb{R}^d)} \int_{\mathbb{R}^d} \log \|\xi\| \left|\mathscr{F}_D(f)(\xi)\right|^2 d\gamma_k(\xi) - 2C_k(0) \int_{\mathbb{R}^{2d}} \log \|t\| \, |\mathscr{G}^D_h(f)(t,\nu)|^2 d\mu_k(t,\nu) \\
&-C'_k(0) \int_{\mathbb{R}^{2d}} \left|\mathscr{G}^D_h(f)(t,\nu)\right|^2 d\mu_k(t,\nu) \leq 0.
\end{aligned} \tag{3.15}
$$

Applying Plancherel's formula (2.30) and the obtained estimate (3.12) of $C'_k(0)$, we get

$$
\begin{aligned}
-2\|h\|^2_{L^2_k(\mathbb{R}^d)} \int_{\mathbb{R}^d} \log \|\xi\| \left|\mathscr{F}_D(f)(\xi)\right|^2 d\gamma_k(\xi) - 2 \int_{\mathbb{R}^{2d}} \log \|t\| \left|\mathscr{G}^D_h(f)(t,\nu)\right|^2 d\mu_k(t,\nu) \\
+2\Big[\log 2 + \frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})}\Big] \|h\|^2_{L^2_k(\mathbb{R}^d)} \|f\|^2_{L^2_k(\mathbb{R}^d)} \leq 0
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
&\int_{\mathbb{R}^{2d}} \log \|t\| \left|\mathscr{G}^D_h(f)(t,\nu)\right|^2 d\mu_k(t,\nu) + \|h\|^2_{L^2_k(\mathbb{R}^d)} \int_{\mathbb{R}^d} \log \|\xi\| \, |\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi) \geq \\
&\Big[\frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})} + \log 2\Big] \|h\|^2_{L^2_k(\mathbb{R}^d)} \|f\|^2_{L^2_k(\mathbb{R}^d)}.
\end{aligned} \tag{3.16}
$$

Inequality (3.16) is the desired Beckner's uncertainty principle for the Dunkl Gabor transform in arbitrary space dimensions. □

4 Beckner's type inequalities for the Dunkl Gabor transforms

Dunkl Beckner's inequality [17] is given by

$$\int_{\mathbb{R}^d} \log \|t\| \, |f(t)|^2 d\gamma_k(t) + \int_{\mathbb{R}^d} \log \|\xi\| \, |\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi) \geq \Big[\frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})} + \log 2\Big] \int_{\mathbb{R}^d} |f(t)|^2 d\gamma_k(t) \tag{4.1}$$

for all $f \in \mathscr{S}(\mathbb{R}^d)$. This inequality is related to the Heisenberg uncertainty principle and for that reason it is often referred as the logarithmic uncertainty principle. Considerable attention has been paid to this inequality for its various generalizations, improvements, analogues, and their applications [20].

We now present an alternate proof of Theorem 3.2. The strategy of the proof is different of given in the previous section and is obtained directly from Dunkl Beckner's inequality (4.1).

Proof. of Theorem 3.2. We replace f in (4.1) with $\mathscr{G}_h^D(f)(., \nu)$, so that

$$\begin{aligned} &\int_{\mathbb{R}^d} \log \|t\| \, |\mathscr{G}_h^D(f)(t,\nu)|^2 d\gamma_k(t) + \int_{\mathbb{R}^d} \log \|\xi\| \Big|\mathscr{F}_D[\mathscr{G}_h^D(f)(.,\nu)](\xi)\Big|^2 d\gamma_k(\xi) \geq \\ &\Big[\frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})} + \log 2\Big] \int_{\mathbb{R}^d} \Big|\mathscr{G}_h^D(f)(t,\nu)\Big|^2 d\gamma_k(t), \ \text{for all } \nu \in \mathbb{R}^d. \end{aligned} \tag{4.2}$$

Integrating (4.2) with respect to the measure $d\gamma_k(\nu)$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} \log \|t\| \, |\mathscr{G}_h^D(f)(t,\nu)|^2 d\mu_k(t,\nu) + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \log \|\xi\| \Big|\mathscr{F}_D[\mathscr{G}_h^D(f)(.,\nu)](\xi)\Big|^2 d\mu_k(\xi,\nu) \geq \\ &\Big[\frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})} + \log 2\Big] \int_{\mathbb{R}^{2d}} \Big|\mathscr{G}_h^D(f)(t,\nu)\Big|^2 d\mu_k(t,\nu). \end{aligned} \tag{4.3}$$

Using Plancherel's formula (2.30), we get

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} \log \|t\| \, |\mathscr{G}_h^D(f)(t,\nu)|^2 d\mu_k(t,\nu) + \int_{\mathbb{R}^{2d}} \log \|\xi\| \, |\mathscr{F}_D[\mathscr{G}_h^D(f)(.,\nu)](\xi)|^2 d\mu_k(\xi,\nu) \geq \\ &\Big[\frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})} + \log 2\Big] \|h\|^2_{L^2_k(\mathbb{R}^d)} \|f\|^2_{L^2_k(\mathbb{R}^d)}. \end{aligned} \tag{4.4}$$

We shall now simplify the second integral of (4.4). By using Lemma 3.1 and relation (2.18) we infer that

$$\begin{aligned} \int_{\mathbb{R}^{2d}} \log \|\xi\| \, |\mathscr{F}_D[\mathscr{G}_h^D(f)(.,\nu)](\xi)|^2 d\mu_k(\xi,\nu) &= \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \log \|\xi\| \Big|\mathscr{F}_D[\mathscr{G}_h^D(f)(.,\nu)](\xi)\Big|^2 d\gamma_k(\xi) \Big) d\gamma_k(\nu) \\ &= \|h\|^2_{L^2_k(\mathbb{R}^d)} \Big(\int_{\mathbb{R}^d} \log \|\xi\| \, |\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi) \Big). \end{aligned} \tag{4.5}$$

Plugging the estimate (4.5) in (4.4) gives the desired inequality for the Dunkl Gabor transforms as

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} \log \|t\| \, |\mathscr{G}_h^D(f)(t,\nu)|^2 d\mu_k(t,\nu) + \|h\|^2_{L^2_k(\mathbb{R}^d)} \int_{\mathbb{R}^d} \log \|\xi\| \, |\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi) \geq \\ &\Big[\frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})} + \log 2\Big] \|h\|^2_{L^2_k(\mathbb{R}^d)} \|f\|^2_{L^2_k(\mathbb{R}^d)}. \end{aligned}$$

This completes the second proof of Theorem 3.2. □

Corollary 4.1. *Let $h \in L^2_{k,rad}(\mathbb{R}^d) \bigcap L^\infty_k(\mathbb{R}^d)$ such that $\|h\|_{L^2_k(\mathbb{R}^d)} = 1$. For any function $f \in S(\mathbb{R}^d)$, the following inequality holds:*

$$\Big\{\int_{\mathbb{R}^{2d}} \|t\|^2 \left|\mathscr{G}^D_h(f)(t,\nu)\right|^2 d\mu_k(t,\nu)\Big\}^{1/2}\Big\{\int_{\mathbb{R}^d} \|\xi\|^2 |\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi)\Big\}^{1/2}$$
$$\geq \exp\Big(\frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})} + \log 2\Big)\|f\|^2_{L^2_k(\mathbb{R}^d)}.$$

Proof. Using Jensen's inequality in (3.8) and the fact that $\|h\|_{L^2_k(\mathbb{R}^d)} = 1$, we obtain an analogue of the classical Heisenberg's uncertainty inequality for the Dunkl Gabor transforms as

$$\begin{aligned}
&\log\Big\{\int_{\mathbb{R}^{2d}} \|y\|^2 \frac{\left|\mathscr{G}^D_h(f)(y,\nu)\right|^2}{\|f\|^2_{L^2_k(\mathbb{R}^d)}} d\mu_k(y,\nu) \int_{\mathbb{R}^d} \|\xi\|^2 \frac{|\mathscr{F}_D(f)(\xi)|^2}{\|f\|^2_{L^2_k(\mathbb{R}^d)}} d\gamma_k(\xi)\Big\}^{1/2}\\
= \;&\log\Big\{\int_{\mathbb{R}^{2d}} \|y\|^2 \frac{\left|\mathscr{G}^D_h(f)(y,\nu)\right|^2}{\|f\|^2_{L^2_k(\mathbb{R}^d)}} d\mu_k(y,\nu)\Big\}^{1/2} + \log\Big\{\int_{\mathbb{R}^d} \|\xi\|^2 \frac{|\mathscr{F}_D(f)(\xi)|^2}{\|f\|^2_{L^2_k(\mathbb{R}^d)}} d\gamma_k(\xi)\Big\}^{1/2}\\
\geq \;&\int_{\mathbb{R}^{2d}} \log\|y\| \frac{\left|\mathscr{G}^D_h(f)(y,\nu)\right|^2}{\|f\|^2_{L^2_k(\mathbb{R}^d)}} d\mu_k(y,\nu) + \int_{\mathbb{R}^d} \log\|\xi\| \frac{|\mathscr{F}_D(f)(\xi)|^2}{\|f\|^2_{L^2_k(\mathbb{R}^d)}} d\gamma_k(\xi)\\
\geq \;&\frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})} + \log 2,
\end{aligned}$$

which upon simplification with yields the result. □

Remark 4.1. *(i) Using the approximation identity*

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} - 2\int_0^\infty \frac{t}{(t^2+z^2)(e^{2\pi t}-1)} dt \tag{4.6}$$

we infer

$$\exp\Big[\frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})} + \log 2\Big] \approx \frac{2\gamma+d}{2} \quad \text{for } 2\gamma+d \gg 1, \tag{4.7}$$

which is the constant of the Heisenberg uncertainty principle for the Dunkl Gabor transform given in [25].

(ii) Proceeding as above in logarithmic uncertainty inequality (4.1) *we deduce the following Heisenberg uncertainty inequality*

$$\Big\{\int_{\mathbb{R}^d} \|t\|^2 |f(t)|^2 d\gamma_k(t)\Big\}^{\frac12}\Big\{\int_{\mathbb{R}^d} \|\xi\|^2 |\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi)\Big\}^{\frac12} \geq \exp\Big(\frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})} + \log 2\Big)\int_{\mathbb{R}^d} |f(t)|^2 d\gamma_k(t). \tag{4.8}$$

(iii) Using the approximation relation (4.6) *we deduce that the constant in the right-hand side of (4.8),*

$$\exp\Big[\frac{\Gamma'(\frac{2\gamma+d}{4})}{\Gamma(\frac{2\gamma+d}{4})} + \log 2\Big] \approx \frac{2\gamma+d}{2} \quad \text{for } 2\gamma+d \gg 1,$$

which is the constant of the Heisenberg uncertainty principle for the Dunkl transform given in [41].

5 Concentration uncertainty principles for the Dunkl Gabor transforms

In this Section, we derive two concentration uncertainty principles for the Dunkl Gabor transforms as an analog of the Benedick-Amrein-Berthier and local uncertainty principles in the time-frequency analysis.

5.1 Benedick-Amrein-Berthier's uncertainty principle for the Dunkl Gabor transforms

Recently Ghobber and Jaming in [16] have proved the Benedicks-Amrein-Berthier uncertainty principle for the Dunkl transform which states that if E_1 and E_2 are two subsets of $\mathbb{R}^d$ with finite measure, then there exist a positive constant $C_k(E_1, E_2)$ such that for any $f \in L^2_k(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} |f(t)|^2 d\gamma_k(t) \leq C_k(E_1, E_2)\Big\{\int_{\mathbb{R}^d\setminus E_1} |f(t)|^2 d\gamma_k(t) + \int_{\mathbb{R}^d\setminus E_2} |\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi)\Big\}. \tag{5.1}$$

In this Subsection, our primary interest is to establish the Benedick-Amrein-Berthier uncertainty principle for the Dunkl Gabor transforms in arbitrary space dimensions by employing the inequality (5.1). In this direction, we have the following main theorem.

Theorem 5.1. *Let $h \in L^2_{k,rad}(\mathbb{R}^d) \bigcap L^\infty_k(\mathbb{R}^d)$. For any arbitrary function $f \in L^2_k(\mathbb{R}^d)$, we have the following uncertainty inequality*

$$\int_{\mathbb{R}^d\setminus E_1}\int_{\mathbb{R}^d} \left|\mathscr{G}^D_h(f)(t,\nu)\right|^2 d\mu_k(t,\nu) + \|h\|^2_{L^2_k(\mathbb{R}^d)} \int_{\mathbb{R}^d\setminus E_2} |\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi) \geq \frac{\|h\|^2_{L^2_k(\mathbb{R}^d)}\|f\|^2_{L^2_k(\mathbb{R}^d)}}{C_k(E_1,E_2)} \tag{5.2}$$

where $C_k(E_1, E_2)$ the constant given in relation (5.1).

Proof. Since for all $\nu \in \mathbb{R}^d$, $\mathscr{G}^D_h(f)(.,\nu) \in L^2_k(\mathbb{R}^d)$ whenever $f \in L^2_k(\mathbb{R}^d)$, so we can replace the function f appearing in (5.1) with $\mathscr{G}^D_h(f)(.,\nu)$ to get

$$\int_{\mathbb{R}^d} \left|\mathscr{G}^D_h(f)(t,\nu)\right|^2 d\gamma_k(t) \leq C_k(E_1,E_2)\Big\{\int_{\mathbb{R}^d\setminus E_1} \left|\mathscr{G}^D_h(f)(t,\nu)\right|^2 d\gamma_k(t) + \int_{\mathbb{R}^d\setminus E_2} \left|\mathscr{F}_D[\mathscr{G}^D_h(f)(.,\nu)](\xi)\right|^2 d\gamma_k(\xi)\Big\}. \tag{5.3}$$

By integrating (5.3) with respect the measure $d\gamma_k(\nu)$, we obtain

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left|\mathscr{G}^D_h(f)(t,\nu)\right|^2 d\mu_k(t,\nu) \leq$$
$$C_k(E_1,E_2)\Big(\int_{\mathbb{R}^d\setminus E_1}\int_{\mathbb{R}^d} \left|\mathscr{G}^D_h(f)(t,\nu)\right|^2 d\mu_k(t,\nu) + \int_{\mathbb{R}^d\setminus E_2}\int_{\mathbb{R}^d} \left|\mathscr{F}_D[\mathscr{G}^D_h(f)(.,\nu)](\xi)\right|^2 d\mu_k(\xi,\nu)\Big).$$

Using Lemma 3.1 together with Plancherel's formula (2.30), the above inequality becomes

$$\int_{\mathbb{R}^d\setminus E_1}\int_{\mathbb{R}^d} \left|\mathscr{G}^D_h(f)(t,\nu)\right|^2 d\mu_k(t,\nu) + \int_{\mathbb{R}^d\setminus E_2}\int_{\mathbb{R}^d} |\mathscr{F}_D(f)(\xi)\sqrt{\tau_\nu|h|^2(-\xi)}|^2 d\mu_k(\xi,\nu) \geq \frac{\|h\|^2_{L^2_k(\mathbb{R}^d)}\|f\|^2_{L^2_k(\mathbb{R}^d)}}{C_k(E_1,E_2)}$$

which further implies

$$\int_{\mathbb{R}^d\setminus E_1}\int_{\mathbb{R}^d} \left|\mathscr{G}^D_h(f)(t,\nu)\right|^2 d\mu_k(t,\nu) + \int_{\mathbb{R}^d\setminus E_2} |\mathscr{F}_D(f)(\xi)|^2 \Big\{\int_{\mathbb{R}^d} \tau_\nu|h|^2(-\xi) d\gamma_k(\nu)\Big\} d\gamma_k(\xi) \geq \frac{\|h\|^2_{L^2_k(\mathbb{R}^d)}\|f\|^2_{L^2_k(\mathbb{R}^d)}}{C_k(E_1,E_2)}.$$

Thus using the fact that $h \in L^2_{k,rad}(\mathbb{R}^d) \bigcap L^\infty_k(\mathbb{R}^d)$ and relation (2.18) we obtain

$$\int_{\mathbb{R}^d\setminus E_1}\int_{\mathbb{R}^d}\left|\mathscr{G}_h^D(f)(t,\nu)\right|^2 d\mu_k(t,\nu) + \|h\|^2_{L^2_k(\mathbb{R}^d)}\int_{\mathbb{R}^d\setminus E_2}|\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi) \geq \frac{\|h\|^2_{L^2_k(\mathbb{R}^d)}\|f\|^2_{L^2_k(\mathbb{R}^d)}}{C_k(E_1,E_2)}$$

which is the desired Benedick-Amrein-Berthier's uncertainty principle for the Dunkl Gabor transforms in arbitrary space dimensions. □

Theorem 5.1 allows as to obtain a general form of Heisenberg-type uncertainty inequality for the Dunkl Gabor transforms.

Corollary 5.1. *Let $p, q > 0$. Then there exist a positive constant $\mathscr{C}_k(p,q)$ such that for any arbitrary function $f \in L^2_k(\mathbb{R}^d)$, we have the following uncertainty inequality*

$$\Big(\int_{\mathbb{R}^{2d}}\|y\|^{2p}\left|\mathscr{G}_h^D(f)(y,\nu)\right|^2 d\mu_k(y,\nu)\Big)^{\frac{q}{2}}\Big(\int_{\mathbb{R}^d}\|\xi\|^{2q}|\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi)\Big)^{\frac{p}{2}} \geq \mathscr{C}_k(p,q)\|h\|^q_{L^2_k(\mathbb{R}^d)}\|f\|^{p+q}_{L^2_k(\mathbb{R}^d)}. \tag{5.4}$$

Proof. Let $p, q > 0$ and let $f \in L^2_k(\mathbb{R}^d)$. Take $E_1 = E_2 = B_d(0,1)$ the unit ball in $\mathbb{R}^d$. Then by (5.2)

$$\int_{B^c_d(0,1)}\int_{\mathbb{R}^d}\left|\mathscr{G}_h^D(f)(y,\nu)\right|^2 d\mu_k(y,\nu) + \|h\|^2_{L^2_k(\mathbb{R}^d)}\int_{B^c_d(0,1)}|\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi) \geq \frac{\|h\|^2_{L^2_k(\mathbb{R}^d)}\|f\|^2_{L^2_k(\mathbb{R}^d)}}{C(k)}.$$

Here $C(k) := C_k(E_1, E_2)$.

It follows that

$$\int_{\mathbb{R}^{2d}}\|y\|^{2p}\left|\mathscr{G}_h^D(f)(y,\nu)\right|^2 d\mu_k(y,\nu) + \|h\|^2_{L^2_k(\mathbb{R}^d)}\int_{\mathbb{R}^d}\|\xi\|^{2q}|\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi) \geq \frac{\|h\|^2_{L^2_k(\mathbb{R}^d)}\|f\|^2_{L^2_k(\mathbb{R}^d)}}{C(k)}.$$

Now replacing f by f_λ and h by $h_{\frac{1}{\lambda}}$, we get by (2.28)

$$\int_{\mathbb{R}^{2d}}\|y\|^{2p}\left|\mathscr{G}_h^D(f)(\frac{y}{\lambda},\lambda\nu)\right|^2 d\mu_k(y,\nu)+\lambda^{2\gamma+d}\|h\|^2_{L^2_k(\mathbb{R}^d)}\int_{\mathbb{R}^d}\|\xi\|^{2q}|\mathscr{F}_D(f)(\lambda\xi)|^2 d\gamma_k(\xi) \geq \frac{\|h\|^2_{L^2_k(\mathbb{R}^d)}\|f\|^2_{L^2_k(\mathbb{R}^d)}}{C(k)}.$$

Thus

$$\lambda^{2p}\int_{\mathbb{R}^{2d}}\|y\|^{2p}\left|\mathscr{G}_h^D(f)(y,\nu)\right|^2 d\mu_k(y,\nu)+\lambda^{-2q}\|h\|^2_{L^2_k(\mathbb{R}^d)}\int_{\mathbb{R}^d}\|\xi\|^{2q}|\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi) \geq \frac{\|h\|^2_{L^2_k(\mathbb{R}^d)}\|f\|^2_{L^2_k(\mathbb{R}^d)}}{C(k)}.$$

The desired result follows by minimizing the right hand side over $\lambda > 0$. □

5.2 Local-type Uncertainty Principle for the Dunkl Gabor Transforms

We begin this subsection by recalling the local uncertainty principle for the Dunkl Gabor transforms proved in [29, 30].

Theorem 5.2. *We assume that $h \in L^2_{k,rad}(\mathbb{R}^d)$. Let $1 < p \leq 2$, $a > 0$ and a measurable subset $T \subset \mathbb{R}^{2d}$ satisfying $0 < \mu_k(T) := \int_T d\mu_k(x,y) < \infty$. Then for all $f \in L^2_k(\mathbb{R}^d)$, we have*

$$\|1_T \mathscr{G}_h^D(f)\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})} \leq \begin{cases} C_1(a,h)(\mu_k(T))^{\frac{2a}{2\gamma+d}}\Big[\|\,\|y\|^a f\|_{L^2_k(\mathbb{R}^d)} + \|\,\|y\|^a f\|_{L^{2p}_k(\mathbb{R}^d)}\Big], & 0 < a < \frac{2\gamma+d}{2p'}, \\ C_2(a,h)(\mu_k(T))^{\frac{1}{p'}}\|f\|^{1-\frac{2\gamma+d}{2ap'}}_{L^{2p}_k(\mathbb{R}^d)}\|\,\|y\|^a f\|^{\frac{2\gamma+d}{2ap'}}_{L^{2p}_k(\mathbb{R}^d)}, & a > \frac{2\gamma+d}{2p'}, \\ C_3(a,h)(\mu_k(T))^{\frac{1}{2p'}}\Big[\|f\|^{\frac12}_{L^2_k(\mathbb{R}^d)}\|\,\|y\|^a f\|^{\frac12}_{L^2_k(\mathbb{R}^d)} + \|f\|^{\frac12}_{L^{2p}_k(\mathbb{R}^d)}\|\,\|y\|^a f\|^{\frac12}_{L^{2p}_k(\mathbb{R}^d)}\Big], & a = \frac{2\gamma+d}{2p'}, \end{cases}$$

where

$$\begin{aligned} C_1(a,h) &= c_k^{1-\frac{2}{p}-\frac{4a}{2\gamma+d}}\Big(\frac{d_k}{2\gamma+d-2ap'}\Big)^{\frac{a}{2\gamma+d}}\|h\|_{L^2_k(\mathbb{R}^d)}, \\ C_2(a,h) &= \Big(\frac{2ap'}{2p'a-2\gamma-d}\Big)^{\frac{1}{2p}}\Big(\frac{2ap'}{2\gamma+d}-1\Big)^{\frac{2\gamma+d}{4app'}}(C(a,p))^{\frac{1}{2p'}}\frac{\|h\|_{L^2_k(\mathbb{R}^d)}}{c_k}, \\ C_3(a,h) &= \frac{2}{c_k^{\frac1p}}\Big(\frac{2d_k}{2\gamma+d}\Big)^{\frac{1}{4p'}}\|h\|_{L^2_k(\mathbb{R}^d)} \end{aligned}$$

and

$$C(a,p) := \frac{d_k}{2pa}\frac{\Gamma(\frac{2\gamma+d}{2ap})\Gamma(\frac{2p'a-2\gamma-d}{2pa})}{\Gamma(\frac{p'}{p})}. \tag{5.5}$$

Corollary 5.2. *We assume that $h \in L^2_{k,rad}(\mathbb{R}^d)$. Let $1 < p \leq 2$ and $a > 0$. Then for all $f \in L^2_k(\mathbb{R}^d)$, we have*

$$\|\mathscr{G}_h^D(f)\|_{L_{\mu_k}^{\frac{(2\gamma+d)p'}{2\gamma+d-2p'a},p'}(\mathbb{R}^{2d})} \leq C_1(a,h)\Big[\|\,\|y\|^a f\|_{L^2_k(\mathbb{R}^d)} + \|\,\|y\|^a f\|_{L^{2p}_k(\mathbb{R}^d)}\Big], \quad 0 < a < \frac{2\gamma+d}{2p'},$$

$$\|\mathscr{G}_h^D(f)\|_{L^{\infty,p'}_{\mu_k}(\mathbb{R}^{2d})} \leq C_2(a,h)\|f\|^{1-\frac{2\gamma+d}{2ap'}}_{L^{2p}_k(\mathbb{R}^d)}\|\,\|y\|^a f\|^{\frac{2\gamma+d}{2ap'}}_{L^{2p}_k(\mathbb{R}^d)}, \quad a > \frac{2\gamma+d}{2p'},$$

$$\|\mathscr{G}_h^D(f)\|_{L^{2p',p'}_{\mu_k}(\mathbb{R}^{2d})} \leq C_3(a,h)\Big[\|f\|^{\frac12}_{L^2_k(\mathbb{R}^d)}\|\,\|y\|^a f\|^{\frac12}_{L^2_k(\mathbb{R}^d)} + \|f\|^{\frac12}_{L^{2p}_k(\mathbb{R}^d)}\|\,\|y\|^a f\|^{\frac12}_{L^{2p}_k(\mathbb{R}^d)}\Big], \ a = \frac{2\gamma+d}{2p'},$$

where $L^{p,q}_{\mu_k}(\mathbb{R}^{2d})$ is the Lorentz space defined by the following norm

$$\|g\|_{L^{p,q}_{\mu_k}(\mathbb{R}^{2d})} = \sup_{T\subset\mathbb{R}^{2d},\, 0<\mu_k(T)<\infty}\Big[(\mu_k(T))^{\frac1p-\frac1q}\|1_T g\|_{L^q_{\mu_k}(\mathbb{R}^{2d})}\Big], \tag{5.6}$$

and $C_j(a,h)$, j =1-3, the constants given in Theorem 5.2.

Theorem 5.3. *We assume that $h \in L^2_{k,rad}(\mathbb{R}^d)$. Let $a, b > 0$ and $1 < p \leq 2$. Then for all $f \in L^2_k(\mathbb{R}^d)$, we have*

$$\|\mathscr{G}_h^D(f)\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})} \leq \begin{cases} C_1(a,b,h)\Big[\|\,\|y\|^a f\|_{L^2_k(\mathbb{R}^d)} + \|\,\|y\|^a f\|_{L^{2p}_k(\mathbb{R}^d)}\Big]^{\frac{b}{4a+b}}\Big\|\,\|(x,\nu)\|^b\mathscr{G}_h^D(f)\Big\|^{\frac{4a}{4a+b}}_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})}, & 0 < a < \frac{2\gamma+d}{2p'}, \\ C_2(a,b,h)\Big(\|f\|^{1-\frac{2\gamma+d}{2ap'}}_{L^{2p}_k(\mathbb{R}^d)}\Big\|\,\|y\|^a f\Big\|^{\frac{2\gamma+d}{2ap'}}_{L^{2p}_k(\mathbb{R}^d)}\Big)^{\frac{bp'}{4\gamma+2d+bp'}}\Big\|\,\|(x,\nu)\|^b\mathscr{G}_h^D(f)\Big\|^{\frac{4\gamma+2d}{4\gamma+2d+bp'}}_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})}, & a > \frac{2\gamma+d}{2p'}, \\ C_3(a,b,h)\Big[\|f\|^{\frac12}_{L^2_k(\mathbb{R}^d)}\Big\|\|y\|^a f\Big\|^{\frac12}_{L^2_k(\mathbb{R}^d)} + \|f\|^{\frac12}_{L^{2p}_k(\mathbb{R}^d)}\Big\|\,\|y\|^a f\Big\|^{\frac12}_{L^{2p}_k(\mathbb{R}^d)}\Big]^{\frac{b}{2a+b}} & \\ \Big\|\,\|(x,\nu)\|^b\mathscr{G}_h^D(f)\Big\|^{\frac{2a}{2a+b}}_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})}, & a = \frac{2\gamma+d}{2p'}, \end{cases}$$

where

$$
\begin{aligned}
C_1(a,b,h) &= \Big[(\tfrac{b}{4a})^{\frac{4a}{4a+b}} + (\tfrac{4a}{b})^{\frac{b}{4a+b}}\Big]^{\frac{1}{p'}}\Big(C_1(a,h)\sigma_{2d}^{\frac{2a}{2\gamma+d}}\Big)^{\frac{b}{4a+b}},\\
C_2(a,b,h) &= [(\tfrac{bp'}{4\gamma+2d})^{\frac{4\gamma+2d}{4\gamma+2d+bp'}} + (\tfrac{4\gamma+2d}{bp'})^{\frac{bp'}{4\gamma+2d+bp'}}]^{\frac{1}{p'}}\Big(\sigma_{2d}C_2^{p'}(a,h)\Big)^{\frac{b}{4\gamma+2d+bp'}},\\
C_3(a,b,h) &= [(\tfrac{b}{2a})^{\frac{2a}{2a+b}} + (\tfrac{2a}{b})^{\frac{b}{2a+b}}]^{\frac{1}{p'}}\Big(\sigma_{2d}^{\frac{1}{2p'}}C_3(a,h)\Big)^{\frac{b}{2a+b}},
\end{aligned}
$$

and $C_j(a,h)$, j =1-3, the constants given in Theorem 5.2 and σ_{2d} the constant given by

$$\sigma_{2d} := \frac{(d_k\Gamma(\frac{2\gamma+d}{2}))^2}{4(2\gamma+d)\Gamma(2\gamma+d)}.$$

In the remember of this Subsection, we derive some new local uncertainty principles for the Dunkl Gabor. Recently Jaming and Ghobber have proved the following local uncertainty principle for the Dunkl transforms:

Proposition 5.1. *([16]). Let E be a subset of $\mathbb{R}^d$ such that $0 < \gamma_k(E) := \int_E d\gamma_k(x) < \infty$. For $0 < s < \frac{2\gamma+d}{2}$, there exist a positive constant $\mathfrak{C}(k,s)$ such that for any $f \in L^2_k(\mathbb{R}^d)$*

$$\int_E \Big|\mathscr{F}_D(f)(\xi)\Big|^2 d\gamma_k(\xi) \le \mathfrak{C}(k,s)(\gamma_k(E))^{\frac{2s}{2\gamma+d}} \| \|x\|^s f\|^2_{L^2_k(\mathbb{R}^d)}. \tag{5.7}$$

The main objective of this Subsection is to establish the local uncertainty principle for the Dunkl Gabor transforms in arbitrary space dimensions by employing the inequality (5.7).

Theorem 5.4. *Let $h \in L^2_{k,rad}(\mathbb{R}^d) \bigcap L^\infty_k(\mathbb{R}^d)$. Let $E \subset \mathbb{R}^d$ as above with finite measure. Then, for any $f \in L^2_k(\mathbb{R}^d)$, we have*

$$\int_{\mathbb{R}^{2d}} \|t\|^{2s}\Big|\mathscr{G}^D_h(f)(t,\nu)\Big|^2 d\mu_k(t,\nu) \ge \frac{\|h\|^2_{L^2_k(\mathbb{R}^d)}}{\mathfrak{C}(k,s)(\gamma_k(E))^{\frac{2s}{2\gamma+d}}}\int_E |\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi), \quad 0 < s < \frac{2\gamma+d}{2} \tag{5.8}$$

where $\mathfrak{C}(k,s)$ the constant given in the relation (5.7).

Proof. Since $\mathscr{G}^D_h(f)(.,\nu) \in L^2_k(\mathbb{R}^d)$, whenever $f \in L^2_k(\mathbb{R}^d)$, so we can replace the function f appearing in (5.7) with $\mathscr{G}^D_h(f)(.,\nu)$ to get

$$\int_E \Big|\mathscr{F}_D[\mathscr{G}^D_h(f)(.,\nu)](\xi)\Big|^2 d\gamma_k(\xi) \le \mathfrak{C}(k,s)(\gamma_k(E))^{\frac{2s}{2\gamma+d}} \| \|t\|^s \mathscr{G}^D_h(f)(.,\nu)\|^2_{L^2_k(\mathbb{R}^d)}, \text{ for all } \nu \in \mathbb{R}^d. \tag{5.9}$$

For explicit expression of (5.9), we shall integrate this inequality with respect to the measure $d\gamma_k(\nu)$ to get

$$\int_E\int_{\mathbb{R}^d} \Big|\mathscr{F}_D[\mathscr{G}^D_h(f)(.,\nu)](\xi)\Big|^2 d\mu_k(\xi,\nu) \le \mathfrak{C}(k,s)(\gamma_k(E))^{\frac{2s}{2\gamma+d}} \int_{\mathbb{R}^{2d}} \|t\|^{2s}\Big|\mathscr{G}^D_h(f)(t,\nu)|^2 d\mu_k(t,\nu)$$

which together with Lemma 3.1 gives

$$\int_E\int_{\mathbb{R}^d} |\mathscr{F}_D(f)(\xi)|^2 \tau_\nu|h|^2(-\xi) d\gamma_k(\xi)d\gamma_k(\nu) \le \mathfrak{C}(k,s)(\gamma_k(E))^{\frac{2s}{2\gamma+d}} \int_{\mathbb{R}^{2d}} \|t\|^{2s}\Big|\mathscr{G}^D_h(f)(t,\nu)|^2 d\mu_k(t,\nu). \tag{5.10}$$

Using the hypothesis on h, inequality (5.10) reduces to

$$\|h\|^2_{L^2_k(\mathbb{R}^d)}\int_E\left|\mathscr{F}_D(f)(\xi)\right|^2 d\gamma_k(\xi)\le \mathfrak{C}(k,s)(\gamma_k(E))^{\frac{2s}{2\gamma+d}}\int_{\mathbb{R}^{2d}}\|t\|^{2s}\left|\mathscr{G}_h^D(f)(t,\nu)\right|^2 d\mu_k(t,\nu).$$

Or equivalently,

$$\int_{\mathbb{R}^{2d}}\|t\|^{2s}\left|\mathscr{G}_h^D(f)(t,\nu)\right|^2 d\mu_k(t,\nu)\ge \frac{\|h\|^2_{L^2_k(\mathbb{R}^d)}}{\mathfrak{C}(k,s)(\gamma_k(E))^{\frac{2s}{2\gamma+d}}}\int_E|\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi),\quad 0<s<\frac{2\gamma+d}{2}. \tag{5.11}$$

This completes the proof of Theorem 5.4. □

Let E be a subset of $\mathbb{R}^d$. We define the Paley-Wiener space $PW_k(E)$ as follow:

$$PW_k(E):=\left\{f\in L^2_k(\mathbb{R}^d):\ supp\,\mathscr{F}_D(f)\subset E\right\}.$$

Involving Plancherel's formula (2.14), definition of the Paley-Wiener space $PW_k(E)$ and the previous theorem we obtain the following:

Corollary 5.3. *Let E be a subset of* $\mathbb{R}^d$ *such that* $0<\gamma_k(E)<\infty$. *Let* $0<s<\frac{2\gamma+d}{2}$. *For any* $f\in PW_k(E)$, *we have*

$$\|f\|^2_{L^2_k(\mathbb{R}^d)}\le\frac{\mathfrak{C}(k,s)(\gamma_k(E))^{\frac{2s}{2\gamma+d}}}{\|h\|^2_{L^2_k(\mathbb{R}^d)}}\int_{\mathbb{R}^{2d}}\|y\|^{2s}\left|\mathscr{G}_h^D(f)(y,\nu)\right|^2 d\mu_k(y,\nu), \tag{5.12}$$

where $\mathfrak{C}(k,s)$ *the constant given in Proposition 5.1.*

By interchanging the roles of f and $\mathscr{F}_D(f)$ in Proposition 5.1, we get the following:

Corollary 5.4. *Let F be a subset of* $\mathbb{R}^d$ *such that* $0<\gamma_k(F)<\infty$.
For $0<t<\frac{2\gamma+d}{2}$ *and for any* $f\in L^2_k(\mathbb{R}^d)$, *we have*

$$\int_F\left|f(y)\right|^2 d\gamma_k(y)\le\mathfrak{C}(k,t)(\gamma_k(F))^{\frac{2t}{2\gamma+d}}\|\,\|\xi\|^t\mathscr{F}_D(f)\|^2_{L^2_k(\mathbb{R}^d)}, \tag{5.13}$$

where $\mathfrak{C}(k,t)$ *the constant given in Proposition 5.1.*

Involving Corollary 5.4 and using similar ideas given in the proof of Theorem 5.4, we prove the following.

Corollary 5.5. *Let F be a subset of* $\mathbb{R}^d$ *such that* $0<\gamma_k(F)<\infty$. *Let* $0<t<\frac{2\gamma+d}{2}$. *For any* $f\in L^2_k(\mathbb{R}^d)$, *we have*

$$\int_{\mathbb{R}^d}\int_F\left|\mathscr{G}_h^D(f)(y,\nu)\right|^2 d\mu_k(y,\nu)\le\mathfrak{C}(k,t)(\gamma_k(F))^{\frac{2t}{2\gamma+d}}\|h\|^2_{L^2_k(\mathbb{R}^d)}\int_{\mathbb{R}^d}\|\xi\|^{2t}\left|\mathscr{F}_D(f)(\xi)\right|^2 d\gamma_k(\xi), \tag{5.14}$$

where $\mathfrak{C}(k,t)$ *the constant given in Proposition 5.1.*

Let F be a subset of $\mathbb{R}^d$. We define the generalized Paley-Wiener space $GPW_k(F)$ as follow:

$$GPW_k(F):=\left\{f\in L^2_k(\mathbb{R}^d):\forall\,\nu\in\mathbb{R}^d,\ supp\,\mathscr{G}_h^D(f)(.,\nu)\subset F\right\}.$$

Applying Plancherel's formula (2.30), the definition of the generalized Paley-Wiener space $GPW_k(F)$ and the previous corollary we obtain the following:

Corollary 5.6. *Let E and F be two subsets of $\mathbb{R}^d$ such that $0 < \gamma_k(E), \gamma_k(F) < \infty$. Let $0 < s, t < \frac{2\gamma+d}{2}$.*

i*) For any $f \in GPW_k(F)$, we have*

$$\|f\|^2_{L^2_k(\mathbb{R}^d)} \leq \mathfrak{C}(k,t)(\gamma_k(F))^{\frac{2t}{2\gamma+d}} \int_{\mathbb{R}^d} \|\xi\|^{2t} \left|\mathscr{F}_D(f)(\xi)\right|^2 d\gamma_k(\xi). \tag{5.15}$$

ii*) For any $f \in PW_k(E) \bigcap GPW_k(F)$, we have*

$$\begin{aligned} \|f\|^{s+t}_{L^2_k(\mathbb{R}^d)} \leq & \ (\mathfrak{C}(k,t))^{\frac{s}{2}} (\mathfrak{C}(k,s))^{\frac{t}{2}} (\gamma_k(E)\gamma_k(F))^{\frac{2ts}{2\gamma+d}} \\ & \Big(\int_{\mathbb{R}^d} \|\xi\|^{2t} \left|\mathscr{F}_D(f)(\xi)\right|^2 d\gamma_k(\xi) \Big)^{\frac{s}{2}} \Big(\int_{\mathbb{R}^{2d}} \|y\|^{2s} \left|\mathscr{G}^D_h(f)(y,\nu)\right|^2 d\mu_k(y,\nu) \Big)^{\frac{t}{2}}, \end{aligned} \tag{5.16}$$

where $\mathfrak{C}(k,t)$ the constant given in Proposition 5.1.

We finish this Subsection by establishing another version of Heisenberg-type uncertainty inequality for the Dunkl Gabor transforms in arbitrary space dimensions.

Theorem 5.5. *Let $0 < p < \frac{2\gamma+d}{2}$ and $q > 0$. Then for any $f \in L^2_k(\mathbb{R}^d)$, we have*

$$\|f\|^2_{L^2_k(\mathbb{R}^d)} \leq \mathscr{C}(k,p,q) \Big\| \, \|y\|^p \, \mathscr{G}^D_h(f) \Big\|^{\frac{2q}{p+q}}_{L^2_{\mu_k}(\mathbb{R}^{2d})} \Big\| \, \|\xi\|^q \mathscr{F}_D(f) \Big\|^{\frac{2p}{p+q}}_{L^2_k(\mathbb{R}^d)}. \tag{5.17}$$

where

$$\mathscr{C}(k,p,q) = \Big(\frac{(d_k)^{\frac{2p}{2\gamma+d}} \mathfrak{C}(k,p)}{(2\gamma+d)^{\frac{2p}{2\gamma+d}} \|h\|^2_{L^2_k(\mathbb{R}^d)}} \Big)^{\frac{q}{p+q}} \big[(\tfrac{p}{q})^{\frac{q}{p+q}} + (\tfrac{q}{p})^{\frac{p}{p+q}} \big].$$

Proof. Let $0 < p < \frac{2\gamma+d}{2}$, $q > 0$ and $r > 0$. Then

$$\|f\|^2_{L^2_k(\mathbb{R}^d)} = \|\mathscr{F}_D(f)\|^2_{L^2_k(\mathbb{R}^d)} = \int_{B_d(0,r)} |\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi) + \int_{B^c_d(0,r)} |\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi), \tag{5.18}$$

where $B_d(0,r)$ denotes the ball of $\mathbb{R}^d$ of radius r.

From Theorem 5.4 and by simple calculation, we have

$$\int_{B_d(0,r)} |\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi) \leq \frac{(d_k)^{\frac{2p}{2\gamma+d}} \mathfrak{C}(k,p)}{(2\gamma+d)^{\frac{2p}{2\gamma+d}} \|h\|^2_{L^2_k(\mathbb{R}^d)}} r^{2p} \int_{\mathbb{R}^{2d}} \|y\|^{2p} \left|\mathscr{G}^D_h(f)(y,\nu)\right|^2 d\mu_k(y,\nu). \tag{5.19}$$

Moreover it is easy to see that

$$\int_{B^c_d(0,r)} |\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi) \leq r^{-2q} \int_{\mathbb{R}^d} \|\xi\|^{2q} |\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi). \tag{5.20}$$

Combining the relations (5.18), (5.19) and (5.20), we get

$$\|f\|^2_{L^2_k(\mathbb{R}^d)} \leq \frac{(d_k)^{\frac{2p}{2\gamma+d}} \mathfrak{C}(k,p)}{(2\gamma+d)^{\frac{2p}{2\gamma+d}} \|h\|^2_{L^2_k(\mathbb{R}^d)}} r^{2p} \Big\| \, \|y\|^p \left|\mathscr{G}^D_h(f)\right\|^2_{L^2_{\mu_k}(\mathbb{R}^{2d})} + r^{-2q} \| \, \|\xi\|^q \mathscr{F}_D(f)\|^2_{L^2_k(\mathbb{R}^d)}.$$

We choose

$$r = \Big[\frac{q(2\gamma+d)^{\frac{2p}{2\gamma+d}} \|h\|^2_{L^2_k(\mathbb{R}^d)}}{p(d_k)^{\frac{2p}{2\gamma+d}} \mathfrak{C}(k,p)} \Big]^{\frac{1}{2p+2q}} \Big(\frac{\| \, \|\xi\|^q \mathscr{F}_D(f)\|_{L^2_k(\mathbb{R}^d)}}{\Big\| \, \|y\|^p \mathscr{G}^D_h(f)\|_{L^2_{\mu_k}(\mathbb{R}^{2d})}} \Big)^{\frac{1}{p+q}},$$

we obtain the desired inequality. □

6 Dunkl logarithmic Sobolev inequalities and applications

This Section is devoted to establish new Dunkl logarithmic Sobolev inequalities. Next we use these inequalities to obtain Dunkl logarithm Sobolev type uncertainty inequalities for the Dunkl Gabor transform. To facilitate our intention, we start with the following definitions:

Definition 6.1. *(i) The Dunkl transform of a distribution u in* $\mathscr{S}'(\mathbb{R}^d)$ *is defined by*

$$\langle \mathscr{F}_D(u), \varphi\rangle = \langle u, \mathscr{F}_D^{-1}(\varphi)\rangle, \quad \text{for all } \varphi \in \mathscr{S}(\mathbb{R}^d). \tag{6.1}$$

(ii) Let u be in $\mathscr{S}'(\mathbb{R}^d)$*. We recall that*

$$\mathscr{F}_D(T_j u) = i\xi_j \mathscr{F}_D(u), \quad j = 1, ..., d. \tag{6.2}$$

Definition 6.2. *[23] Let* $s \in \mathbb{R}$*. The Dunkl Sobolev space* $H_k^s(\mathbb{R}^d)$ *of order s is defined by*

$$H_k^s(\mathbb{R}^d) = \Big\{ f \in \mathscr{S}'(\mathbb{R}^d) : (1 + \|\xi\|^2)^{\frac{s}{2}} \mathscr{F}_D(f) \in L_k^2(\mathbb{R}^d)\Big\}. \tag{6.3}$$

Remark 6.1. *Using Parseval's formula (2.14) and relation (6.2) we can see that*

$$H_k^1(\mathbb{R}^d) = \Big\{ f \in L_k^2(\mathbb{R}^d) : \nabla_k f \in L_k^2(\mathbb{R}^d)\Big\}, \tag{6.4}$$

where ∇_k *denotes the Dunkl nabla operator given by* $\nabla_k = \Big(T_1, ..., T_d\Big)$*. Fore more details on the Dunkl Sobolev spaces we refer the reader to [26].*

Definition 6.3. *For* $1 \leq p < \infty$ *and* $b > 0$*, the weighted Lebesgue space on* $\mathbb{R}^d$ *is defined by*

$$L_{k,b}^p(\mathbb{R}^d) = \Big\{ f \in L_k^p(\mathbb{R}^d) : \langle t\rangle^b f \in L_k^p(\mathbb{R}^d)\Big\}, \tag{6.5}$$

where $\langle t\rangle$ *is the weight function given by* $\langle t\rangle = (1 + \|t\|^2)^{1/2}$*,* $t \in \mathbb{R}^d$*.*

Now we recall the modified logarithmic Beckner's inequality [33].

Theorem 6.1. *For any* $f \in H_k^1(\mathbb{R}^d) \cap L_{k,1}^1(\mathbb{R}^d)$*,*

$$\frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})}\|f\|^2_{L_k^2(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d} |f(x)|^2 \log(C(k,d)\langle x\rangle^2) d\gamma_k(x) + \int_{\mathbb{R}^d} \log(K(k,d)\|\xi\|)\, |\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi), \tag{6.6}$$

where $K(k,d)$ *is a positive constant and*

$$C(k,d) := \Big(\frac{d_k \Gamma^2(\frac{d+2\gamma}{2})}{2\Gamma(2\gamma+d)}\Big)^{\frac{1}{2\gamma+d}}.$$

In the following we prove another version of uncertainty inequalities for the Dunkl Gabor transform:

Theorem 6.2. *Let* $h \in L_{k,rad}^2(\mathbb{R}^d) \bigcap L_k^\infty(\mathbb{R}^d)$*. For any arbitrary function* $f \in H_k^1(\mathbb{R}^d) \cap L_{k,1}^1(\mathbb{R}^d)$ *we have*

$$\int_{\mathbb{R}^{2d}} \Big|\mathscr{G}_h^D(f)(t,\nu)\Big|^2 \log\Big(C(k,d)(1 + \|t\|^2)\Big) d\mu_k(t,\nu) + \|h\|^2_{L_k^2(\mathbb{R}^d)} \int_{\mathbb{R}^d} |\mathscr{F}_D(f)(\xi)|^2 \log(K(k,d)\|\xi\|) d\gamma_k(\xi) \geq \frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})}\|h\|^2_{L_k^2(\mathbb{R}^d)}\|f\|^2_{L_k^2(\mathbb{R}^d)} \tag{6.7}$$

whenever the L.H.S of (6.7) is defined.

Proof. As a consequence of inequality (6.6), we have

$$\int_{\mathbb{R}^d}\left|\mathscr{G}_h^D(f)(t,\nu)\right|^2\log\Big(C(k,d)(1+\|t\|^2)\Big)d\gamma_k(t)+\int_{\mathbb{R}^d}\left|\mathscr{F}_D[\mathscr{G}_h^D(f)(.,\nu)](\xi)\right|^2\log(K(k,d)\|\xi\|)d\gamma_k(\xi)\geq \frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})}\int_{\mathbb{R}^d}\left|\mathscr{G}_h^D(f)(t,\nu)\right|^2d\gamma_k(t),\ \text{for all}\ \nu\in\mathbb{R}^d$$

which upon integration yields with the measure $d\gamma_k(\nu)$

$$\int_{\mathbb{R}^{2d}}\left|\mathscr{G}_h^D(f)(t,\nu)\right|^2\log\Big(C(k,d)(1+\|t\|^2)\Big)d\mu_k(t,\nu)+\int_{\mathbb{R}^{2d}}\log(K(k,d)\|\xi\|)\left|\mathscr{F}_D[\mathscr{G}_h^D(f)(.,\nu)](\xi)\right|^2d\mu_k(\xi,\nu)\geq \frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})}\int_{\mathbb{R}^{2d}}\left|\mathscr{G}_h^D(f)(t,\nu)\right|^2d\mu_k(t,\nu). \tag{6.8}$$

Using Lemma 3.1, relation (2.18) for the second integral on the L.H.S of (6.8) and invoking (2.30), we get

$$\int_{\mathbb{R}^{2d}}\left|\mathscr{G}_h^D(f)(t,\nu)\right|^2\log\Big(C(k,d)(1+\|t\|^2)\Big)d\mu_k(t,\nu)+\|h\|^2_{L^2_k(\mathbb{R}^d)}\int_{\mathbb{R}^d}|\mathscr{F}_D(f)(\xi)|^2\log(K(k,d)\|\xi\|)d\gamma_k(\xi)\geq \frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})}\|h\|^2_{L^2_k(\mathbb{R}^d)}\|f\|^2_{L^2_k(\mathbb{R}^d)}. \tag{6.9}$$

This completes the proof of Theorem 6.2. □

Based on the Dunkl logarithm Sobolev type uncertainty inequality (6.7), we shall derive another uncertainty principle for the Dunkl Gabor transform in arbitrary space dimensions.

Theorem 6.3. *Let $h\in L^2_{k,rad}(\mathbb{R}^d)\bigcap L^\infty_k(\mathbb{R}^d)$ such that $\|h\|^2_{L^2_k(\mathbb{R}^d)}=1$. Then, for any arbitrary function $f\in H^1_k(\mathbb{R}^d)\cap L^1_{k,1}(\mathbb{R}^d)\setminus\{0\}$, we have*

$$\int_{\mathbb{R}^{2d}}\|t\|^2\left|\mathscr{G}_h^D(f)(t,\nu)\right|^2d\mu_k(t,\nu)\geq \frac{\exp\Big(\frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})}\Big)}{C(k,d)K(k,d)\|\nabla_k f\|_{L^2_k(\mathbb{R}^d)}}\|f\|^3_{L^2_k(\mathbb{R}^d)}-\|f\|^2_{L^2_k(\mathbb{R}^d)}. \tag{6.10}$$

Proof. Let f be in $H^1_k(\mathbb{R}^d)\cap L^1_{k,1}(\mathbb{R}^d)\setminus\{0\}$. For $\|h\|^2_{L^2_k(\mathbb{R}^d)}=1$, we infer from (6.7) that

$$\frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})}\|f\|^2_{L^2_k(\mathbb{R}^d)}\leq\int_{\mathbb{R}^{2d}}\left|\mathscr{G}_h^D(f)(t,\nu)\right|^2\log(C(k,d)(1+\|t\|^2))d\mu_k(t,\nu)+\int_{\mathbb{R}^d}\log(K(k,d)\|\xi\|)\,|\mathscr{F}_D(f)(\xi)|^2d\gamma_k(\xi). \tag{6.11}$$

Using Jensen's inequality in (6.11), we can deduce that

$$\frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})}\leq\log\ C(k,d)\Big(\int_{\mathbb{R}^{2d}}\frac{\left|\mathscr{G}_h^D(f)(t,\nu)\right|^2}{\|f\|^2_{L^2_k(\mathbb{R}^d)}}(1+\|t\|^2)d\mu_k(t,\nu)\Big)+\frac{1}{2}\int_{\mathbb{R}^d}\log(K^2(k,d)\|\xi\|^2)\frac{|\mathscr{F}_D(f)(\xi)|^2}{\|f\|^2_{L^2_k(\mathbb{R}^d)}}d\gamma_k(\xi). \tag{6.12}$$

To obtain a fruitful estimate of the second integral of (6.12), we set

$$d\varrho_k(\xi)=\frac{|\mathscr{F}_D(f)(\xi)|^2}{\|f\|^2_{L^2_k(\mathbb{R}^d)}}d\gamma_k(\xi),\quad\text{so that}\quad\int_{\mathbb{R}^d}d\varrho_k(\xi)=1. \tag{6.13}$$

Again by employing the Jensen's inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d}\log(K^2(k,d)\|\xi\|^2)|\mathscr{F}_D(f)(\xi)|^2d\gamma_k(\xi) &= \|f\|^2_{L^2_k(\mathbb{R}^d)}\int_{\mathbb{R}^d}\log(K^2(k,d)\|\xi\|^2)|d\rho_k(\xi)\\ &\le \|f\|^2_{L^2_k(\mathbb{R}^d)}\log\Big\{K^2(k,d)\int_{\mathbb{R}^d}\|\xi\|^2|d\rho_k(\xi)\Big\}\\ &\le \|f\|^2_{L^2_k(\mathbb{R}^d)}\log\Big\{\frac{K^2(k,d)}{\|f\|^2_{L^2_k(\mathbb{R}^d)}}\int_{\mathbb{R}^d}\|\xi\|^2\|\mathscr{F}_D(f)(\xi)|^2d\gamma_k(\xi)\Big\}\\ &\le \|f\|^2_{L^2_k(\mathbb{R}^d)}\log\Big\{\frac{K^2(k,d)}{\|f\|^2_{L^2_k(\mathbb{R}^d)}}\int_{\mathbb{R}^d}|\nabla_k f(t)|^2d\gamma_k(t)\Big\}. \quad (6.14)\end{aligned}$$

Using the expression (6.14) in (6.12), we infer

$$\frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})} \le \log\Big(\frac{C(k,d)K(k,d)}{\|f\|^3_{L^2_k(\mathbb{R}^d)}}\Big\{\int_{\mathbb{R}^{2d}}\big|\mathscr{G}_h^D(f)(t,\nu)\big|^2(1+\|t\|^2)d\mu_k(t,\nu)\Big\}\|\nabla_k f\|_{L^2_k(\mathbb{R}^d)}\Big). \quad (6.15)$$

Expression (6.15) can be rewritten in a lucid manner as

$$\Big\{\int_{\mathbb{R}^{2d}}\big|\mathscr{G}_h^D(f)(t,\nu)\big|^2\big(1+\|t\|^2\big)d\mu_k(t,\nu)\Big\}\Big\{\int_{\mathbb{R}^d}|\nabla_k f(t)|^2d\gamma_k(t)\Big\}^{1/2} \ge \frac{\exp\Big(\frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})}\Big)}{C(k,d)K(k,d)}\|f\|^3_{L^2_k(\mathbb{R}^d)}. \quad (6.16)$$

Applying Plancherel's formula (2.30) with $\|h\|^2_{L^2_k(\mathbb{R}^d)} = 1$, we get

$$\Big\{\int_{\mathbb{R}^{2d}}\|t\|^2\big|\mathscr{G}_h^D(f)(t,\nu)\big|^2d\mu_k(t,\nu)\Big\}\Big\{\int_{\mathbb{R}^d}|\nabla_k f(t)|^2d\gamma_k(t)\Big\}^{1/2} \ge \frac{\exp\Big(\frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})}\Big)}{C(k,d)K(k,d)}\|f\|^3_{L^2_k(\mathbb{R}^d)} -\|f\|^2_{L^2_k(\mathbb{R}^d)}\|\nabla_k f\|_{L^2_k(\mathbb{R}^d)},$$

which upon simplification gives the desired inequality

$$\int_{\mathbb{R}^{2d}}\|t\|^2\big|\mathscr{G}_h^D(f)(t,\nu)\big|^2d\mu_k(t,\nu) \ge \frac{\exp\Big(\frac{\Gamma'(\frac{2\gamma+d}{2})}{\Gamma(\frac{2\gamma+d}{2})}\Big)}{C(k,d)K(k,d)\|\nabla_k f\|_{L^2_k(\mathbb{R}^d)}}\|f\|^3_{L^2_k(\mathbb{R}^d)} - \|f\|^2_{L^2_k(\mathbb{R}^d)}.$$

This completes the proof of the theorem. □

Remark 6.2. *1) Let h be in* $L^2_{k,rad}(\mathbb{R}^d)$. *We proceed as in [7], we define the modulation of h by ν otherwise, as follow:*

$$\mathscr{M}_\nu(h) := \mathscr{F}_D\big(\sqrt{\tau_\nu(|\mathscr{F}_D(h)|^2)}\big). \quad (6.17)$$

Subsequently, we define the generalized Gabor transform $\mathscr{V}_h^D$ *as follow:*

$$\forall\,(y,\nu)\in\mathbb{R}^{2d},\ \mathscr{V}_h^D(f)(y,\nu) := \frac{1}{c_k}\int_{\mathbb{R}^d} f(x)\tau_{-y}(\overline{\check{\mathscr{M}_\nu(h)}})(y)d\gamma_k(x) = f *_D \overline{\check{\mathscr{M}_\nu(h)}}(y). \quad (6.18)$$

It is clear that

$$\mathscr{V}_h^D = \mathscr{G}^D_{\mathscr{F}_D(h)}. \quad (6.19)$$

Thus, by involving Plancherel's formula (2.14)*, we derive that the two integral transforms are equivalent and then all results proved for one are valuables for the second. So, I reclame that* *all results proved in [29, 30] and in this paper for the Dunkl Gabor transform $\mathscr{G}_h^D$ are valuables for the integral transform $\mathscr{V}_h^D$ and it is suffice to replace h by $\mathscr{F}_D(h)$ to derive the analogues results. Finally, I note and I insist that any adaptation of results proved for the Dunkl Gabor transform $\mathscr{G}_h^D$ in the context of the transformation $\mathscr{V}_h^D$ is a plagiarism (in particular results proved in [29, 30] and in the current paper), since I mentioned that the two transformations coincide modulo the formulas (6.19) and* (2.14).

2) We note that we have studied these types of uncertainty principles and others for some integral transforms as the k-Hankel Gabor transform, the (k, a)-generalized wavelet transform, the generalized Stockwell transforms, the deformed Hankel Gabor transform, the q-Dunkl wavelet transform, the q-Bessel Gabor transform, and others integral transforms. These studies have given some papers. We cite as examples [31, 32, 33, 34, 35, 36, 37].

3) When $W = \mathbb{Z}_2^d$ all results of this paper for the Dunkl Gabor transform $\mathscr{G}_h^D$ are true without the assumption that the function h is radial. It suffices to choose a function h such that $\tau_\nu(|h|^2) \geq 0$.

7 Open Problem

In the present paper, we have successfully studied new quantitative uncertainty principles associated with the Dunkl Gabor transforms. The obtained results have a novelty and contribution to the literature. It is our hope that this work motivate the researchers to study the qualitative uncertainty principles as Hardy's, Morgan's, Beurling's and Miyachi's uncertainty principles associated with the Dunkl Gabor transform.

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