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# Deformed Gabor transform and applications

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**Abstract**

***In this paper, we investigate the deformed Gabor transform for some problems of time-frequency analysis and reproducing kernel theory. Firstly, we present for this transform the main theorems of harmonic analysis as Plancherel's, Lieb's and inversion formulas. Next, we formulate some quantitative uncertainty principles including the Heisenberg uncertainty principles, Benedick-Amrein-Berthier's uncertainty principle, local uncertainty principles and Shapiro's uncertainty principle. In sequel, we derive for the deformed Gabor transform some applications of the Tikhonov regularization on the generalized Sobolev spaces.***

**Keywords:** *Generalized translation operator. Deformed Hankel transform. Deformed Gabor transform. Heisenberg's uncertainty principle. Shapiro's uncertainty principle. Generalized Sobolev spaces. Extremal functions. Tikhonov regularization. Time-frequency concentration. Reproducing kernel.*

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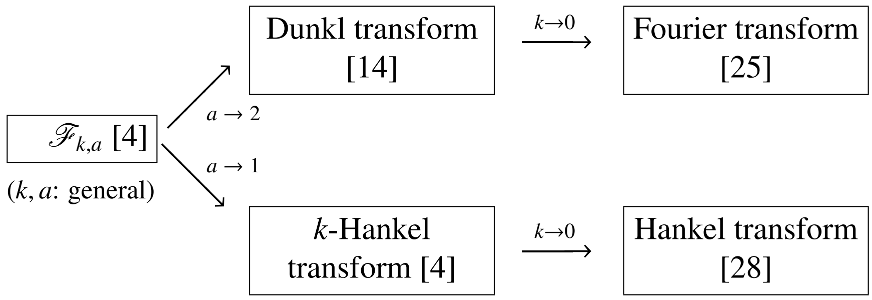
# 1 Introduction

Recently, Ben Saïd and all in [4], have given a foundation of the deformation theory of the classical situation, by constructing a generalization $\mathscr{F}_k$ of the Fourier transform, and the holomorphic semigroup $\mathscr{I}_{k,a}(z)$ with infinitesimal generator

$$\mathscr{L}_{k,a} := \|x\|^{2-a}\triangle_k - \|x\|^a, \quad a > 0,$$

acting on a concrete Hilbert space deforming $L^2(\mathbb{R}^d)$. Here $\triangle_k$ is the Dunkl Laplacian See [13]. The deformation parameters consists of a real parameter $a > 0$ coming from the interpolation of the minimal unitary representations of two different reductive groups and a parameter $k$ coming from Dunkl's theory of differential difference operators [13].

As it turned out, various known integral transforms are covered by $\mathscr{F}_{k,a}$ :




As of now, the $(k, a)$-generalized Fourier transform $\mathscr{F}_{k,a}$ has witnessed an ample amount of research in the realm of harmonic analysis, which include study of the kernel of the $(k, a)$-generalized Fourier transform [9], Pitt's inequalities [21], uncertainty principles [21, 26], the $(k, a)$-generalized wavelet multipliers [35, 36], the $(k, a)$-generalized wavelet transform [37, 38, 40], and many more.

One of the aims of the Fourier transform, is the study of the time-frequency analysis. In the sixties the time-frequency analysis has emerged with the works of Gabor [19] who provided an interesting way to study the local frequency spectrum of signals by introducing many time-frequency representations, as, for instance, the short-time Fourier transform, the continuous wavelet transform or also the Wigner distribution where all of these representations have a same common point, that is the simultaneous representation of the space and the frequency variables in a same set called the time-frequency plane.

The Gabor transform has been successfully used to analyse signals in numerous applications, such as seismic recordings, ground vibrations, geophysics, medical imaging, hydrology, gravitational waves, power system analysis and many other areas.

In this paper, we consider the case $a = \frac{2}{n}$, $n \in \mathbb{N}$ and $d = 1$. We shall call the generalized Fourier transform $\mathscr{F}_{k,\frac{2}{n}}$ the deformed *Hankel transform* and we will denote it (simply) by $\mathscr{F}_{k,n}$.

We recall that in [44] we have studied the generalized translation operator on the deformed Hankel setting. In particular, we have proved its positivity on suitable space of functions. Profiting of this positivity in [45] we have introduced the generalized Gabor transform in the setting of the deformed Hankel transform and we have studied its harmonic analysis. In the same paper we have derived some quantitative uncertainty principles for this transform as the Heisenberg's uncertainty principles, the Beckner uncertainty principle, the Pitt uncertainty inequalities, the Benedick-Amrein-Berthier uncertainty principle and the local-type uncertainty principles.

The purpose of this document is threefold:

- On one hand, we want to prove a new inversion formula for the deformed Gabor transform.

- The second aim of this paper is to derive some novels versions of quantitative uncertainty principles for this transform. It is worth mentioning that quantitative uncertainty principles have a long and rich history; we refer the reader to the survey [18], the book [24] and the references [5, 21, 26, 27, 56] for numerous versions of uncertainty principles for the Fourier transform in different settings. To date, several generalizations, modifications and variations of the harmonic based uncertainty principles have appeared in the open literature, for instance, Benedick's uncertainty principle, Amrein and Berthier's uncertainty principles, Slepian and Pollak's uncertainty principles, Donoho and Stark's uncertainty principles and much more [1, 12, 53]. In the classical setting, Wilczok in [57] is the first who introduced and studied the notion of the quantitative uncertainty principles for the Gabor transform. Later on, similar results appeared for several extended Gabor transforms in different setups (see, e.g., [2, 3, 16, 17, 30, 31, 32, 39, 41]).

- Keeping in view the fact that the reproducing kernel theory for the deformed Gabor transforms is yet to be investigated exclusively, our third endeavour is to investigate some problems of the reproducing kernel theory associated with this transform. We note that in [31], we have the first who introduced and studied the notion of the reproducing kernel theory for the Gabor transform in the quantum theory setting. Later on, similar results appeared for several extended Gabor transforms in different setups (see, e.g., [32, 39]).

The remainder of this paper is arranged as follows.

In §2, we recall the main results about the deformed Hankel transform and the generalized translation operators. In §3, we investigate the harmonic analysis associated with the deformed Gabor transform. More precisely Plancherel's, Lieb's and inversion formulas are established. §4 is devoted to study the Shapiro's uncertainty principle for the deformed Gabor transform. Next, in §5 we establish the Heisenberg, Benedicks and Donoho-Stark's type uncertainty principles for the deformed Gabor transform. Finally, the last section is devoted to introducing the generalized Sobolev spaces $W^s_{k,n}(\mathbb{R})$ associated with the deformed Gabor transform. Afterwards, we give some applications of the general theory of reproducing kernels to the Tikhonov regularization, which gives the best approximation of the deformed Gabor transform on these generalized Sobolev spaces.

# 2 Preliminaries

This section gives an introduction to the harmonic analysis associated with the deformed Hankel transform. Main references are [4, 6, 44].

## 2.1 Deformed Hankel transform

**Notation**. Let us denote by

For $p \in [1, \infty]$, $p'$ denotes as in all that follows, the conjugate exponent of $p$.

$M_{k,n} := \frac{n^{\frac{n(2k-1)}{2}}}{2^{\frac{n(2k-1)+2}{2}}\Gamma(\frac{n(2k-1)+2}{2})}$, $n \in \mathbb{N}$.

$d\gamma_{k,n}(x) := M_{k,n}|x|^{\frac{(2k-2)n+2}{n}}dx$, $k \geq \frac{n-1}{n}$.

$L^p_{k,n}(\mathbb{R})$, $1 \leq p \leq \infty$, the space of measurable functions $f$ on $\mathbb{R}$ such that

$$\begin{aligned} \|f\|_{L^p_{k,n}(\mathbb{R})} &:= \Big(\int_{\mathbb{R}} |f(x)|^p d\gamma_{k,n}(x)\Big)^{\frac{1}{p}} < \infty, \quad \text{if} \quad 1 \leq p < \infty, \\ \|f\|_{L^\infty_{k,n}(\mathbb{R})} &:= \text{ess} \sup_{x\in\mathbb{R}} |f(x)| < \infty. \end{aligned}$$

For $p = 2$, we provide this space with the scalar product

$$\langle f, g\rangle_{L^2_{k,n}(\mathbb{R})} := \int_{\mathbb{R}} f(x)\overline{g(x)}d\gamma_{k,n}(x).$$

For $k \geq \frac{n-1}{n}$, and $f \in L^1_{k,n}(\mathbb{R})$, the deformed Hankel transform is defined by

$$\mathscr{F}_{k,n}(f)(\lambda) = \int_{\mathbb{R}} f(x)B_{k,n}(\lambda, x)d\gamma_{k,n}(x), \quad \text{for all } \lambda \in \mathbb{R}, \tag{2.1}$$

where $B_{k,n}(\lambda, x)$ is the deformed Hankel kernel given by

$$B_{k,n}(\lambda, x) = j_{nk-\frac{n}{2}}(n|\lambda x|^{\frac{1}{n}}) + (-i)^n(\frac{n}{2})^n \frac{\Gamma(nk - \frac{n}{2} + 1)}{\Gamma(nk + \frac{n}{2} + 1)} \lambda x j_{nk+\frac{n}{2}}(n|\lambda x|^{\frac{1}{n}}). \tag{2.2}$$

Here

$$j_\alpha(u) := \Gamma(\alpha + 1)\Big(\frac{u}{2}\Big)^{-\alpha} J_\alpha(u) = \Gamma(\alpha + 1) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(\alpha + m + 1)}\Big(\frac{u}{2}\Big)^{2m} \tag{2.3}$$

denotes the normalized Bessel function of index $\alpha$.

Next, we give some properties of the deformed Hankel kernel.

**Proposition 2.1.** i) *For $z, t \in \mathbb{R}$, we have*

$$B_{k,n}(z, t) = B_{k,n}(t, z), \quad B_{k,n}(z, 0) = 1, \quad \overline{B_{k,n}(z, t)} = B_{k,n}((-1)^n z, t)$$

*and $B_{k,n}(\lambda z, t) = B_{k,n}(z, \lambda t)$ for all $\lambda \in \mathbb{R}$.*

ii) *There exists a finite positive constant $C$ only depends on $n$ and $k$, such that for all $x, y \in \mathbb{R}$ we have*

$$|B_{k,n}(x, y)| \leq C.$$

**Convention:**([26]). We shall replace $B_{k,n}$ by the rescaled version $B_{k,n}/C$ but continue to use the same symbol $B_{k,n}$ and we obtain

$$\forall\, x, y \in \mathbb{R}, \quad |B_{k,n}(x, y)| \leq 1. \tag{2.4}$$

**Remark 2.1.** *(i) We note that the previous inequality implies that the deformed Hankel transform is bounded on the space $L^1_{k,n}(\mathbb{R})$, and we have*

$$\|\mathscr{F}_{k,n}(f)\|_{L^\infty_{k,n}(\mathbb{R})} \leq \|f\|_{L^1_{k,n}(\mathbb{R})}, \tag{2.5}$$

*for all $f$ in $L^1_{k,n}(\mathbb{R})$.*

*(ii) The deformed Hankel transform $\mathscr{F}_{k,n}$ provides a natural generalization of the Hankel transform. Indeed, if we set*

$$\begin{aligned} B^{even}_{k,n}(x, y) &= \tfrac{1}{2}(B_{k,n}(x, y) + B_{k,n}(x, -y)) \\ &= j_{nk-\frac{n}{2}}(n|xy|^{\frac{1}{n}}). \end{aligned}$$

*Then, the deformed Hankel transform $\mathscr{F}_{k,n}$ of an even function $f$ on the real line specializes to a Hankel type transform on $\mathbb{R}_+$. In fact, when $f(x) = F(|x|)$ is an even function on $\mathbb{R}$ and belongs to $L^1_{k,n}(\mathbb{R})$, we have*

$$\forall \xi \in \mathbb{R},\ \mathscr{F}_{k,n}(f)(\xi) = \frac{(\frac{n}{2})^{(\frac{2nk-n}{2})}}{\Gamma(\frac{2nk+2-n}{2})} \int_0^\infty F(r) j_{\frac{2nk-n}{2}}\Big(n(r|\xi|)^{\frac{1}{n}}\Big) r^{\frac{2}{n}(\frac{2nk+2-n}{2})-1} dr. \tag{2.6}$$

The authors in [4] have proved the following.

**Proposition 2.2.** i*) Plancherel's theorem for* $\mathscr{F}_{k,n}$.
*The deformed Hankel transform* $f \mapsto \mathscr{F}_{k,n}(f)$ *is an isometric isomorphism on* $L^2_{k,n}(\mathbb{R})$ *and we have*

$$\int_{\mathbb{R}} |f(x)|^2 d\gamma_{k,n}(x) = \int_{\mathbb{R}} |\mathscr{F}_{k,n}(f)(\lambda)|^2 d\gamma_{k,n}(\lambda). \tag{2.7}$$

ii*) Parseval's formula for* $\mathscr{F}_{k,n}$.
*For all* $f, g$ *in* $L^2_{k,n}(\mathbb{R})$ *we have*

$$\int_{\mathbb{R}} f(x)\overline{g(x)} d\gamma_{k,n}(x) = \int_{\mathbb{R}} \mathscr{F}_{k,n}(f)(\lambda)\overline{\mathscr{F}_{k,n}(g)(\lambda)} d\gamma_{k,n}(\lambda). \tag{2.8}$$

iii*) Inversion formula.*
*The deformed Hankel transform is an involutive unitary operator on* $L^1_{k,n}(\mathbb{R})$*, i.e., we have*

$$\mathscr{F}^{-1}_{k,n}(f)(x) = \mathscr{F}_{k,n}(f)((-1)^n x) \quad x \in \mathbb{R}. \tag{2.9}$$

**Proposition 2.3.** *Let* $f$ *be in* $L^p_{k,n}(\mathbb{R})$, $p \in [1,2]$. *Then* $\mathscr{F}_{k,n}(f)$ *belongs to* $L^{p'}_{k,n}(\mathbb{R})$ *and we have*

$$\left\|\mathscr{F}_{k,n}(f)\right\|_{L^{p'}_{k,n}(\mathbb{R})} \leqslant \|f\|_{L^p_{k,n}(\mathbb{R})}.$$

**Definition 2.1.** *Let* $U, V$ *be two measurable subsets of* $\mathbb{R}$*. Then:*

*(1) We say that the pair* $(U, V)$ *is weakly annihilating, if* $supp\, f \subset U$ *and* $supp\, \mathscr{F}_{k,n}(f) \subset V$ *implies* $f = 0$.

*(2) We say that the pair* $(U, V)$ *is strongly annihilating, if there exists a positive constant* $C := C_{k,n}(U, V)$ *such that for every function* $f$ *in* $L^2_{k,n}(\mathbb{R})$,

$$C(\|\mathscr{F}_{k,n}(f)\|^2_{L^2_{k,n}(V^c)} + \|f\|^2_{L^2_{k,n}(U^c)}) \geq \|f\|^2_{L^2_{k,n}(\mathbb{R})}. \tag{2.10}$$

*Here* $A^c := \mathbb{R}\backslash A$ *is the complement of* $A$*. The constant* $C_{k,n}(U, V)$ *will be called the annihilation constant of* $(U, V)$.

Now, we recall the following Benedicks-type uncertainty principle for the deformed Hankel transform proved by Johansen in [[26], Theorem 9.1].

**Proposition 2.4.** *Let* $U, V$ *be two measurable subsets of* $\mathbb{R}$ *with*

$$\gamma_{k,n}(U) := \int_U d\gamma_{k,n}(x) < \infty \quad \text{and} \quad \gamma_{k,n}(V) := \int_V d\gamma_{k,n}(x) < \infty.$$

*Then the pair* $(U, V)$ *is a strongly annihilating pair.*

## 2.2 Generalized translation operator

**Definition 2.2.** *([37]) Let* $x \in \mathbb{R}$*. We define the generalized translation operator* $f \mapsto \tau^{k,n}_x f$ *on* $L^2_{k,n}(\mathbb{R})$ *by*

$$\mathscr{F}_{k,n}(\tau^{k,n}_x f) = \overline{B_{k,n}(.,x)}\mathscr{F}_{k,n}(f). \tag{2.11}$$

It is useful to have a class of functions in which (2.11) holds pointwise. One such class is given by the generalized Wigner space $\mathscr{W}_{k,n}(\mathbb{R})$ given by

$$\mathscr{W}_{k,n}(\mathbb{R}) := \Big\{f \in L^1_{k,n}(\mathbb{R}) : \mathscr{F}_{k,n}(f) \in L^1_{k,n}(\mathbb{R})\Big\}.$$

On the follow we recall several properties of the generalized translation operator.

**Proposition 2.5.** *([37, 44]) (i) Let $f$ be in $L^2_{k,n}(\mathbb{R})$, we have*

$$\|\tau^{k,n}_x f\|_{L^2_{k,n}(\mathbb{R})} \leq \|f\|_{L^2_{k,n}(\mathbb{R})}, \quad \forall x \in \mathbb{R}. \tag{2.12}$$

*(ii) For all $f$ in $\mathscr{W}_{k,n}(\mathbb{R})$ we have*

$$\tau^{k,n}_x f(y) = \int_{\mathbb{R}} B_{k,n}((-1)^n x, \xi) B_{k,n}((-1)^n y, \xi)\mathscr{F}_{k,n}(f)(\xi) d\gamma_{k,n}(\xi), \quad \forall x, y \in \mathbb{R}. \tag{2.13}$$

*(iii) For all $f$ in $L^2_{k,n}(\mathbb{R})$ and for all $x, y \in \mathbb{R}$, we have*

$$\tau^{k,n}_x f(y) = \tau^{k,n}_y (f)(x). \tag{2.14}$$

*(iv) For all $f$ in $\mathscr{W}_{k,n}(\mathbb{R})$ and $g \in L^1_{k,n}(\mathbb{R}) \cap L^\infty_{k,n}(\mathbb{R})$, we have*

$$\forall\, x \in \mathbb{R}, \quad \int_{\mathbb{R}} \tau^{k,n}_x f(y) g(y) d\gamma_{k,n}(y) = \int_{\mathbb{R}} f(y) \tau^{k,n}_{(-1)^n x} g(y) d\gamma_{k,n}(y). \tag{2.15}$$

Recently, the authors in [6] have proved for the generalized translation operators the following results.

**Theorem 2.1.** *(i) Let $x \in \mathbb{R}$ and let $f \in L^p_{k,n}(\mathbb{R})$, $1 \leq p \leq \infty$. For $k \geq \frac{n-1}{n}$, the generalized translation operator $\tau^{k,n}_x$ is given by*

$$\tau^{k,n}_x f(y) = \int_{\mathbb{R}} f(z) d\zeta^{k,n}_{x,y}(z), \tag{2.16}$$

*here*

$$d\zeta^{k,n}_{x,y}(z) = \begin{cases} \mathscr{K}_{k,n}(x, y, z) d\gamma_{k,n}(z), & \text{if} \quad xy \neq 0, \\ d\delta_x(z), & \text{if} \quad y = 0, \\ d\delta_y(z), & \text{if} \quad x = 0, \end{cases}$$

*where $\mathscr{K}_{k,n}(x, y, .)$ is given explicitly in [6], and is supported on the set*

$$\Big\{z \in \mathbb{R} : \quad \big|\,|x|^{\frac{1}{n}} - |y|^{\frac{1}{n}}\,\big| < |z|^{\frac{1}{n}} < |x|^{\frac{1}{n}} + |y|^{\frac{1}{n}}\Big\}.$$

*(ii) For all $f \in L^p_{k,n}(\mathbb{R})$, $1 \leq p \leq \infty$, we have*

$$\forall\, x \in \mathbb{R}, \quad \|\tau^{k,n}_x f\|_{L^p_{k,n}(\mathbb{R})} \leq 4\|f\|_{L^p_{k,n}(\mathbb{R})}. \tag{2.17}$$

On the follows we recall the "trigonometric" form of the generalized translation operator.

**Theorem 2.2.** *([44]) For $f \in L^1_{k,n}(\mathbb{R})$ write $f = f_e + f_o$ as a sum of even and odd functions. Then*

$$\begin{aligned}\tau_x^{k,n} f(y) &= \tfrac{M_{k,n}}{2n}\Big[\int_0^{\pi} f_e(\langle\!\langle x,y\rangle\!\rangle_{\varphi,n})\Big\{1+(-1)^n\frac{n!\mathrm{sgn}(xy)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}(\cos\varphi)\Big\}(\sin\varphi)^{2nk-n}d\varphi\\ &+ \int_0^{\pi} f_o(\langle\!\langle x,y\rangle\!\rangle_{\varphi,n})\Big\{\frac{n!\mathrm{sgn}(x)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}\Big(\frac{|x|^{\frac{1}{n}}-|y|^{\frac{1}{n}}\cos\varphi}{\langle\!\langle x,y\rangle\!\rangle_{\varphi,n}^{\frac{1}{n}}}\Big)\\ &+ \tfrac{n!\mathrm{sgn}(y)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}\Big(\tfrac{|y|^{\frac{1}{n}}-|x|^{\frac{1}{n}}\cos\varphi}{\langle\!\langle x,y\rangle\!\rangle_{\varphi,n}^{\frac{1}{n}}}\Big)\Big\}(\sin\varphi)^{2nk-n}d\varphi\Big],\end{aligned}$$

*where $C_n^{nk-\frac{n}{2}}$ is the Gegenbauer polynomials and*

$$\langle\!\langle x,y\rangle\!\rangle_{\varphi,n} := (|x|^{\frac{2}{n}} + |y|^{\frac{2}{n}} - 2|xy|^{\frac{1}{n}}\cos\varphi)^{\frac{n}{2}}. \tag{2.18}$$

Below we will recall the positivity of the generalized translation operator on even functions in $L^1_{k,n}(\mathbb{R})$, which is far from being obvious. This result will be crucial for the rest of the paper. To do so, we will give an explicit expression of the translation operator acting on such functions. Let $L^p_{k,n,e}(\mathbb{R})$ be the space of even functions in $L^p_{k,n}(\mathbb{R})$.

**Corollary 2.1.** *([44]) For all $f$ in $L^p_{k,n,e}(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, we have*

$$\tau_x^{k,n} f(y) = \frac{M_{k,n}}{2n}\int_0^{\pi} f(\langle\!\langle x,y\rangle\!\rangle_{\varphi,n})\Big\{1+(-1)^n\frac{n!\mathrm{sgn}(xy)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}(\cos\varphi)\Big\}(\sin\varphi)^{2nk-n}d\varphi.$$

Now, let us go back to the properties of the generalized translation operator.

**Proposition 2.6.** *([44]) Let $f$ be an nonnegative even function of $\mathscr{W}_{k,n}(\mathbb{R})$. Then*
*(i) For any $x \in \mathbb{R}$, we have $\tau_x^{k,n} f \geq 0$.*
*(ii) For every $x \in \mathbb{R}$, we have $\tau_x^{k,n} f \in L^1_{k,n}(\mathbb{R})$ and*

$$\int_{\mathbb{R}} \tau_x^{k,n} f(y)d\gamma_{k,n}(y) = \int_{\mathbb{R}} f(y)d\gamma_{k,n}(y). \tag{2.19}$$

We close the notion of the generalized translation operators by giving the following results.

**Theorem 2.3.** *([44]) (i) For all nonnegative $f$ in $L^1_{k,n,e}(\mathbb{R})$, we have*

$$\forall\, x \in \mathbb{R}, \quad \tau_x^{k,n} f \geq 0, \quad \tau_x^{k,n} f \in L^1_{k,n}(\mathbb{R})$$

*and*

$$\int_{\mathbb{R}} \tau_x^{k,n} f(y)d\gamma_{k,n}(y) = \int_{\mathbb{R}} f(y)d\gamma_{k,n}(y). \tag{2.20}$$

*(ii) For all $f$ in $L^p_{k,n,e}(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, we have*

$$\forall\, x \in \mathbb{R}, \quad \|\tau_x^{k,n} f\|_{L^p_{k,n}(\mathbb{R})} \leqslant \|f\|_{L^p_{k,n}(\mathbb{R})}. \tag{2.21}$$

By means of the generalized translation operator, we define the generalized convolution product of two suitable functions $f$ and $g$ by

$$f *_{k,n} g(x) = \int_{\mathbb{R}} \tau_x^{k,n} f((-1)^n y)g(y)d\gamma_{k,n}(y). \tag{2.22}$$

Now, let us go back to the properties of the generalized convolution product.

**Proposition 2.7.** *([6]) The following statements hold true.*

(i) *The generalized convolution product is both commutative and associative.*

(ii) *Let $f \in L^2_{k,n}(\mathbb{R})$ and $g \in L^1_{k,n}(\mathbb{R})$. Then the function $f *_{k,n} g$ defined almost everywhere on $\mathbb{R}$ by*

$$f *_{k,n} g(x) = \int_{\mathbb{R}} \tau_x^{k,n} f((-1)^n y) g(y) d\gamma_{k,n}(y)$$

*belongs to $L^2_{k,n}(\mathbb{R})$.*

(iii) *Assume that $1 \leqslant p, q, r \leqslant \infty$ satisfy $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$. Then, for every $f \in L^p_{k,n}(\mathbb{R})$ and $g \in L^q_{k,n}(\mathbb{R})$, the convolution product $f *_{k,n} g$ belongs to $L^r_{k,n}(\mathbb{R})$ and*

$$\|f *_{k,n} g\|_{L^r_{k,n}(\mathbb{R})} \leqslant 4\|f\|_{L^p_{k,n}(\mathbb{R})}\|g\|_{L^q_{k,n}(\mathbb{R})}. \tag{2.23}$$

(iv) *For $f \in L^2_{k,n}(\mathbb{R})$ and $g \in L^1_{k,n}(\mathbb{R})$, we have*

$$\mathscr{F}_{k,n}(f *_{k,n} g) = \mathscr{F}_{k,n}(f)\mathscr{F}_{k,n}(g). \tag{2.24}$$

Involving Theorem 2.3 we improve the estimate given in Proposition 2.7 iii). More precisely, we have:

**Corollary 2.2.** *Assume that $1 \leqslant p, q, r \leqslant \infty$ satisfy $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$. Then, for every $f \in L^p_{k,n,e}(\mathbb{R})$ and $g \in L^q_{k,n}(\mathbb{R})$, the convolution product $f *_{k,n} g$ belongs to $L^r_{k,n}(\mathbb{R})$ and*

$$\|f *_{k,n} g\|_{L^r_{k,n}(\mathbb{R})} \leqslant \|f\|_{L^p_{k,n}(\mathbb{R})}\|g\|_{L^q_{k,n}(\mathbb{R})}. \tag{2.25}$$

We close this section by recalling the following result which will play a significant role.

**Proposition 2.8.** *[37, 44] Let $f$ and $g$ in $L^2_{k,n}(\mathbb{R})$. Then $f *_{k,n} g \in L^2_{k,n}(\mathbb{R})$ if and only if $\mathscr{F}_{k,n}(f)\mathscr{F}_{k,n}(g)$ belongs to $L^2_{k,n}(\mathbb{R})$, and in this case we have*

$$\mathscr{F}_{k,n}(f *_{k,n} g) = \mathscr{F}_{k,n}(f)\mathscr{F}_{k,n}(g).$$

An immediate consequence of Proposition 2.8 and the Plancherel formula (2.7) is the following statement which will be used later.

**Proposition 2.9.** *Let $f$ and $g$ be in $L^2_{k,n}(\mathbb{R})$. Then, we have*

$$\int_{\mathbb{R}} |f *_{k,n} g(x)|^2 d\gamma_{k,n}(x) = \int_{\mathbb{R}} |\mathscr{F}_{k,n}(f)(\xi)|^2 |\mathscr{F}_{k,n}(g)(\xi)|^2 d\gamma_{k,n}(\xi) \tag{2.26}$$

*whenever both sides are finite.*

## 3 Deformed Gabor transform

For $1 \leqslant p \leqslant \infty$, let $L^p_{\mu_{k,n}}(\mathbb{R}^2)$ be the space of measurable functions $f$ on $\mathbb{R}^2$ such that

$$\begin{aligned}\|f\|_{L^p_{\mu_{k,n}}(\mathbb{R}^2)} &:= \left(\int_{\mathbb{R}^2}|f(x,y)|^p d\mu_{k,n}(x,y)\right)^{\frac{1}{p}} < \infty, \qquad 1\leqslant p<\infty\\ \|f\|_{L^\infty_{\mu_{k,n}}(\mathbb{R}^2)} &:= \operatorname*{ess\,sup}_{(x,y)\in\mathbb{R}^2}|f(x,y)| < \infty,\end{aligned}$$

where $d\mu_{k,n}(x,y) := d\gamma_{k,n}(x)d\gamma_{k,n}(y)$.

**Definition 3.1.** *For any function $h$ in $L^2_{k,n,e}(\mathbb{R})$ and any $\nu \in \mathbb{R}$, we define the modulation of $h$ by $\nu$ as :*

$$h_\nu := \mathscr{F}_{k,n}\Big(\sqrt{\tau^{k,n}_\nu(|h|^2)}\Big), \tag{3.1}$$

*where $\tau^{k,n}_\nu$, $\nu \in \mathbb{R}$, are the generalized translation operators.*

**Remark 3.1.** *(i) Using the positivity of the generalized translation operator on even functions given by Theorem 2.3, we see that the formula (3.1) is well defined.*

*(ii) Using Plancherel's formula (2.7) and relation (2.19), we get for all $h$ in $L^2_{k,n,e}(\mathbb{R})$*

$$\|h_\nu\|_{L^2_{k,n}(\mathbb{R})} = \|h\|_{L^2_{k,n}(\mathbb{R})}. \tag{3.2}$$

We consider the family $h_{\nu,y}$, $\nu, y \in \mathbb{R}$ defined by

$$h_{\nu,y}(x) = \tau^{k,n}_{(-1)^n y} h_\nu(x), \;\; x \in \mathbb{R}.$$

We note that we have

$$\forall\, y,\nu \in \mathbb{R}, \quad \|h_{\nu,y}\|_{L^2_{k,n}(\mathbb{R})} \le \|h\|_{L^2_{k,n}(\mathbb{R})}. \tag{3.3}$$

**Definition 3.2.** *Let $h$ be in $L^2_{k,n,e}(\mathbb{R})$. For any function $f$ in $L^2_{k,n}(\mathbb{R})$ we define its deformed Gabor transform by*

$$\mathscr{G}^{k,n}_h(f)(y,\nu) := \int_{\mathbb{R}} f(x)\overline{h_{\nu,y}(x)}d\gamma_{k,n}(x), \tag{3.4}$$

*which can also be written in the form*

$$\mathscr{G}^{k,n}_h(f)(y,\nu) \;\; := \;\; f *_{k,n} \overline{r_n(h_\nu)}(y), \tag{3.5}$$

*where $r_n(g)(t) := g((-1)^n t)$.*

**Remark 3.2.** *By a standard computation it is easy to see that, for every $f \in L^2_{k,n}(\mathbb{R})$ and any $h$ in $L^2_{k,n,e}(\mathbb{R})$, for all $\lambda > 0$ and for all $(y,\nu) \in \mathbb{R}^2$, we have*

$$\mathscr{G}^{k,n}_{h_{\frac{1}{\lambda}}}(f_\lambda)(y,\nu) = \mathscr{G}^{k,n}_h(f)(\frac{y}{\lambda},\lambda\nu), \tag{3.6}$$

*where*

$$\forall\; t>0,\;\; \forall\; x\in\mathbb{R},\;\; g_t(x) := \frac{1}{t^{\frac{(2k-1)n+2}{n}}}g(\frac{x}{t}).$$

**Proposition 3.1.** *For $f$ in $L^2_{k,n}(\mathbb{R})$ and $h$ in $L^2_{k,n,e}(\mathbb{R})$ we have*

$$\|\mathscr{G}_h^{k,n}(f)\|_{L^\infty_{\mu_{k,n}}(\mathbb{R}^2)} \leq \|f\|_{L^2_{k,n}(\mathbb{R})}\|h\|_{L^2_{k,n}(\mathbb{R})}. \tag{3.7}$$

*Proof.* The result is immediately from (3.4), the Cauchy-Schwarz inequality and (3.3). □

**Proposition 3.2.** *(Plancherel's formula) Let $h$ be in $L^2_{k,n,e}(\mathbb{R})$. Then, for all $f$ in $L^2_{k,n}(\mathbb{R})$, we have*

$$\|\mathscr{G}_h^{k,n}(f)\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} = \|h\|_{L^2_{k,n}(\mathbb{R})}\|f\|_{L^2_{k,n}(\mathbb{R})}. \tag{3.8}$$

*Proof.* Using relations (3.5), (2.26), Fubini's theorem, relations (2.9), (3.1), Plancherel's formula (2.7) and relation (2.19), we get

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} |f *_{k,n} \overline{r_n(h_\nu)}|^2(y)d\gamma_{k,n}(y)d\gamma_{k,n}(\nu) &= \int_{\mathbb{R}}\int_{\mathbb{R}} |\mathscr{F}_{k,n}(f)(\xi)|^2|\mathscr{F}_{k,n}(\overline{r_n(h_\nu)})(\xi)|^2 d\gamma_{k,n}(\xi)d\gamma_{k,n}(\nu)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} |\mathscr{F}_{k,n}(f)(\xi)|^2\tau_\nu^{k,n}(|h|^2)((-1)^n\xi)d\gamma_{k,n}(\xi)d\gamma_{k,n}(\nu)\\
&= \int_{\mathbb{R}} |\mathscr{F}_{k,n}(f)(\xi)|^2\Big(\int_{\mathbb{R}} \tau^{k,n}_{(-1)^n\xi}(|h|^2)(\nu)d\gamma_{k,n}(\nu)\Big)d\gamma_{k,n}(\xi)\\
&= \|f\|^2_{L^2_{k,n}(\mathbb{R})}\|h\|^2_{L^2_{k,n}(\mathbb{R})}.
\end{aligned}$$

□

As in the classical case, the continuous deformed Gabor transform preserves the orthogonality relation. However, we have the following result.

**Corollary 3.1.** *Let $h$ be in $L^2_{k,n,e}(\mathbb{R})$. Then, for all $f, g$ in $L^2_{k,n}(\mathbb{R})$, we have*

$$\int_{\mathbb{R}^2} \mathscr{G}_h^{k,n}(f)(y,\nu)\overline{\mathscr{G}_h^{k,n}(g)(y,\nu)}d\mu_{k,n}(y,\nu) = \|h\|^2_{L^2_{k,n}(\mathbb{R})}\int_{\mathbb{R}} f(x)\overline{g(x)}d\gamma_{k,n}(x). \tag{3.9}$$

**Proposition 3.3.** *Let $h$ be in $L^2_{k,n,e}(\mathbb{R})$. Then for any $f$ be in $L^2_{k,n}(\mathbb{R})$ and any $p \in [2,\infty)$, we have*

$$\|\mathscr{G}_h^{k,n}(f)\|_{L^p_{\mu_{k,n}}(\mathbb{R}^2)} \leqslant \|f\|_{L^2_{k,n}(\mathbb{R})}\|h\|_{L^2_{k,n}(\mathbb{R})}. \tag{3.10}$$

*Proof.* Using Proposition 3.1 and Proposition 3.2 the result follows by applying the Riesz-Thorin interpolation theorem. □

By simple calculations we prove the following:

**Lemma 3.1.** *Let $h \in L^2_{k,n,e}(\mathbb{R}) \bigcap L^\infty_{k,n}(\mathbb{R})$, then for any $f \in L^2_{k,n}(\mathbb{R})$, we have*

$$\mathscr{F}_{k,n}\Big(\mathscr{G}_h^{k,n}(f)(.,\nu)\Big)(\xi) = \mathscr{F}_{k,n}(f)(\xi)\sqrt{\tau_\nu^{k,n}|h|^2((-1)^n\xi)}. \tag{3.11}$$

Henceforth, the function $h$ will denote an arbitrary nonzero element in $L^2_{k,n,e}(\mathbb{R})$.
Now, we will prove a new inversion formula for the deformed Gabor transform.

**Theorem 3.1.** *(Inversion formula). For any function $f$ in $L^1_{k,n}(\mathbb{R}) \cap L^2_{k,n}(\mathbb{R})$ such that $\mathscr{F}_{k,n}(f)$ belongs to $L^1_{k,n}(\mathbb{R})$, we have*

$$f(y) = \frac{1}{\|h\|^2_{L^2_{k,n}(\mathbb{R})}} \int_{\mathbb{R}} \Big( \int_{\mathbb{R}} \mathscr{G}^{k,n}_h(f)(x,\nu)\tau^{k,n}_y h_\nu(x) d\gamma_{k,n}(x) \Big) d\gamma_{k,n}(\nu), \quad a.e.\ y \in \mathbb{R}. \tag{3.12}$$

To prove this theorem we need the following lemma.

**Lemma 3.2.** ($L^2_{k,n}$ *inversion formula). Retain the assumption of Theorem 3.1. For any function $f$ in $L^1_{k,n}(\mathbb{R}) \cap L^2_{k,n}(\mathbb{R})$, we have*

$$f(y) = \frac{1}{\|h\|^2_{L^2_{k,n}(\mathbb{R})}} \lim_{j\to\infty} \int_{-j}^{j} \int_{\mathbb{R}} \mathscr{F}_{k,n}(\mathscr{G}^{k,n}_h(f)(.,\nu))(\xi) \mathscr{F}_{k,n}(\tau^{k,n}_y h_\nu)(\xi) d\mu_{k,n}(\xi,\nu), \tag{3.13}$$

*where the limit is in* $L^2_{k,n}(\mathbb{R})$.

*Proof.* As $f$ is in $L^1_{k,n}(\mathbb{R}) \cap L^2_{k,n}(\mathbb{R})$ then from (3.5) we have

$$\mathscr{G}^{k,n}_h(f)(x,\nu) = f *_{k,n} \overline{r_n(h_\nu)}(x).$$

Thus using (2.24) we get

$$\mathscr{F}_{k,n}(\mathscr{G}^{k,n}_h(f)(.,\nu))(\xi) = \mathscr{F}_{k,n}(f)(\xi)\mathscr{F}_{k,n}(\overline{r_n(h_\nu)})(\xi),\ \xi \in \mathbb{R}. \tag{3.14}$$

From (3.1) we have

$$\mathscr{F}_{k,n}(h_\nu)(\xi) = \sqrt{\tau^{k,n}_\nu(|h|^2)((-1)^n\xi)},\ \xi \in \mathbb{R}. \tag{3.15}$$

On the other hand using (2.11), and the fact that the function $h$ is even and $|h|^2 \in L^1_{k,n}(\mathbb{R})$, we deduce that the function $\lambda \mapsto \mathscr{F}_{k,n}(\tau^{k,n}_\nu |h|^2)(\lambda)$ is continuous on $\mathbb{R}$ and we have

$$\forall \lambda \in \mathbb{R}, \quad \mathscr{F}_{k,n}(\tau^{k,n}_\xi |h|^2)(\lambda) = \overline{B_{k,n}(\lambda,\xi)}\mathscr{F}_{k,n}(|h|^2)(\lambda). \tag{3.16}$$

By taking $\lambda = 0$ and using the relation (2.14), we obtain

$$\int_{\mathbb{R}} \tau^{k,n}_\nu(|h|^2)((-1)^n\xi) d\gamma_{k,n}(\nu) = \|h\|^2_{L^2_{k,n}(\mathbb{R})}. \tag{3.17}$$

Thus from (3.15) and (3.17) we get

$$\int_{\mathbb{R}} |\mathscr{F}_{k,n}(h_\nu)(\xi)|^2 d\gamma_{k,n}(\nu) = \int_{\mathbb{R}} \tau^{k,n}_\nu(|h|^2)((-1)^n\xi) d\gamma_{k,n}(\nu) = \|h\|^2_{L^2_{k,n}(\mathbb{R})}. \tag{3.18}$$

This relation shows that for $\xi \in \mathbb{R}$ :

- The function $\nu \mapsto \mathscr{F}_{k,n}(h_\nu)(\xi)$ is in $L^2_{k,n}(\mathbb{R})$.
- The function $\nu \mapsto \tau^{k,n}_\nu(|h|^2)((-1)^n\xi)$ belongs to $L^1_{k,n}(\mathbb{R})$.

From (3.14), (3.15) and (2.9) we have for $\xi \in \mathbb{R}$

$$\begin{aligned} \int_{\mathbb{R}} \mathscr{F}_{k,n}(\mathscr{G}^{k,n}_h(f)(.,\nu))(\xi)\mathscr{F}_{k,n}(h_\nu)(\xi) d\gamma_{k,n}(\nu) &= \mathscr{F}_{k,n}(f)(\xi) \int_{\mathbb{R}} \mathscr{F}_{k,n}(\overline{r_n(h_\nu)})(\xi)\mathscr{F}_{k,n}(h_\nu)(\xi) d\gamma_{k,n}(\nu) \\ &= \Big( \int_{\mathbb{R}} \tau^{k,n}_\nu(|h|^2)((-1)^n\xi) d\gamma_{k,n}(\nu) \Big) \mathscr{F}_{k,n}(f)(\xi). \end{aligned} \tag{3.19}$$

Using this relation and (3.18) we get

$$\mathscr{F}_{k,n}(f)(\xi) = \frac{1}{\|h\|^2_{L^2_{k,n}(\mathbb{R})}} \int_{\mathbb{R}} \mathscr{F}_{k,n}(\mathscr{G}^{k,n}_h(f)(.,\nu))(\xi)\mathscr{F}_{k,n}(h_\nu)(\xi)d\gamma_{k,n}(\nu). \tag{3.20}$$

Thus using this relation and the relation (2.11) we obtain

$$\frac{1}{\|h\|^2_{L^2_{k,n}(\mathbb{R})}} \int_{-j}^{j} ( \int_{\mathbb{R}} \mathscr{F}_{k,n}(\mathscr{G}^{k,n}_h(f)(.,\nu))(\xi)\mathscr{F}_{k,n}(\tau^{k,n}_y h_\nu)(\xi)d\gamma_{k,n}(\nu))d\gamma_{k,n}(\xi) = \int_{-j}^{j} \mathscr{F}_{k,n}(f)(\xi)B_{k,n}(\xi,y)d\gamma_{k,n}(\xi). \tag{3.21}$$

But as $f$ is in $L^2_{k,n}(\mathbb{R})$, we have

$$\lim_{j\to\infty} \int_{-j}^{j} \mathscr{F}_{k,n}(f)(\xi)B_{k,n}((-1)^n\xi,y)d\gamma_{k,n}(\xi) = f(y), \ \ y \in \mathbb{R} \tag{3.22}$$

the limit is in $L^2_{k,n}(\mathbb{R})$.
Thus this relation and (3.21) imply that for $y \in \mathbb{R}$ :

$$f(y) = \frac{1}{\|h\|^2_{L^2_{k,n}(\mathbb{R})}} \lim_{j\to\infty} \int_{-j}^{j} \Big( \int_{\mathbb{R}} \mathscr{F}_{k,n}(\mathscr{G}^{k,n}_h(f)(.,\nu))(\xi)\mathscr{F}_{k,n}(\tau^{k,n}_y h_\nu)(\xi)d\gamma_{k,n}(\nu)\Big)d\gamma_{k,n}(\xi). \tag{3.23}$$

The limit is in $L^2_{k,n}(\mathbb{R})$. □

Proof of Theorem 3.1

Using the relation (3.23) we deduce that for almost every $y \in \mathbb{R}$

$$f(y) = \frac{1}{\|h\|^2_{L^2_{k,n}(\mathbb{R})}} \lim_{j\to\infty} \int_{-j}^{j} ( \int_{\mathbb{R}} \mathscr{F}_{k,n}(\mathscr{G}^{k,n}_h(f)(.,\nu))(\xi)\mathscr{F}_{k,n}(\tau^{k,n}_y h_\nu)(\xi)d\gamma_{k,n}(\nu))d\gamma_{k,n}(\xi).$$

On the other hand from (3.14), (2.9) and (3.18) we deduce that

$$\begin{aligned} & \int_{-j}^{j} \int_{\mathbb{R}} |\mathscr{F}_{k,n}(\mathscr{G}^{k,n}_h(f)(.,\nu))(\xi)\mathscr{F}_{k,n}(\tau^{k,n}_y h_\nu)(\xi)|d\gamma_{k,n}(\nu)d\gamma_{k,n}(\xi) \\ = & \int_{-j}^{j} \int_{\mathbb{R}} |\mathscr{F}_{k,n}(\overline{r_n(h_\nu)})(\xi)\mathscr{F}_{k,n}(\tau^{k,n}_y h_\nu)(\xi)|d\gamma_{k,n}(\nu) \\ \leq & \ \|\mathscr{F}_{k,n}(f)\|_{L^1_{k,n}(\mathbb{R})}\|h\|^2_{L^2_{k,n}(\mathbb{R})} < \infty. \end{aligned} \tag{3.24}$$

Then by applying Fubini's theorem, we obtain for almost every $y \in \mathbb{R}$

$$f(y) = \lim_{j\to\infty} \int_{\mathbb{R}} \Big( \int_{-j}^{j} \mathscr{F}_{k,n}(\mathscr{G}^{k,n}_h(f)(.,\nu))(\xi)\mathscr{F}_{k,n}(\tau^{k,n}_y h_\nu)(\xi)d\gamma_{k,n}(\xi)\Big)\frac{d\gamma_{k,n}(\nu)}{\|h\|^2_{L^2_{k,n}(\mathbb{R})}}. \tag{3.25}$$

We consider for $y \in \mathbb{R}$, the sequence $U_j$ given by

$$U_j(\nu) = \int_{-j}^{j} \mathscr{F}_{k,n}(\mathscr{G}^{k,n}_h(f)(.,\nu))(\xi)\mathscr{F}_{k,n}(\tau^{k,n}_y h_\nu)(\xi)d\gamma_{k,n}(\xi).$$

This sequence satisfies the following

$$\forall\ v \in \mathbb{R}, \quad \lim_{j\to\infty} U_j(v) = \int_{\mathbb{R}} \mathscr{F}_{k,n}(\mathscr{G}_h^{k,n}(f)(.,v))(\xi)\mathscr{F}_{k,n}(\tau_y^{k,n}h_v)(\xi)d\gamma_{k,n}(\xi).$$

On the other hand for all $v \in \mathbb{R}$, we have

$$|U_j(v)| \leq \int_{\mathbb{R}} | \mathscr{F}_{k,n}(\mathscr{G}_h^{k,n}(f)(.,v))(\xi)\mathscr{F}_{k,n}(\tau_y^{k,n}h_v)(\xi) | \, d\gamma_{k,n}(\xi).$$

By making the same calculus as for (3.24) we deduce that the function

$$v \mapsto \int_{\mathbb{R}} | \mathscr{F}_{k,n}(\mathscr{G}_h^{k,n}(f)(.,v))(\xi)\mathscr{F}_{k,n}(\tau_y^{k,n}h_v)(\xi) | \, d\gamma_{k,n}(\xi)$$

is integrable on $\mathbb{R}$ with respect to the measure $d\gamma_{k,n}(v)$. Then by applying the dominated convergence theorem to the relation (3.25), we obtain for almost every $y \in \mathbb{R}$

$$f(y) = \frac{1}{||h||^2_{L^2_{k,n}(\mathbb{R})}} \int_{\mathbb{R}} \Big( \int_{\mathbb{R}} \mathscr{F}_{k,n}(\mathscr{G}_h^{k,n}(f)(.,v))(\xi)\mathscr{F}_{k,n}(\tau_y^{k,n}h_v)(\xi)d\gamma_{k,n}(\xi)\Big)d\gamma_{k,n}(v). \tag{3.26}$$

By applying (2.9) and Parseval's formula (2.8) to the second integral of the second member of the formula (3.26), we get

$$f(y) = \frac{1}{||h||^2_{L^2_{k,n}(\mathbb{R})}} \int_{\mathbb{R}} \Big( \int_{\mathbb{R}} \mathscr{G}_h^{k,n}(f)(x,v)\tau_y^{k,n}h_v(x)d\gamma_{k,n}(x)\Big)d\gamma_{k,n}(v), \ a.e.\ y \in \mathbb{R}.$$

# 4 Shapiro's dispersion theorem

In this section we will assume that $h$ is a fixed function in $L^2_{k,n,e}(\mathbb{R})$ such that $||h||_{L^2_{k,n}(\mathbb{R})} = 1$.

We denote by $B(L^2_{k,n}(\mathbb{R}))$, the space of bounded operators from $L^2_{k,n}(\mathbb{R})$ into itself.

**Definition 4.1.** *(i) The singular values $(s_j(A))_{j\in\mathbb{N}}$ of a compact operator $A$ in $B(L^2_{k,n}(\mathbb{R}))$ are the eigenvalues of the positive self-adjoint operator $|A| = \sqrt{A^*A}$.*

*(ii) For $1 \leq p < \infty$, the Schatten class $S_p$ is the space of all compact operators whose singular values lie in $l^p(\mathbb{N})$. The space $S_p$ is equipped with the norm*

$$||A||_{S_p} := \Big( \sum_{j=1}^{\infty} (s_j(A))^p \Big)^{\frac{1}{p}}. \tag{4.1}$$

*(iii) We define $S_\infty := B(L^2_{k,n}(\mathbb{R}))$, equipped with the norm,*

$$||A||_{S_\infty} := \sup_{v\in L^2_{k,n}(\mathbb{R}):||v||_{L^2_{k,n}(\mathbb{R})}=1} ||Av||_{L^2_{k,n}(\mathbb{R})}. \tag{4.2}$$

**Definition 4.2.** *The trace of an operator $A$ in $S_1$ is defined by*

$$tr(A) = \sum_{j=1}^{\infty} \langle Av_j, v_j\rangle_{L^2_{k,n}(\mathbb{R})} \tag{4.3}$$

*where $(v_j)_{j\in\mathbb{N}}$ is any orthonormal basis of $L^2_{k,n}(\mathbb{R})$.*

**Remark 4.1.** *If A is positive, then*

$$tr(A) = \|A\|_{S_1}. \tag{4.4}$$

*Moreover, a compact operator A on the Hilbert space $L^2_{k,n}(\mathbb{R})$ is Hilbert-Schmidt, if the positive operator $A^*A$ is in the space of trace class $S_1$. Then*

$$\|A\|^2_{HS} := \|A\|^2_{S_2} = \|A^*A\|_{S_1} = tr(A^*A) = \sum_{j=1}^{\infty} \|Av_j\|^2_{L^2_{k,n}(\mathbb{R})} \tag{4.5}$$

*for any orthonormal basis $(v_j)_{j\in\mathbb{N}}$ of $L^2_{k,n}(\mathbb{R})$.*

The proof of the statement bellow requires the following notation:

- Let $P_h$ be the orthogonal projection from $L^2_{\mu_{k,n}}(\mathbb{R}^2)$ onto the space $\mathcal{G}^{k,n}_h(L^2_{k,n}(\mathbb{R}))$.
- Let $P_U$ be the orthogonal projection from $L^2_{\mu_{k,n}}(\mathbb{R}^2)$ onto the subspace of function in $L^2_{\mu_{k,n}}(\mathbb{R}^2)$ supported in the subset $U \subset \mathbb{R}^2$ where $0 < \mu_{k,n}(U) := \int_U d\mu_{k,n}(x,v) < \infty$.
- We put

$$\|P_U P_h\| := \sup\left\{\|P_U P_h v\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} : \ v \in L^2_{\mu_{k,n}}(\mathbb{R}^2), \ \|v\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} = 1\right\}.$$

**Definition 4.3.** *Let $0 < \varepsilon < 1$ and $U \subset \mathbb{R}^2$ be a measurable subset. For $f \in L^2_{k,n}(\mathbb{R})$, we say that $\mathcal{G}^{k,n}_h(f)$ is $\varepsilon$-concentrated on U if*

$$\left\|\mathcal{G}^{k,n}_h(f)\right\|_{L^2_{\mu_{k,n}}(U^c)} \leqslant \varepsilon \left\|\mathcal{G}^{k,n}_h(f)\right\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)},$$

*where $U^c$ is the complement of U in $\mathbb{R}^2$.*

**Proposition 4.1.** *Let $\left(\varphi_j\right)_{j\in\mathbb{N}}$ be an orthonormal sequence in $L^2_{k,n}(\mathbb{R})$ and U be a measurable subset of $\mathbb{R}^2$ such that $0 < \mu_{k,n}(U) < \infty$. For every nonempty finite subset $\mathcal{E} \subset \mathbb{N}$, we have*

$$\sum_{j\in\mathcal{E}} \left(1 - \left\|\mathbb{1}_{U^c}\mathcal{G}^{k,n}_h(\varphi_j)\right\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}\right) \leqslant \mu_{k,n}(U).$$

*Proof.* Since $\left(\varphi_j\right)_{j\in\mathbb{N}}$ is an orthonormal sequence in $L^2_{k,n}(\mathbb{R})$, by (3.8) we deduce that $\left(\mathcal{G}^{k,n}_h(\varphi_j)\right)_{j\in\mathbb{N}}$ is an orthonormal sequence in $L^2_{\mu_{k,n}}(\mathbb{R}^2)$. Moreover, since the operator $P_U P_h$ is of Hilbert-Schmidt type, then, by (4.5) and (4.3), it is easy to see that

$$\begin{aligned}\sum_{j\in\mathcal{E}} \langle P_U \mathcal{G}^{k,n}_h(\varphi_j), \mathcal{G}^{k,n}_h(\varphi_j)\rangle_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} &= \sum_{j\in\mathcal{E}} \langle P_h P_U P_h \mathcal{G}^{k,n}_h(\varphi_j), \mathcal{G}^{k,n}_h(\varphi_j)\rangle_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \\ &\leqslant \mathrm{tr}(P_h P_U P_h) \\ &= \|P_U P_h\|^2_{HS}.\end{aligned}$$

Further, proceeding as in [30], we get

$$\|P_U P_h\|_{HS} \leqslant \sqrt{\mu_{k,n}(U)}.$$

Thus,

$$\sum_{j\in\mathscr{E}}\langle P_U\mathscr{G}_h^{k,n}(\varphi_j),\mathscr{G}_h^{k,n}(\varphi_j)\rangle_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}\leqslant\mu_{k,n}(U). \tag{4.6}$$

On the other hand, by Cauchy-Schwarz's inequality we have for every $j\in\mathscr{E}$,

$$\begin{aligned}\langle P_U\mathscr{G}_h^{k,n}(\varphi_j),\mathscr{G}_h^{k,n}(\varphi_j)\rangle_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}&=1-\langle P_{U^c}\mathscr{G}_h^{k,n}(\varphi_j),\mathscr{G}_h^{k,n}(\varphi_j)\rangle_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}\\&\geqslant 1-\|\mathbb{1}_{U^c}\mathscr{G}_h^{k,n}(\varphi_j)\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}.\end{aligned}$$

In particular, by relation (4.6), we obtain

$$\sum_{j\in\mathscr{E}}\left(1-\|\mathbb{1}_{U^c}\mathscr{G}_h^{k,n}(\varphi_j)\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}\right)\leqslant\sum_{j\in\mathscr{E}}\langle P_U\mathscr{G}_h^{k,n}(\varphi_j),\mathscr{G}_h^{k,n}(\varphi_j)\rangle_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}\leqslant\mu_{k,n}(U).$$

□

Next, we shall use Proposition 4.1 to prove that if the deformed Gabor transform of an orthonormal sequence is $\varepsilon$-concentrated on a given centered ball in $\mathbb{R}^2$, then a such sequence is necessary finite

**Proposition 4.2.** *Let $\varepsilon$ and $\delta$ be two positive real numbers such that $0<\varepsilon<1$. Let $\mathscr{E}\subset\mathbb{N}$ be a nonempty subset and $\left(\varphi_j\right)_{j\in\mathscr{E}}$ be an orthonormal sequence in $L^2_{k,n}(\mathbb{R})$. If, for every $j\in\mathscr{E}$, $\mathscr{G}_h^{k,n}(\varphi_j)$ is $\varepsilon$-concentrated on the ball*

$$B_2(0,\delta):=\left\{(x,\nu)\in\mathbb{R}^2:\|(x,\nu)\|\leqslant\delta\right\},$$

*then the set $\mathscr{E}$ is finite and*

$$\mathrm{Card}(\mathscr{E})\leqslant\frac{(\Gamma(\frac{(2k-1)n+2}{2n}))^2}{\Gamma(\frac{2kn+2}{n})(1-\varepsilon)}\delta^{\frac{2(2k-1)n+4}{n}}. \tag{4.7}$$

*Proof.* Let $\mathscr{M}\subset\mathscr{E}$ be a nonempty finite subset, then by Proposition 4.1, we deduce that

$$\sum_{n\in\mathscr{M}}\left(1-\|\mathbb{1}_{B_2(0,\delta)^c}\mathscr{G}_h^{k,n}(\varphi_j)\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}\right)\leqslant\mu_{k,n}(B_2(0,\delta)). \tag{4.8}$$

However, for every $j\in\mathscr{M}$, we have

$$\|\mathbb{1}_{B_2(0,\delta)^c}\mathscr{G}_h^{k,n}(\varphi_j)\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}\leqslant\varepsilon\quad\text{and}\quad\mu_{k,n}(B_2(0,\delta))=\frac{(\Gamma(\frac{(2k-1)n+2}{2n}))^2}{\Gamma(\frac{2kn+2}{n})}\delta^{\frac{2(2k-1)n+4}{n}}. \tag{4.9}$$

Hence, by combining relations (4.8) and (4.9), we deduce that

$$\mathrm{Card}(\mathscr{M})\leqslant\frac{(\Gamma(\frac{(2k-1)n+2}{2n}))^2}{\Gamma(\frac{2kn+2}{n})(1-\varepsilon)}\delta^{\frac{2(2k-1)n+4}{n}},$$

which means that $\mathscr{E}$ is finite and satisfies relation (4.7). □

For a positive real number $p$, the generalized $p^{\text{th}}$ time-frequency dispersion of $\mathscr{G}_h^{k,n}(f)$ is defined by

$$\rho_p(\mathscr{G}_h^{k,n}(f))=\left(\int_{\mathbb{R}^2}\|(x,\nu)\|^p\left|\mathscr{G}_h^{k,n}(f)(x,\nu)\right|^2d\mu_{k,n}(x,\nu)\right)^{\frac{1}{p}}.$$

**Corollary 4.1.** *Let $A$ and $p$ be two positive real numbers. Let $\mathscr{E} \subset \mathbb{N}$ be a nonempty subset and $\left(\varphi_j\right)_{j\in\mathscr{E}}$ be an orthonormal sequence in $L^2_{k,n}(\mathbb{R})$. Assume that for every $j \in \mathscr{E}$,*

$$\rho_p(\mathscr{G}_h^{k,n}(\varphi_j)) \leqslant A.$$

*Then $\mathscr{E}$ is finite and*

$$\mathrm{Card}(\mathscr{E}) \leqslant M(k,n,p)A^{\frac{2(2k-1)n+4}{n}},$$

*where*

$$M(k,n,p) = 2^{\frac{8kn+(p-4)n+8}{np}} \frac{\left(\Gamma(\frac{(2k-1)n+2}{2n})\right)^2}{\Gamma(\frac{2kn+2}{n})}.$$

*Proof.* Since $\rho_p(\mathscr{G}_h^{k,n}(\varphi_j)) \leqslant A$, for every $j \in \mathscr{E}$, it follows

$$\int_{B_2^c(0,A2^{\frac{2}{p}})} |\mathscr{G}_h^{k,n}(\varphi_j)(x,\nu)|^2 d\mu_{k,n}(x,\nu) \leqslant \frac{1}{\left(A2^{\frac{2}{p}}\right)^p}\rho_p^p(\mathscr{G}_h^{k,n}(\varphi_j)) \leqslant \frac{1}{4}. \tag{4.10}$$

The inequality (4.10) means that for every $j \in \mathscr{E}$, $\mathscr{G}_h^{k,n}(\varphi_j)$ is $\frac{1}{2}$-concentrated in the ball $B_2(0, A2^{\frac{2}{p}})$. According to Proposition 4.2, we deduce that $\mathscr{E}$ is finite and

$$\mathrm{Card}(\mathscr{E}) \leqslant M(k,n,p)A^{\frac{2(2k-1)n+4}{n}}.$$

□

**Lemma 4.1.** *Let $p$ be a positive real number. If $\left(\varphi_j\right)_{j\in\mathbb{N}}$ is an orthonormal sequence in $L^2_{k,n}(\mathbb{R})$, then there exists $j_0 \in \mathbb{Z}$ such that*

$$\rho_p^p(\mathscr{G}_h^{k,n}(\varphi_j)) \geqslant 2^{p(j_0-1)}, \qquad \forall j \in \mathbb{N}.$$

*Proof.* Proceeding as in [41], using the assumptions $\|h\|_{L^2_{k,n}(\mathbb{R})} = 1$ and the fact that $\left(\varphi_j\right)_{j\in\mathbb{N}}$ is an orthonormal sequence in $L^2_{k,n}(\mathbb{R})$, we infer that there exist a positive constant $C_1(k,n,p)$ such that

$$\rho_p^p(\mathscr{G}_h^{k,n}(\varphi_j)) \geqslant \frac{1}{(C_1(k,n,p))^2}.$$

Moreover it is easy to see that there exists $j_0 \in \mathbb{Z}$ such that

$$\frac{1}{(C_1(k,n,p))^2} \geq 2^{p(j_0-1)}.$$

Thus the desired result is proved. □

**Theorem 4.1** (Shapiro's dispersion theorem for $\mathscr{G}_h^{k,n}$)**.** *Let $\left(\varphi_j\right)_{j\in\mathbb{N}}$ be an orthonormal sequence in $L^2_{k,n}(\mathbb{R})$. For every positive real numbers $p$ and for every nonempty finite subset $\mathscr{E} \subset \mathbb{N}$, we have*

$$\sum_{j\in\mathscr{E}} \left(\rho_p(\mathscr{G}_h^{k,n}(\varphi_j))\right)^p \geqslant \frac{1}{2}\left(\frac{3}{2^{\frac{8kn-3n+8}{n}}M(k,n,p)}\right)^{\frac{np}{2(2k-1)n+4}} (\mathrm{Card}(\mathscr{E}))^{1+\frac{np}{2(2k-1)n+4}}. \tag{4.11}$$

*Proof.* For every $j \in \mathbb{Z}$, let

$$P_j = \left\{ m \in \mathbb{N} \ : \ \rho_p(\mathscr{G}_h^{k,n}(\varphi_m)) \in [2^{j-1}, 2^j) \right\}.$$

Then, for every $m \in P_j$,

$$\int_{\mathbb{R}^2} \|(x,\nu)\|^p \left| \mathscr{G}_h^{k,n}(\varphi_m)(x,\nu) \right|^2 d\mu_{k,n}(x,\nu) \leqslant 2^{jp}.$$

That is the sequence $(\varphi_m)_{m \in P_j}$ satisfies the conditions of Corollary 4.1, and therefore $P_j$ is finite with

$$\mathrm{Card}(P_j) \leqslant M(k,n,p) 2^{(\frac{2(2k-1)n+4}{n})j}. \tag{4.12}$$

For $m \in \mathbb{Z},\ m \geqslant j_0$, we denote by $Q_m := \bigcup_{j=j_0}^{m} P_j$. According to (4.12), we have

$$\mathrm{Card}\,(Q_m) = \sum_{j=j_0}^{m} \mathrm{Card}(P_j) \leqslant \frac{M(k,n,p) 2^{\frac{2(2k-1)n+4}{n}}}{3} 2^{(\frac{2(2k-1)n+4}{n})m}.$$

Now, if $\mathrm{Card}(\mathscr{E}) > \dfrac{M(k,n,p) 2^{\frac{4kn-2n+4}{n}}}{3} 2^{(\frac{2(2k-1)n+4}{n})j_0}$, then we can choose an integer $m > j_0$ such that

$$\frac{M(k,n,p) 2^{\frac{4kn-2n+4}{n}}}{3} 2^{(\frac{2(2k-1)n+4}{n})(m-1)} < \mathrm{Card}(\mathscr{E}) \leqslant \frac{M(k,n,p) 2^{\frac{4kn-2n+4}{n}}}{3} 2^{(\frac{2(2k-1)n+4}{n})m}. \tag{4.13}$$

Thus, by (4.13), we get

$$\sum_{j \in \mathscr{E}} \left( \rho_p(\mathscr{G}_h^{k,n}(\varphi_j)) \right)^p \geqslant \frac{\mathrm{Card}(\mathscr{E})}{2} 2^{(m-1)p} \geqslant \frac{1}{2} (\mathrm{Card}(\mathscr{E}))^{1+\frac{np}{2(2k-1)n+4}} \left( \frac{3}{2^{\frac{8kn-3n+8}{n}} M(k,n,p)} \right)^{\frac{np}{2(2k-1)n+4}}.$$

Finally, if $\mathrm{Card}(\mathscr{E}) \leqslant \dfrac{M(k,n,p) 2^{\frac{4kn-2n+4}{n}}}{3} 2^{(\frac{2(2k-1)n+4}{n})j_0}$, then

$$\sum_{j \in \mathscr{E}} \left( \rho_p(\mathscr{G}_h^{k,n}(\varphi_j)) \right)^p \geqslant \mathrm{Card}(\mathscr{E}) 2^{(j_0-1)p} \geqslant (\mathrm{Card}(\mathscr{E}))^{1+\frac{p}{4k}} \left( \frac{3}{2^{\frac{8kn-3n+8}{n}} M(k,n,p)} \right)^{\frac{np}{2(2k-1)n+4}}.$$

□

**Remark 4.2.** *By taking* $\mathrm{Card}(\mathscr{E}) = 1$*, relation (4.11) appears as a general version of Heisenberg-Pauli-Weyl inequality for the the deformed Gabor transform including the* $p^{\text{th}}$ *dispersion.*

**Corollary 4.2.** *Let* $p > 0$ *and let* $\left(\varphi_j\right)_{j \in \mathbb{N}}$ *be an orthonormal sequence in* $L^2_{k,n}(\mathbb{R})$*. Then for every* $\mathscr{E} \subset \mathbb{N}$

$$\sum_{j \in \mathscr{E}} \left( \left\| |\nu|^{\frac{p}{2}} \mathscr{G}_h^{k,n}(\varphi_j) \right\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} + \left\| |x|^{\frac{p}{2}} \mathscr{G}_h^{k,n}(\varphi_j) \right\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \right)$$
$$\geq \frac{1}{2} \left( \frac{3}{M(k,n,p) 2^{\frac{12kn-5n+12}{n}}} \right)^{\frac{np}{2(2k-1)n+4}} (\mathrm{Card}(\mathscr{E}))^{1+\frac{np}{2(2k-1)n+4}}. \tag{4.14}$$

*Proof.* The result is an immediate consequence of Theorem 4.1 together with the fact that

$$\|(x,\nu)\|^p \leqslant 2^p(|\nu|^p + |x|^p).$$

□

The dispersion inequality (4.14) implies that there is no infinite sequence $\left(\varphi_j\right)_{j\in\mathscr{E}}$ in $L^2_{k,n}(\mathbb{R})$ such that both sequences

$$\left\| |\nu|^{\frac{p}{2}} \mathscr{G}_h^{k,n}(\varphi_j) \right\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \quad \text{and} \quad \left\| |x|^{\frac{p}{2}} \mathscr{G}_h^{k,n}(\varphi_j) \right\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}$$

are bounded. More precisely:

**Corollary 4.3.** *Let $p > 0$ and let $\left(\varphi_j\right)_{j\in\mathbb{N}}$ be an orthonormal sequence in $L^2_{k,n}(\mathbb{R})$. For every $\mathscr{E} \subset \mathbb{N}$, we have*

$$\sup_{j\in\mathscr{E}} \left( \left\| |\nu|^{\frac{p}{2}} \mathscr{G}_h^{k,n}(\varphi_j) \right\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}, \left\| |x|^{\frac{p}{2}} \mathscr{G}_h^{k,n}(\varphi_j) \right\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \right) \geq \frac{1}{4} \left( \frac{3}{M(k,n,p) 2^{\frac{12kn-5n+12}{n}}} \right)^{\frac{np}{2(2k-1)n+4}} (\mathrm{Card}(\mathscr{E}))^{\frac{np}{2(2k-1)n+4}}. \quad (4.15)$$

*In particular,*

$$\sup_{j\in\mathscr{E}} \left( \left\| |\nu|^{\frac{p}{2}} \mathscr{G}_h^{k,n}(\varphi_j) \right\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} + \left\| |x|^{\frac{p}{2}} \mathscr{G}_h^{k,n}(\varphi_j) \right\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \right) = \infty.$$

**Theorem 4.2** (Shapiro's Umbrella theorem for $\mathscr{G}_h^{k,n}$)**.** *Let $\mathscr{E} \subset \mathbb{N}$ be a nonempty subset and $\left(\varphi_j\right)_{j\in\mathscr{E}}$ be an orthonormal sequence in $L^2_{k,n}(\mathbb{R})$. If there is a positive function $g \in L^2_{\mu_{k,n}}(\mathbb{R}^2)$ such that*

$$|\mathscr{G}_h^{k,n}(\varphi_j)(x,\nu)| \leqslant g(x,\nu)$$

*for every $j \in \mathscr{E}$ and for almost every $(x,\nu) \in \mathbb{R}^2$, then $\mathscr{E}$ is finite.*

*Proof.* Following the idea of Malinnikova [29], for every positive real number $0 < \varepsilon < 1$, there is a subset $\Delta_{g,\varepsilon} \subset \mathbb{R}^2$ such that

$$\mu_{k,n}(\Delta_{g,\varepsilon}) = \inf\left\{ \mu_{k,n}(U) : \int_{U^c} |g(x,\nu)|^2 \, d\mu_{k,n}(x,\nu) \leqslant \varepsilon^2 \right\},$$

and

$$\int_{\Delta^c_{g,\varepsilon}} |g(x,\nu)|^2 \, d\mu_{k,n}(x,\nu) = \varepsilon^2.$$

Hence, according to the hypothesis, for every $n \in \mathscr{E}$ we have

$$\int_{\Delta^c_{g,\varepsilon}} \left| \mathscr{G}_h^{k,n}\left(\varphi_j\right)(x,\nu) \right|^2 d\mu_{k,n}(x,\nu) \leqslant \varepsilon^2,$$

and by Proposition 4.1, we get $\mathrm{Card}(\mathscr{E})(1-\varepsilon) \leqslant \mu_{k,n}(\Delta_{g,\varepsilon})$. □

# 5 Concentration-based uncertainty principles

## 5.1 Heisenberg's-type uncertainty principles

**Proposition 5.1.** *Let h be in* $L^2_{k,n,e}(\mathbb{R}) \cap L^\infty_{k,n}(\mathbb{R})$. *Then,* $\mathscr{G}^{k,n}_h(L^2_{k,n}(\mathbb{R}))$ *is a reproducing kernel Hilbert space in* $L^2_{k,n}(\mathbb{R})$ *with kernel function*

$$\mathscr{K}_h(\nu', x'; \nu, x) := \frac{1}{\|h\|^2_{L^2_{k,n}(\mathbb{R})}} \int_{\mathbb{R}} \overline{h_{\nu',x'}(y)} h_{\nu,x}(y) d\gamma_{k,n}(y). \tag{5.1}$$

*The kernel is pointwise bounded:*

$$|\mathscr{K}_h(\nu', x'; \nu, x)| \leq 1; \quad \forall\, (x', \nu'),\ (x, \nu) \in \mathbb{R}^2. \tag{5.2}$$

*Proof.* Let $f \in L^2_{k,n}(\mathbb{R})$. We have

$$\mathscr{G}^{k,n}_h(f)(x, \nu) = \int_{\mathbb{R}} f(y) \overline{h_{\nu,x}(y)} d\gamma_{k,n}(y),\ \ (x, \nu) \in \mathbb{R}^2.$$

Using Parseval's relation (3.9), we obtain

$$\mathscr{G}^{k,n}_h(f)(x, \nu) = \frac{1}{\|h\|^2_{L^2_{k,n}(\mathbb{R})}} \int_{\mathbb{R}^2} \mathscr{G}^{k,n}_h(f)(x', \nu') \overline{\mathscr{G}^{k,n}_h(h_{\nu,x})(x', \nu')} d\mu_{k,n}(x', \nu').$$

On the other hand, using Proposition 2.8, one can easily see that for every $(x, \nu), (x', \nu') \in \mathbb{R}^2$ the function

$$x' \mapsto \frac{1}{\|h\|^2_{L^2_{k,n}(\mathbb{R})}} \mathscr{G}^{k,n}_h(h_{\nu,x})(x', \nu') = \frac{1}{\|h\|^2_{L^2_{k,n}(\mathbb{R})}} \int_{\mathbb{R}} \overline{h_{\nu',x'}(y)} h_{\nu,x}(y) d\gamma_{k,n}(y)$$

belongs to $L^2_{k,n}(\mathbb{R})$. Therefore, the result is obtained. □

In the following we will prove the concentration of $\mathscr{G}^{k,n}_h(f)$ in small sets.

**Proposition 5.2.** *Let h be in* $L^2_{k,n,e}(\mathbb{R})$ *and* $U \subset \mathbb{R}^2$ *with*

$$0 < \mu_{k,n}(U) < 1.$$

*Then, for all* $f \in L^2_{k,n}(\mathbb{R})$ *we have*

$$\|\mathscr{G}^{k,n}_h(f) - \chi_U \mathscr{G}^{k,n}_h(f)\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \geq \sqrt{1 - \mu_{k,n}(U)} \|h\|_{L^2_{k,n}(\mathbb{R})} \|f\|_{L^2_{k,n}(\mathbb{R})}, \tag{5.3}$$

*where* $\chi_U$ *denotes the characteristic function of* $U$.

*Proof.* From Plancherel's formula (3.8) we have

$$\begin{aligned} \|h\|^2_{L^2_{k,n}(\mathbb{R})} \|f\|^2_{L^2_{k,n}(\mathbb{R})} &= \|\mathscr{G}^{k,n}_h(f)\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \\ &= \|\mathscr{G}^{k,n}_h(f)\|^2_{L^2_{\mu_{k,n}}(U)} + \|\mathscr{G}^{k,n}_h(f)\|^2_{L^2_{\mu_{k,n}}(U^c)}. \end{aligned} \tag{5.4}$$

On the other hand from the relation (3.7) we have

$$\begin{aligned} \int_U |\mathscr{G}^{k,n}_h(f)(x, \nu)|^2 d\mu_{k,n}(x, \nu) &\leq \|\mathscr{G}^{k,n}_h(f)\|^2_{L^\infty_{\mu_{k,n}}(\mathbb{R}^2)} \mu_{k,n}(U) \\ &\leq \mu_{k,n}(U) \|f\|^2_{L^2_{k,n}(\mathbb{R})} \|h\|^2_{L^2_{k,n}(\mathbb{R})}. \end{aligned} \tag{5.5}$$

Thus the result follows immediately by integrating (5.4) in (5.5). □

**Remark 5.1.** *We assume that* $0 < \mu_{k,n}(U) < 1$. *If* $\mathscr{G}_h^{k,n}(f)$ *is supported in* $U$, *then* $f = 0$.

**Proposition 5.3.** *Let* $h$ *be in* $L^2_{k,n,e}(\mathbb{R})$ *such that* $\|h\|_{L^2_{k,n}(\mathbb{R})} = 1$.
*Let* $s > 0$. *Then the following uncertainty inequalities hold.*

1. *A Heisenberg-type uncertainty inequalities for the deformed Gabor transform:*
   - *(i) There exists a constant* $C_1(k,n,s) > 0$ *such that, for all* $f$ *in* $L^2_{k,n}(\mathbb{R})$, *we have*
$$\Big\| \|(x,\nu)\|^s \mathscr{G}_h^{k,n}(f)(x,\nu)\Big\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \geq C_1(k,n,s)\|f\|_{L^2_{k,n}(\mathbb{R})}. \tag{5.6}$$
   - *(ii) There exists a constant* $C_2(k,n,s) > 0$ *such that, for all* $f$ *in* $L^2_{k,n}(\mathbb{R})$, *we have*
$$\Big\| |x|^s \mathscr{G}_h^{k,n}(f)(x,\nu)\Big\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \Big\| |\nu|^s \mathscr{G}_h^{k,n}(f)(x,\nu)\Big\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \geq C_2(k,n,s)\|f\|^2_{L^2_{k,n}(\mathbb{R})}. \tag{5.7}$$
2. *A Faris Local uncertainty inequality for the deformed Gabor transform:*
   *There exists a constant* $C_3(k,n,s) > 0$ *such that, for all* $f$ *in* $L^2_{k,n}(\mathbb{R})$, *and every subset* $U \subset \mathbb{R}^2$ *such that* $0 < \mu_{k,n}(U) < \infty$, *we have*
$$\|\mathscr{G}_h^{k,n}(f)\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \leq C_3(k,n,s)\sqrt{\mu_{k,n}(U)}\Big\| \|(x,\nu)\|^s \mathscr{G}_h^{k,n}(f)(x,\nu)\Big\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}. \tag{5.8}$$

*Proof.* (1) Let $r > 0$ such that $0 < \mu_{k,n}(B_2(0,r)) < 1$ where $B_2(0,r)$ is the open ball of $\mathbb{R}^2$ defined by
$$B_2(0,r) = \Big\{(x,\nu) \in \mathbb{R}^2 : \|(x,\nu)\| < r\Big\}.$$
By applying the relation (5.3) with $U = B_2(0,r)$ we obtain
$$\begin{aligned}(1-\mu_{k,n}(U))\|f\|^2_{L^2_{k,n}(\mathbb{R})} &\leq \int_{B_2(0,r)^c} |\mathscr{G}_h^{k,n}(f)(x,\nu)|^2 d\mu_k(x,\nu)\\ &\leq \tfrac{1}{r^{2s}}\int_{\|(x,\nu)\|\geq r} \|(x,\nu)\|^{2s}|\mathscr{G}_h^{k,n}(f)(x,\nu)|^2 d\mu_k(x,\nu)\\ &\leq \tfrac{1}{r^{2s}}\Big\| \|(x,\nu)\|^s \mathscr{G}_h^{k,n}(f)(x,\nu)\Big\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}.\end{aligned}$$
Thus we obtain the relation (5.6) with $C_1(k,n,s) := r^s\sqrt{1-\mu_{k,n}(U)}$.

(ii) By applying the inequality $\|(x,\nu)\|^s \leq 2^s(|\nu|^s + |x|^s)$ in (5.6), we get
$$\Big\| |x|^s \mathscr{G}_h^{k,n}(f)\Big\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} + \| |\nu|^s \mathscr{G}_h^{k,n}(f)\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \geq \frac{(C_1(k,n,s))^2}{2^{2s}}\|f\|^2_{L^2_{k,n}(\mathbb{R})}. \tag{5.9}$$
We replace $f$ by $f_t$, in the relation (5.9), we apply (3.6) and next we make a change of variables in each term, we obtain the following relation:
$$t^{2s}\| |x|^s \mathscr{G}_h^{k,n}(f)\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} + t^{-2s}\| |\nu|^s \mathscr{G}_h^{k,n}(f)\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \geq \frac{(C_1(k,n,s))^2}{2^{2s}}\|f\|^2_{L^2_{k,n}(\mathbb{R})}.$$
Then (5.7) follows by minimizing the left hand side of this inequality, with respect $t > 0$.

(2) Using the fact that

$$\|\mathscr{G}_h^{k,n}(f)\|_{L^2_{\mu_{k,n}}(U)} \leq \sqrt{\mu_{k,n}(U)}\|\mathscr{G}_h^{k,n}(f)\|_{L^\infty_{\mu_{k,n}}(\mathbb{R}^2)},$$

and the fact that

$$\|\mathscr{G}_h^{k,n}(f)\|_{L^\infty_{\mu_{k,n}}(\mathbb{R}^2)} \leq \|f\|_{L^2_{k,n}(\mathbb{R})},$$

then we get

$$\|\mathscr{G}_h^{k,n}(f)\|_{L^2_{\mu_{k,n}}(U)} \leq \sqrt{\mu_{k,n}(U)}\|f\|_{L^2_{k,n}(\mathbb{R})}.$$

Finally, we obtain the result from (5.6). □

## 5.2 Benedicks's-type uncertainty principle

In the following we will prove the concentration of $\mathscr{G}_h^{k,n}(f)$ in arbitrary sets of finite measures.

**Theorem 5.1.** *Let h be in $L^2_{k,n,e}(\mathbb{R})$ and $U \subset \mathbb{R}^2$ with $0 < \mu_{k,n}(U) < \infty$.*

*If $P_h(L^2_{\mu_{k,n}}(\mathbb{R}^2)) \cap P_U(L^2_{\mu_{k,n}}(\mathbb{R}^2)) = \{0\}$. Then, there exists a constant $C := C_{k,n}(h, U) > 0$ such that for all $f \in L^2_{k,n}(\mathbb{R})$, we have*

$$\|\mathscr{G}_h^{k,n}(f) - \chi_U\mathscr{G}_h^{k,n}(f)\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \geq C\|f\|_{L^2_{k,n}(\mathbb{R})}. \tag{5.10}$$

For the proof of this theorem, we need the following lemma.

**Lemma 5.1.** *([57]). Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two closed subspaces of a Hilbert space $\mathscr{H}$ satisfying $\mathscr{H}_1 \bigcap \mathscr{H}_2 = \{0\}$. Let $P_{\mathscr{H}_1}$ and $P_{\mathscr{H}_2}$ denote the corresponding orthogonal projections, and assume the product $P_{\mathscr{H}_1}P_{\mathscr{H}_2}$ to be a compact operator. Then, there exists a constant $C > 0$ such that for $f \in \mathscr{H}$*

$$\|P_{\mathscr{H}_1^\perp}f\|_{\mathscr{H}} + \|P_{\mathscr{H}_2^\perp}f\|_{\mathscr{H}} \geq C\|f\|_{\mathscr{H}}. \tag{5.11}$$

*Proof.* of Theorem 5.1. Defining $\mathscr{H}_1$ and $\mathscr{H}_2$ by

$$\mathscr{H}_1 := P_U(L^2_{\mu_{k,n}}(\mathbb{R}^2)), \quad \mathscr{H}_2 := P_h(L^2_{\mu_{k,n}}(\mathbb{R}^2)).$$

We proceed as in [30], we prove that

$$\|P_UP_h\|_{HS} := \Big(\int_{\mathbb{R}^2\times\mathbb{R}^2} |\chi_U(x,\nu)|^2|\mathscr{K}_h(\nu',x';\nu,x)|^2 d\mu_{k,n}(x',\nu')d\mu_{k,n}(x,\nu)\Big)^{\frac{1}{2}} \leq \|h\|^2_{L^2_{k,n}(\mathbb{R})}\sqrt{\mu_{k,n}(U)} < \infty.$$

Hence, $P_UP_h$ is a Hilbert-Schmidt operator and, therefore, compact. Now, Lemma 5.1 implies the existence of a constant $C > 0$ such that (5.11) holds for $P_{\mathscr{H}_1} := P_U$ and $P_{\mathscr{H}_2} := P_h$. Since

$$P_{\mathscr{H}_2^\perp}(\mathscr{G}_h^{k,n}(f)) = (Id - P_h)\mathscr{G}_h^{k,n}(f) = 0,$$

this leads to (5.10). □

**Definition 5.1.** *Let $h$ be in $L^2_{k,n,e}(\mathbb{R})$ and $U \subset \mathbb{R}^2$ such that $0 < \mu_{k,n}(U) < \infty$.*

*(1) We say that $U$ is weakly annihilating, if any function $f \in L^2_{k,n}(\mathbb{R})$ vanishes when its deformed Gabor transform $\mathscr{G}_h^{k,n}(f)$ is supported in $U$.*

*(2) We say that $U$ is strongly annihilating, if there exists a constant $\mathfrak{C}_{k,n}(U,h) > 0$ such that for every function $f \in L^2_{k,n}(\mathbb{R})$,*

$$\|\mathscr{G}_h^{k,n}(f) - \chi_U \mathscr{G}_h^{k,n}(f)\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \geq \mathfrak{C}_{k,n}(U,h)\|f\|_{L^2_{k,n}(\mathbb{R})}. \tag{5.12}$$

*The constant $\mathfrak{C}_{k,n}(U,h)$ will be called the annihilation constant of $U$.*

**Remark 5.2.** *(1) It is clear that, every strongly annihilating set is also a weakly.*

*(2) From Proposition 5.2, we see that any set $U \subset \mathbb{R}^2$ with $0 < \mu_{k,n}(U) < 1$, is strongly annihilating.*

*(3) As the operator $P_U P_h$ is Hilbert-Schmidt hence is compact, then from [24] we have if $U$ is weakly annihilating, it is also strongly annihilating.*

*(4) If $\|P_U P_h\| < 1$, then for all $f \in L^2_{k,n}(\mathbb{R})$*

$$\frac{1}{\sqrt{1 - \|P_U P_h\|^2}} \|\mathscr{G}_h^{k,n}(f) - \chi_U \mathscr{G}_h^{k,n}(f)\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \geq \|f\|_{L^2_{k,n}(\mathbb{R})}\|h\|_{L^2_{k,n}(\mathbb{R})}. \tag{5.13}$$

*(5) Following the result established in a general context in [24] p.88, we have if $U$ is strongly annihilating, then $\|P_U P_h\| < 1$.*

In the next, we give Benedicks-type uncertainty principle for the deformed Gabor transform .

**Theorem 5.2.** *Let $r, R > 0$ and let $0 \neq h$ be in $L^2_{k,n,e}(\mathbb{R}) \bigcap L^\infty_{k,n}(\mathbb{R})$ such that*

$$supp\, h \subset (-r, r). \tag{5.14}$$

*For any subset $U := I \times (-R, R) \subset \mathbb{R}^2$ such that*

$$0 < \gamma_{k,n}(I) < \infty,$$

*we have*

$$P_h(L^2_{\mu_{k,n}}(\mathbb{R}^2)) \cap P_U(L^2_{\mu_{k,n}}(\mathbb{R}^2)) = \{0\}. \tag{5.15}$$

*Proof.* Let $F$ be a non-trivial function in $P_h(L^2_{\mu_{k,n}}(\mathbb{R}^2)) \cap P_U(L^2_{\mu_{k,n}}(\mathbb{R}^2))$, then there exists a function $f \in L^2_{k,n}(\mathbb{R})$ such that $F = \mathscr{G}_h^{k,n}(f)$ and $supp\, F \subset U$.
Let $\nu \in (-R, R)$, and let $\psi_\nu$ be the function defined on $\mathbb{R}$ by

$$\psi_\nu(y) = \mathscr{F}_{k,n}(f)((-1)^n y)\sqrt{\tau_\nu^{k,n}|h|^2(y)}.$$

Then for all $(x, \nu) \in U$

$$F(x, \nu) = \mathscr{F}_{k,n}(\psi_\nu)(x).$$

Thus

$$supp\, \mathscr{F}_{k,n}(\psi_\nu) \subset I, \text{ with } \gamma_{k,n}(I) < \infty.$$

Moreover, since $supp\, h \subset (-r, r)$ we deduce that

$$supp\, \psi_\nu \subset supp\, \tau_\nu^{k,n}|h|^2 \subset (-r - R, r + R).$$

Thus, using Proposition 2.4, we deduce that for every $\nu \in \mathbb{R}$, $\psi_\nu = 0$, which implies that $F = 0$. □

Consequently, we obtain the following improvement.

**Corollary 5.1.** *Let $r, R > 0$ and let $0 \neq h$ be in $L^2_{k,n,e}(\mathbb{R}) \bigcap L^\infty_{k,n}(\mathbb{R})$ such that*

$$supp\, h \subset (-r, r). \tag{5.16}$$

*Let $U := I \times (-R, R) \subset \mathbb{R}^2$ such that*

$$0 < \gamma_{k,n}(I) < \infty.$$

*Then, there exists a constant $C := C_{k,n}(h, U) > 0$ such that for all $f \in L^2_{k,n}(\mathbb{R})$, we have*

$$\|\mathscr{G}^{k,n}_h(f) - \chi_U \mathscr{G}^{k,n}_h(f)\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \geq C\|f\|_{L^2_{k,n}(\mathbb{R})}\|h\|_{L^2_{k,n}(\mathbb{R})}. \tag{5.17}$$

## 5.3 Donoho-Stark's uncertainty principe

Now we will derive a sufficient condition by means of which one can recover a signal $F$ belongs to $L^2_{\mu_{k,n}}(\mathbb{R}^2)$ from the knowledge of a truncated version of it, following the Donoho-Stark criterion [12].

Let $h$ be in $L^2_{k,n,e}(\mathbb{R})$. A signal $F \in L^2_{\mu_{k,n}}(\mathbb{R}^2)$ is transmitted to a receiver who knows that $F \in \mathscr{G}^{k,n}_h(L^2_{k,n}(\mathbb{R}))$. Suppose that the observation of F is corrupted by a noise $\mathscr{N} \in L^2_{\mu_{k,n}}(\mathbb{R}^2)$ (which is nonetheless assumed to be small) and unregistered values on $U \in \mathbb{R}^2$. Thus, the observable function $r$ satisfies

$$r(x, \nu) = \begin{cases} F(x, \nu) + \mathscr{N}(x, \nu) & \text{if} \quad (x, \nu) \in U^c \\ 0 & \text{if} \quad (x, \nu) \in U. \end{cases} \tag{5.18}$$

Here we have assumed without loss of generality that $\mathscr{N} = 0$ on $U$. Equivalently,

$$r = (Id - P_U)F + \mathscr{N}. \tag{5.19}$$

We say that $F$ can be stably reconstructed from $r$, if there exists a linear operator

$$L_{U,h} : L^2_{\mu_{k,n}}(\mathbb{R}^2) \to L^2_{\mu_{k,n}}(\mathbb{R}^2)$$

and a constant $\mathscr{C}_{k,n}(U, h)$ such that

$$\|F - L_{U,h}(r)\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \leq \mathscr{C}_{k,n}(U, h)\|\mathscr{N}\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}. \tag{5.20}$$

**Theorem 5.3.** *Let $r, R > 0$ and let $0 \neq h$ be in $L^2_{k,n,e}(\mathbb{R}) \bigcap L^\infty_{k,n}(\mathbb{R})$ such that*

$$supp\, h \subset (-r, r)$$

*and let $U := I \times (-R, R) \subset \mathbb{R}^2$ such that*

$$0 < \gamma_{k,n}(I) < \infty.$$

*Then $F$ can be stably reconstructed from $r$. Moreover, the constant $\mathscr{C}_{k,n}(U, h)$ in (5.20) is not larger than $(1 - \|P_U P_h\|)^{-1}$.*

*Proof.* We apply the same arguments that used in [12]. From Corollary 5.1, $U$ is strongly annihilating, then from Remark 5.2 we have $\|P_U P_h\| < 1$. Therefore $I - P_U P_h$ is invertible. Let

$$L_{U,h} = (Id - P_U P_h)^{-1}.$$

As $F \in \mathscr{G}_h^{k,n}(L^2_{k,n}(\mathbb{R}))$, then $(I - P_U)F = (I - P_U P_h)F$. Thus by simple calculations we see that

$$F - L_{U,h} r = -L_{U,h}\mathscr{N}.$$

So that

$$\begin{aligned}\|F - L_{U,h} r\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} = \|L_{U,h}\mathscr{N}\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} &\leq \|(Id - P_U P_h)^{-1}\| \, \|\mathscr{N}\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \\ &\leq (1 - \|P_U P_h\|)^{-1} \, \|\mathscr{N}\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)},\end{aligned}$$

which allows to conclude. □

**Remark 5.3.** *(An algorithm for computing $L_{U,h} r$)*
*The identity*

$$L_{U,h} = (Id - P_U P_h)^{-1} = \sum_{j=0}^{\infty} (P_U P_h)^j$$

*suggest an algorithm for computing $L_{U,h} r$. Using the similar method given in [12], we give an algorithm for computing $L_{U,h} r$. Indeed, put*

$$F^{(m)} = \sum_{j=0}^{m} (P_U P_h)^j r,$$

*then $F^{(m)} \to L_{U,h}(r)$ as $m \to \infty$. Now*

$$\begin{aligned} F^{(0)} &= r \\ F^{(1)} &= r + P_U P_h F^{(0)} \\ F^{(2)} &= r + P_U P_h F^{(1)} \\ &\cdots \end{aligned} \tag{5.21}$$

*and so on. The iteration converges at a geometric rate to the fixed point*

$$F = r + P_U P_h F.$$

*Algorithms of type (5.21), have been applied to a host of problems in signal recovery see [12], and others.*

# 6 Practical real inversion formulas for $\mathscr{G}_h^{k,n}$

## 6.1 Tikhonov regularization

Nowadays, the general theory of reproducing kernels have found many applications to Integral transforms, Inverse problems, Integral equations, Inversions for a family of bounded linear operators, Sampling theory, Linear differential equations with variable coefficients, Approximations of functions. Arguing from these point of view, many works were done on them, we refer in particular to the papers of Saitoh et al. [51, 52].

Before the applications to the Tiknohov regularization, we shall examine the concept of the Moore-Penrose generalized inverses from the viewpoint of the theory of reproducing kernels. Here, we will be able to realize the natural and powerful method of the theory of reproducing kernels for the best approximation problems that lead to the Moore-Penrose generalized inverses.

Let $E$ be an arbitrary set and let $H_{k,n}$ be a reproducing kernel Hilbert space admitting the reproducing kernel $K$ on $E$. For any Hilbert space $\mathscr{H}$ we consider a bounded linear operator $L$ from $H_{k,n}$ to $\mathscr{H}$. Then the following problem is a classical and fundamental problem which is known as best approximate mean square norm problems: For any member $d$ of $\mathscr{H}$

$$\inf_{f\in H_{k,n}} \|Lf - d\|_{\mathscr{H}}. \tag{6.1}$$

This problem carries, however, a complicated structure, when the Hilbert spaces are infinite dimensions and the problem leads to the generalized inverse in the sense of Moore-Penrose. However, this extremal problem is involved in both the existence of the extremal functions in (6.1) and their representations. So, we shall consider its Tikhonov regularization. We start it with the following fundamental theorem.

**Theorem 6.1.** *( [51].) Let $H_K$ be a Hilbert space admitting the reproducing kernel $K(p,q)$ on a set $E$ and $\mathscr{H}$ an Hilbert space. Let $L : H_K \to \mathscr{H}$ be a bounded linear operator. For $r > 0$, we introduce the inner product in $H_K$ and we call it $H_{K_r}$ as*

$$\langle f_1, f_2\rangle_{H_{K_r}} = r\langle f_1, f_2\rangle_{H_K} + \langle Lf_1, Lf_2\rangle_{\mathscr{H}}.$$

*Then:*

i*) $H_{K_r}$ is a Hilbert space with the reproducing kernel $K_r(p,q)$ on $E$ and satisfying the equation*

$$K_r(.,q) = (rI + L^*L)K(.,q),$$

*where $L^*$ is the adjoint operator of $L : H_K \to \mathscr{H}$.*

ii*) For any $r > 0$ and for any $h$ in $\mathscr{H}$, the infimum*

$$\inf_{f\in H_K} \left\{ r\|f\|^2_{H_K} + \|Lf - h\|^2_{\mathscr{H}} \right\}$$

*is attained by a unique function $f^*_{r,h}$ in $H_K$ and this extremal function is given by*

$$f^*_{r,h}(p) = \langle h, LK_r(.,p)\rangle_{\mathscr{H}}. \tag{6.2}$$

In this Section by applying the general theory of reproducing kernels and in particular the Tikhonov regularization, we shall consider the practical constructions of approximate solutions for bounded linear operator equations involving the deformed Gabor transform. The functional spaces used in our analyse are the generalized Sobolev spaces that are built from the deformed Hankel transform and deformed Gabor transform and that are the typical Hilbert spaces in our setting.

## 6.2 Reproducing kernels

**Notation**. Let us denote by

$\mathscr{S}(\mathbb{R}) := \{f \in C^\infty(\mathbb{R},\mathbb{C}) : \forall\, j, m \in \mathbb{N}^d,\ \|f\|_{j,m} < \infty\}$, where $C^\infty(\mathbb{R},\mathbb{C})$ is the function space of smooth functions from $\mathbb{R}$ into $\mathbb{C}$ and $\|f\|_{j,m} := \sup_{x\in\mathbb{R}} |x^j D^m f(x)|$. This space is known as the Schwartz space.

$\mathscr{S}'(\mathbb{R})$ the topological dual of the Schwartz space $\mathscr{S}(\mathbb{R})$.

**Remark 6.1.** *We note that Johansen in [[26], Lemma 2.12] has proved that the Schwartz space is invariant under the deformed Hankel transform.*

**Definition 6.1.** *The deformed Hankel transform of a distribution* $\tau$ *in* $\mathscr{S}'(\mathbb{R})$ *is defined by*

$$\langle \mathscr{F}_{k,n}(\tau), \varphi\rangle = \langle \tau, \mathscr{F}_{k,n}^{-1}(\varphi)\rangle, \quad \text{for all } \varphi \in \mathscr{S}(\mathbb{R}). \tag{6.3}$$

**Definition 6.2.** *Let* $s \in \mathbb{R}$*, we define the generalized Sobolev space* $W^s_{k,n}(\mathbb{R})$ *as*

$$\Big\{u \in \mathscr{S}'(\mathbb{R}) : (1 + |\xi|^2)^{\frac{s}{2}} \mathscr{F}_{k,n}(u) \in L^2_{k,n}(\mathbb{R})\Big\}.$$

We provided this space with inner product $\langle ., .\rangle_{W^s_{k,n}(\mathbb{R})}$ given by:

$$\langle f, g\rangle_{W^s_{k,n}(\mathbb{R})} = \int_{\mathbb{R}} (1 + |\xi|^2)^s \mathscr{F}_{k,n}(f)(\xi)\overline{\mathscr{F}_{k,n}(g)(\xi)} d\gamma_{k,n}(\xi), \quad \text{for all } f, g \in W^s_{k,n}(\mathbb{R}). \tag{6.4}$$

The norm associated to the inner product is defined by:

$$\|f\|_{W^s_{k,n}(\mathbb{R})} := \Big( \int_{\mathbb{R}} (1 + |\xi|^2)^s |\mathscr{F}_{k,n}(f)(\xi)|^2 d\gamma_{k,n}(\xi)\Big)^{\frac{1}{2}}.$$

**Proposition 6.1.** *For* $s > \frac{(2k-1)n+2}{2n}$*, the generalized Sobolev space* $W^s_{k,n}(\mathbb{R})$ *admits the following reproducing kernel:*

$$K_s(x, y) = \int_{\mathbb{R}} \frac{B_{k,n}((-1)^n\xi, x)B_{k,n}(\xi, y)}{(1 + |\xi|^2)^s} d\gamma_{k,n}(\xi),$$

*(i) for all* $y \in \mathbb{R}$*, the function* $x \mapsto K_s(x, y)$ *belongs to* $W^s_{k,n}(\mathbb{R})$*,*
*(ii) the reproducing property: for all* $f \in W^s_{k,n}(\mathbb{R})$ *and* $y \in \mathbb{R}$*,*

$$f(y) = \langle f, K_s(x, y)\rangle_{W^s_{k,n}(\mathbb{R})}.$$

*Proof.* i) Let $y$ be in $\mathbb{R}$. It is easy to see that the function

$$\Upsilon_y : \xi \mapsto \frac{B_{k,n}(\xi, y)}{(1 + |\xi|^2)^s}$$

belongs to $L^1_{k,n}(\mathbb{R}) \cap L^2_{k,n}(\mathbb{R})$ when $s > \frac{(2k-1)n+2}{2n}$. Thus the function $K_s$ is well defined and we can write

$$K_s(x, y) = \mathscr{F}_{k,n}^{-1}(\Upsilon_y)(x), \quad \text{for all } x \in \mathbb{R}.$$

Moreover, from Proposition 2.2, we can see that the function $K_s(., y)$ belongs to $L^2_{k,n}(\mathbb{R})$, and we have

$$\mathscr{F}_{k,n}(K_s(., y))(\xi) = \frac{B_{k,n}(\xi, y)}{(1 + |\xi|^2)^s}. \tag{6.5}$$

As $B_{k,n}(\xi, y)$ is bounded, we obtain

$$|\mathscr{F}_{k,n}(K_s(., y))(\xi)| \leq \frac{1}{(1 + |\xi|^2)^s}$$

and

$$\|K_s(., y)\|_{W^s_{k,n}(\mathbb{R})} \leq C(k, n, s) := \Big( \int_{\mathbb{R}} \frac{d\gamma_{k,n}(\xi)}{(1 + |\xi|^2)^s}\Big)^{\frac{1}{2}} = \Big(\frac{\Gamma(\frac{(2k-1)n+2}{2n})\Gamma(s - \frac{(2k-1)n+2}{2n})}{\Gamma(s)}\Big)^{\frac{1}{2}} < \infty. \tag{6.6}$$

This proves that for all $y \in \mathbb{R}$ the function $K_s(.,y)$ belongs to $W^s_{k,n}(\mathbb{R})$.

(ii) Let $f$ be in $W^s_{k,n}(\mathbb{R})$ and $y$ in $\mathbb{R}$. From (6.4) and (6.5), we have

$$\langle f, K_s(.,y)\rangle_{W^s_{k,n}(\mathbb{R})} = \int_{\mathbb{R}} \mathscr{F}_{k,n}(f)(\xi) B_{k,n}(y,(-1)^n\xi) d\gamma_{k,n}(\xi), \tag{6.7}$$

and from inversion formula, we obtain the reproducing property

$$f(y) = \langle f, K_s(x,y)\rangle_{W^s_{k,n}(\mathbb{R})}.$$

This completes the proof of the proposition. □

**Corollary 6.1.** *For $s > \frac{(2k-1)n+2}{2n}$, the generalized Sobolev space $W^s_{k,n}(\mathbb{R})$ is embedded in $C(\mathbb{R})$.*

## 6.3 Extremal function associated with the deformed Gabor transform

Let $r > 0$, $s \geq 0$ and $h$ be in $L^2_{k,n,e}(\mathbb{R})$. We introduce the inner product in the space $W^s_{k,n}(\mathbb{R})$

$$\langle f, g\rangle_{\mathscr{G}^{k,n}_h, r, W^s_{k,n}(\mathbb{R})} = r\langle f, g\rangle_{W^s_{k,n}(\mathbb{R})} + \langle \mathscr{G}^{k,n}_h(f), \mathscr{G}^{k,n}_h(g)\rangle_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}, \quad f, g \in W^s_{k,n}(\mathbb{R}).$$

The norm associated to the inner product is defined by:

$$\|f\|^2_{\mathscr{G}^{k,n}_h, r, W^s_{k,n}(\mathbb{R})} := r\|f\|^2_{W^s_{k,n}(\mathbb{R})} + \|\mathscr{G}^{k,n}_h(f)\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}.$$

From (3.9), the inner product $\langle .,,\rangle_{\mathscr{G}^{k,n}_h, r, W^s_{k,n}(\mathbb{R})}$ can be written as

$$\langle f, g\rangle_{\mathscr{G}^{k,n}_h, r, W^s_{k,n}(\mathbb{R})} = r\langle f, g\rangle_{W^s_{k,n}(\mathbb{R})} + \|h\|^2_{L^2_{k,n}(\mathbb{R})}\langle f, g\rangle_{L^2_{k,n}(\mathbb{R})}. \tag{6.8}$$

**Remark 6.2.** *Simple calculations give that the norms $\|.\|_{W^s_{k,n}(\mathbb{R})}$ and $\|.\|_{\mathscr{G}^{k,n}_h, r, W^s_{k,n}(\mathbb{R})}$ are equivalent for $r > 0$.*

**Proposition 6.2.** *Let $s > \frac{(2k-1)n+2}{2n}$ and $h$ be in $L^2_{k,n,e}(\mathbb{R})$. Then the generalized Sobolev space $(W^s_{k,n}(\mathbb{R}), \langle .,.\rangle_{\mathscr{G}^{k,n}_h, r, W^s_{k,n}(\mathbb{R})})$, possesses the reproducing kernel $\mathfrak{K}^{k,n}_{r,h}$ satisfying the following identity*

$$\mathfrak{K}^{k,n}_{r,h}(x,y) = \left(rI + (\mathscr{G}^{k,n}_h)^* \mathscr{G}^{k,n}_h\right)^{-1} K_s(.,y),$$

*where*

$$(\mathscr{G}^{k,n}_h)^* : L^2_{\mu_{k,n}}(\mathbb{R}^2) \longrightarrow W^s_{k,n}(\mathbb{R})$$

*is the adjoint operator of $\mathscr{G}^{k,n}_h$ given by*

$$\langle \mathscr{G}^{k,n}_h(f), g\rangle_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} = \langle f, (\mathscr{G}^{k,n}_h)^* g\rangle_{W^s_{k,n}(\mathbb{R})}, \quad f \in W^s_{k,n}(\mathbb{R}),\ g \in L^2_{\mu_{k,n}}(\mathbb{R}^2).$$

*(i) $\|\mathfrak{K}^{k,n}_{r,h}(.,y)\|_{W^s_{k,n}(\mathbb{R})} \leq \frac{C(k,n,s)}{r}$, for all $y \in \mathbb{R}$.*

*(ii) $\|\mathscr{G}^{k,n}_h(\mathfrak{K}^{k,n}_{r,h}(.,y))\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \leq \frac{C(k,n,s)}{\sqrt{2r}}$, for all $y \in \mathbb{R}$.*

*(iii) $\|(\mathscr{G}^{k,n}_h)^* \mathscr{G}^{k,n}_h(\mathfrak{K}^{k,n}_{r,h}(.,y))\|_{W^s_{k,n}(\mathbb{R})} \leq C(k,n,s)$, for all $y \in \mathbb{R}$, where $C(k,n,s)$ is the constant given by (6.6).*

*Proof.* From Corollary 6.1 and Remark 6.2, we deduce that the map $u \mapsto u(y)$, $y \in \mathbb{R}$, is a continuous linear functional on $(W^s_{k,n}(\mathbb{R}), \langle ., . \rangle_{\mathscr{G}^{k,n}_h, r, W^s_{k,n}(\mathbb{R})})$. Thus from [51], $(W^s_{k,n}(\mathbb{R}), \langle ., . \rangle_{\mathscr{G}^{k,n}_h, r, W^s_{k,n}(\mathbb{R})})$ has a reproducing kernel denoted by $\mathfrak{R}^{k,n}_{r,h}$. On the other hand, we have

$$\begin{aligned} f(y) &= \langle f, \mathfrak{R}^{k,n}_{r,h}(., y)\rangle_{\mathscr{G}^{k,n}_h, r, W^s_{k,n}(\mathbb{R})} \\ &= r\langle f, \mathfrak{R}^{k,n}_{r,h}(., y)\rangle_{W^s_{k,n}(\mathbb{R})} + \langle \mathscr{G}^{k,n}_h(f), \mathscr{G}^{k,n}_h(\mathfrak{R}^{k,n}_{r,h}(., y))\rangle_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \\ &= \langle f, \left(rI + (\mathscr{G}^{k,n}_h)^* \mathscr{G}^{k,n}_h\right)\mathfrak{R}^{k,n}_{r,h}(., y)\rangle_{W^s_{k,n}(\mathbb{R})}. \end{aligned}$$

Thus,

$$(rI + (\mathscr{G}^{k,n}_h)^* \mathscr{G}^{k,n}_h)\mathfrak{R}^{k,n}_{r,h}(., y) = K_s(., y). \tag{6.9}$$

Furthermore, the previous identity implies that

$$r^2\|\mathfrak{R}^{k,n}_{r,h}(., y)\|^2_{W^s_{k,n}(\mathbb{R})} + 2r\|\mathscr{G}^{k,n}_h(\mathfrak{R}^{k,n}_{r,h}(., y))\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} + \|(\mathscr{G}^{k,n}_h)^*\mathscr{G}^{k,n}_h(\mathfrak{R}^{k,n}_{r,h}(., y))\|^2_{W^s_{k,n}(\mathbb{R})} = \|K_s(., y)\|^2_{W^s_{k,n}(\mathbb{R})}.$$

From this relation and using the fact that

$$\|K_s(., y)\|_{W^s_{k,n}(\mathbb{R})} \leq C(k, n, s),$$

we obtain the properties (i), (ii), and (iii). □

**Remark 6.3.** *Using similar ideas as in Proposition 6.1, we prove*

$$\mathfrak{R}^{k,n}_{r,h}(x, y) = \int_{\mathbb{R}} \frac{B_{k,n}((-1)^n\xi, x)B_{k,n}(\xi, y)}{r(1 + |\xi|^2)^s + \|h\|^2_{L^2_{k,n}(\mathbb{R})}} d\gamma_{k,n}(\xi). \tag{6.10}$$

We can now state the main result of this paragraph.

**Theorem 6.2.** *Let* $s > \frac{(2k-1)n+2}{2n}$ *and* $h$ *be in* $L^2_{k,n,e}(\mathbb{R})$.

i) *For any* $g \in L^2_{\mu_{k,n}}(\mathbb{R}^2)$ *and for any* $r > 0$, *the best approximate function* $f^*_{r,g}$ *in the sense*

$$\inf_{f \in W^s_{k,n}(\mathbb{R})} \left\{ r\|f\|^2_{W^s_{k,n}(\mathbb{R})} + \|g - \mathscr{G}^{k,n}_h(f)\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \right\} = r\|f^*_{r,g}\|^2_{W^s_{k,n}(\mathbb{R})} + \|g - \mathscr{G}^{k,n}_h(f^*_{r,g})\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \tag{6.11}$$

*exists uniquely and* $f^*_{r,g}$ *is defined by*

$$f^*_{r,g}(x) = \langle g, \mathscr{G}^{k,n}_h(\mathfrak{R}^{k,n}_{r,h}(., x))\rangle_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}.$$

ii) *The best approximate function* $f^*_{r,g}$ *is represented by*

$$f^*_{r,g}(x) = \int_{\mathbb{R}^2} g(\nu, y)\overline{Q^{k,n}_{r,h}(\nu, x, y)} d\mu_{k,n}(\nu, y),$$

*where*

$$Q^{k,n}_{r,h}(\nu, x, y) = \int_{\mathbb{R}} \frac{\sqrt{\tau^{k,n}_\nu(|h|^2)((-1)^n\xi)} B_{k,n}((-1)^n\xi, x)B_{k,n}(\xi, y)}{r(1 + \|\xi\|^2)^s + \|h\|^2_{L^2_{k,n}(\mathbb{R})}} d\gamma_{k,n}(\xi).$$

*(iii) The extremal function* $f^*_{r,g}$ *satisfies the following inequality:*

$$\forall\, y \in \mathbb{R}, \quad |f^*_{r,g}(y)| \leq \frac{C(k, n, s)}{\sqrt{2r}} \|g\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}.$$

*Proof.* (i) The existence and uniqueness of extremal function $f^*_{r,g}$ satisfying (6.11) is given by [52], and the extremal function $f^*_{r,g}$ is represented by

$$f^*_{r,g}(y) = \langle g, \mathscr{G}_h^{k,n}(\mathfrak{R}^{k,n}_{r,h}(.,y))\rangle_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}, \quad y \in \mathbb{R}.$$

(ii) Involving Lemma 3.1 we have

$$\mathscr{G}_h^{k,n}(f)(x,\nu) = \int_{\mathbb{R}} \sqrt{\tau_\nu^{k,n}(|h|^2)((-1)^n\xi)}\mathscr{F}_{k,n}(f)(\xi)B_{k,n}((-1)^n\xi, x)d\gamma_{k,n}(\xi), \quad \text{for all } x \in \mathbb{R}.$$

Using the properties of the kernel $\mathfrak{R}^{k,n}_{r,h}$ and the definition of the deformed Gabor transform, we get

$$\mathscr{G}_h^{k,n}\Big(\mathfrak{R}^{k,n}_{r,h}(.,y)\Big)(x,\nu) = \int_{\mathbb{R}} \frac{\sqrt{\tau_\nu^{k,n}(|h|^2)((-1)^n\xi)}B_{k,n}((-1)^n\xi, x)B_{k,n}(\xi,y)}{r(1+|\xi|^2)^s + \|h\|^2_{L^2_{k,n}(\mathbb{R})}}d\gamma_{k,n}(\xi) = Q^{k,n}_{r,h}(\nu,x,y).$$

This gives the result.

(iii) From Proposition 6.2 (ii), we have

$$|f^*_{r,g}(y)| \leq \|g\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}\|\mathscr{G}_h^{k,n}(\mathfrak{R}^{k,n}_{r,h}(.,y))\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \leq \frac{C(k,n,s)}{\sqrt{2r}}\|g\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}.$$

Thus the theorem is proved. □

**Corollary 6.2.** *Let $s > \frac{(2k-1)n+2}{2n}$ and $r > 0$. If $f$ is in $W^s_{k,n}(\mathbb{R})$ and $g = \mathscr{G}_h^{k,n}(f)$. Then*

i*) $\{f^*_{r,g}\}_{r>0}$ converges uniformly to $f$ as $r \to 0^+$.*

*(ii) For all $y \in \mathbb{R}$, $f(y) = \lim_{r\to 0^+} f^*_{r,g}(y)$.*

*(iii) For all $y \in \mathbb{R}$, $|f(y) - f^*_{r,g}(y)| \leq C(k,n,s)\|f\|_{W^s_{k,n}(\mathbb{R})}$.*

*(iv) For all $y \in \mathbb{R}$, $|f^*_{r,g}(y)| \leq C(k,n,s)\|f\|_{W^s_{k,n}(\mathbb{R})}$.*

*Proof.* Let $f$ be in $W^s_{k,n}(\mathbb{R})$.

(i) Then

$$\forall\, y \in \mathbb{R}, \quad f^*_{r,g}(y) = \langle f, (\mathscr{G}_h^{k,n})^*\mathscr{G}_h^{k,n}(\mathfrak{R}^{k,n}_{r,h}(.,y))\rangle_{W^s_{k,n}(\mathbb{R})}. \tag{6.12}$$

But from (6.9), we have

$$\forall\, y \in \mathbb{R}, \quad \lim_{r\to 0^+}(\mathscr{G}_h^{k,n})^*\mathscr{G}_h^{k,n}(\mathfrak{R}^{k,n}_{r,h}(.,y)) = K_s(.,y).$$

Thus

$$\lim_{r\to 0^+} f^*_{r,g}(y) = \langle f, K_s(.,y)\rangle_{W^s_{k,n}(\mathbb{R})} = f(y).$$

(ii) From (6.9) and (6.12), the extremal function $f^*_{r,g}$ satisfies

$$\forall\, y \in \mathbb{R}, \quad f^*_{r,g}(y) = f(y) - r\langle f, \mathfrak{R}^{k,n}_{r,h}(.,y)\rangle_{W^s_{k,n}(\mathbb{R})}.$$

Thus by Proposition 6.2 (i) we obtain

$$\forall\, y \in \mathbb{R}, \quad |f^*_{r,g}(y) - f(y)| \leq r\|f\|_{W^s_{k,n}(\mathbb{R})}\|\mathfrak{R}^{k,n}_{r,h}(.,y)\|_{W^s_{k,n}(\mathbb{R})} \leq C(k,n,s)\|f\|_{W^s_{k,n}(\mathbb{R})}.$$

(iii) From (6.12) and Proposition 6.2 (iii), the extremal function $f^*_{r,g}$ satisfies

$$\forall\, y \in \mathbb{R}, \quad |f^*_{r,g}(y)| \leq \|f\|_{W^s_{k,n}(\mathbb{R})}\|(\mathscr{G}_h^{k,n})^*\mathscr{G}_h^{k,n}(\mathfrak{R}^{k,n}_{r,h}(.,y))\|_{W^s_{k,n}(\mathbb{R})} \leq C(k,n,s)\|f\|_{W^s_{k,n}(\mathbb{R})}.$$

□

**Remark 6.4.** *Let $s > \frac{(2k-1)n+2}{2n}$ and $r > 0$.*

*If $\mathscr{G}_h^{k,n}$ is isometry (i.e. $(\mathscr{G}_h^{k,n})^*\mathscr{G}_h^{k,n} = Id$) then*

*(i) $\langle ., .\rangle_{\mathscr{G}_h^{k,n},r,W^s_{k,n}(\mathbb{R})} = (r+1)\langle ., .\rangle_{W^s_{k,n}(\mathbb{R})}$.*

*(ii) $\mathfrak{K}^{k,n}_{r,h}(x,y) = \frac{1}{r+1}K_s(x,y)$, for all $x, y \in \mathbb{R}$.*

*(iii) For all $y \in \mathbb{R}$, $f^*_{r,g}(y) = \frac{1}{r+1}(\mathscr{G}_h^{k,n})^*g(y)$, $g \in L^2_{\mu_{k,n}}(\mathbb{R}^2)$.*

*(iv) For all $y \in \mathbb{R}$, $f^*_{r,\mathscr{G}_h^{k,n}(u)}(y) = \frac{1}{r+1}u(y)$, $u \in W^s_{k,n}(\mathbb{R})$.*

## 6.4 Extremal functions associated with the partial deformed Gabor transform

**Notation.** Let $h$ be in $L^2_{k,n,e}(\mathbb{R})$ and let $\nu \in \mathbb{R}$. We denote by $\mathscr{P}^{k,n}_{h,\nu}$ the partial deformed Gabor transform defined by

$$\mathscr{P}^{k,n}_{h,\nu}(f) := \mathscr{G}_h^{k,n}(f)(., \nu), \quad \text{for all } f \in L^2_{k,n}(\mathbb{R}).$$

**Proposition 6.3.** *Let $h$ be in $L^2_{k,n,e}(\mathbb{R}) \cap L^\infty_{k,n}(\mathbb{R})$ and let $\nu \in \mathbb{R}$. The transformation $\mathscr{P}^{k,n}_{h,\nu}$ is a bounded linear operator from $W^s_{k,n}(\mathbb{R})$, $s \geq 0$, into $L^2_{k,n}(\mathbb{R})$, and there exist a positive constant $C_{k,n}(\nu, h)$ such that we have*

$$\|\mathscr{P}^{k,n}_{h,\nu}(f)\|_{L^2_{k,n}(\mathbb{R})} \leq C_{k,n}(\nu, h)\|f\|_{W^s_{k,n}(\mathbb{R})}, \quad f \in W^s_{k,n}(\mathbb{R}).$$

*Proof.* Using the relation (3.11) and Proposition 2.9 we obtain the result. □

Let $r > 0$, $s \geq 0$, $\nu \in \mathbb{R}$ and $h$ be in $L^2_{k,n,e}(\mathbb{R}) \cap L^\infty_{k,n}(\mathbb{R})$. We introduce the inner product in the space $W^s_{k,n}(\mathbb{R})$

$$\langle f, g\rangle_{\mathscr{P}^{k,n}_{h,\nu},r,W^s_{k,n}(\mathbb{R})} = r\langle f, g\rangle_{W^s_{k,n}(\mathbb{R})} + \langle \mathscr{P}^{k,n}_{h,\nu}(f), \mathscr{P}^{k,n}_{h,\nu}(g)\rangle_{L^2_{k,n}(\mathbb{R})}, \quad f, g \in W^s_{k,n}(\mathbb{R}).$$

The norm associated to the inner product is defined by:

$$\|f\|^2_{\mathscr{P}^{k,n}_{h,\nu},r,W^s_{k,n}(\mathbb{R})} := r\|f\|^2_{W^s_{k,n}(\mathbb{R})} + \|\mathscr{P}^{k,n}_{h,\nu}(f)\|^2_{L^2_{k,n}(\mathbb{R})}.$$

**Remark 6.5.** *Simple calculations give that the norms $\|.\|_{W^s_{k,n}(\mathbb{R})}$ and $\|.\|_{\mathscr{P}^{k,n}_{h,\nu},r,W^s_{k,n}(\mathbb{R})}$ are equivalent for $r > 0$ and $\nu \in \mathbb{R}$.*

**Proposition 6.4.** *Let $r > 0$, $\nu \in \mathbb{R}$, $s > \frac{(2k-1)n+2}{2n}$ and $h$ be in $L^2_{k,n,e}(\mathbb{R}) \cap L^\infty_{k,n}(\mathbb{R})$. Then the generalized Sobolev space $(W^s_{k,n}(\mathbb{R}), \langle ., .\rangle_{\mathscr{P}^{k,n}_{h,\nu},r,W^s_{k,n}(\mathbb{R})})$, possesses a reproducing kernel $\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}$ satisfying the identity*

$$\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}(., y) = \left(rI + (\mathscr{P}^{k,n}_{h,\nu})^*\mathscr{P}^{k,n}_{h,\nu}\right)^{-1}K_s(., y) \tag{6.13}$$

*where $(\mathscr{P}^{k,n}_{h,\nu})^* : L^2_{k,n}(\mathbb{R}) \longrightarrow W^s_{k,n}(\mathbb{R})$ is the adjoint operator of $\mathscr{P}^{k,n}_{h,\nu}$ given by*

$$\langle \mathscr{P}^{k,n}_{h,\nu}(f), g\rangle_{L^2_{k,n}(\mathbb{R})} = \langle f, (\mathscr{P}^{k,n}_{h,\nu})^*g\rangle_{W^s_{k,n}(\mathbb{R})}, \quad f \in W^s_{k,n}(\mathbb{R}),\ g \in L^2_{k,n}(\mathbb{R}).$$

*Moreover the kernel $\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}$ satisfies the following properties*

*(i) $\|\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}(., y)\|_{W^s_{k,n}(\mathbb{R})} \leq \frac{C(k,n,s)}{r}$, for all $y \in \mathbb{R}$.*

*(ii) $\|\mathscr{P}^{k,n}_{h,\nu}(\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}(., y))\|_{L^2_{k,n}(\mathbb{R})} \leq \frac{C(k,n,s)}{\sqrt{2r}}$, for all $y \in \mathbb{R}$.*

*(iii) $\|(\mathscr{P}^{k,n}_{h,\nu})^*\mathscr{P}^{k,n}_{h,\nu}(\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}(., y))\|_{W^s_{k,n}(\mathbb{R})} \leq C(k,n,s)$, for all $y \in \mathbb{R}$, where $C(k,n,s)$ is the constant given by (6.6).*

*Proof.* From Corollary 6.1, Proposition 6.3 and Remark 6.5, we deduce that the map $u \mapsto u(y)$, $y \in \mathbb{R}$, is a continuous linear functional on $(W^s_{k,n}(\mathbb{R}), \langle .,. \rangle_{\mathscr{P}^{k,n}_{h,\nu},r,W^s_{k,n}(\mathbb{R})})$. Thus from [51], $(W^s_{k,n}(\mathbb{R}), \langle .,. \rangle_{\mathscr{P}^{k,n}_{h,\nu},r,W^s_{k,n}(\mathbb{R})})$ has a reproducing kernel denoted by $\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}$. On the other hand, we have

$$\begin{aligned} f(y) &= \langle f, \mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}(.,y)\rangle_{\mathscr{P}^{k,n}_{h,\nu},r,W^s_{k,n}(\mathbb{R})} \\ &= r\langle f, \mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}(.,y)\rangle_{W^s_{k,n}(\mathbb{R})} + \langle \mathscr{P}^{k,n}_{h,\nu}(f), \mathscr{P}^{k,n}_{h,\nu}(\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}(.,y))\rangle_{L^2_{k,n}(\mathbb{R})} \\ &= \langle f, \Big(rI + (\mathscr{P}^{k,n}_{h,\nu})^* \mathscr{P}^{k,n}_{h,\nu}\Big)\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}(.,y)\rangle_{W^s_{k,n}(\mathbb{R})}. \end{aligned}$$

Thus,

$$(rI + (\mathscr{P}^{k,n}_{h,\nu})^* \mathscr{P}^{k,n}_{h,\nu})\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}(.,y) = K_s(.,y). \tag{6.14}$$

Furthermore, the previous identity implies that

$$\begin{aligned} &r^2 \|\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}(.,y)\|^2_{W^s_{k,n}(\mathbb{R})} + 2r\|\mathscr{P}^{k,n}_{h,\nu}(\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}(.,y))\|^2_{L^2_{k,n}(\mathbb{R})} + \|(\mathscr{P}^{k,n}_{h,\nu})^* \mathscr{P}^{k,n}_{h,\nu}(\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}(.,y))\|^2_{W^s_{k,n}(\mathbb{R})} \\ &= \|K_s(.,y)\|^2_{W^s_{k,n}(\mathbb{R})}. \end{aligned}$$

From this relation and using the fact that

$$\|K_s(.,y)\|_{W^s_{k,n}(\mathbb{R})} \leq C(k,n,s),$$

we obtain the properties (i), (ii) and (iii). □

**Remark 6.6.** *Using similar ideas as in Proposition 6.1, we prove that*

$$\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}(x,y) = \int_{\mathbb{R}} \frac{B_{k,n}((-1)^n\xi, x)B_{k,n}(\xi, y)}{r(1+|\xi|^2)^s + \tau^{k,n}_\nu(|h|^2)((-1)^n\xi)} d\gamma_{k,n}(\xi).$$

We can now state the main result of this paragraph.

**Theorem 6.3.** *Let $s > \frac{(2k-1)n+2}{2n}$ and $h$ be in $L^2_{k,n,e}(\mathbb{R}) \cap L^\infty_{k,n}(\mathbb{R})$.*

*(i) For any $g \in L^2_{k,n}(\mathbb{R})$ and for any $r > 0$, $\nu \in \mathbb{R}$ the best approximate function $f^*_{r,\nu,g}$ in the sense*

$$\inf_{f \in W^s_{k,n}(\mathbb{R})} \Big\{ r\|f\|^2_{W^s_{k,n}(\mathbb{R})} + \|g - \mathscr{P}^{k,n}_{h,\nu}(f)\|^2_{L^2_{k,n}(\mathbb{R})} \Big\} \tag{6.15}$$

*exists uniquely and it is represented by*

$$\forall\, y \in \mathbb{R}, \quad f^*_{r,\nu,g}(y) = \langle g, \mathscr{P}^{k,n}_{h,\nu}(\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}(.,y))\rangle_{L^2_{k,n}(\mathbb{R})}. \tag{6.16}$$

*(ii) The extremal function $f^*_{r,\nu,g}$ satisfies the following inequality:*

$$\forall\, y \in \mathbb{R}, \quad |f^*_{r,\nu,g}(y)| \leq \frac{C(k,n,s)}{\sqrt{2r}} \|g\|_{L^2_{k,n}(\mathbb{R})}.$$

*Proof.* (i) The existence and uniqueness of extremal function $f^*_{r,\nu,g}$ satisfying (6.15) is given by [52], and the extremal function $f^*_{r,\nu,g}$ is represented by

$$f^*_{r,\nu,g}(y) = \langle g, \mathscr{P}^{k,n}_{h,\nu}(\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}(.,y))\rangle_{L^2_{k,n}(\mathbb{R})}, \quad y \in \mathbb{R}.$$

(ii) From Proposition 6.4 (ii), we have

$$|f^*_{r,\nu,g}(y)| \leq \|g\|_{L^2_{k,n}(\mathbb{R})} \|\mathscr{P}^{k,n}_{h,\nu}(\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}(.,y))\|_{L^2_{k,n}(\mathbb{R})} \leq \frac{C(k,n,s)}{\sqrt{2r}} \|g\|_{L^2_{k,n}(\mathbb{R})}.$$

Thus the theorem is proved. □

**Corollary 6.3.** *Let $s > \frac{(2k-1)n+2}{2n}$ and $r > 0$, $\nu \in \mathbb{R}$. If $f$ is in $W^s_{k,n}(\mathbb{R})$ and $g = \mathscr{P}^{k,n}_{h,\nu}(f)$. Then*
*(i) $f(y) = \lim_{r\to 0^+} f^*_{r,\nu,g}(y)$, for all $y \in \mathbb{R}$.*
*(ii) $|f(y) - f^*_{r,\nu,g}(y)| \le C(k,n,s)\|f\|_{W^s_{k,n}(\mathbb{R})}$, for all $y \in \mathbb{R}$.*
*(iii) $|f^*_{r,\nu,g}(y)| \le C(k,n,s)\|f\|_{W^s_{k,n}(\mathbb{R})}$, for all $y \in \mathbb{R}$.*

*Proof.* Let $f$ be in $W^s_{k,n}(\mathbb{R})$.
(i) Then

$$\forall\, y \in \mathbb{R}, \quad f^*_{r,\nu,g}(y) = \langle f, (\mathscr{P}^{k,n}_{h,\nu})^* \mathscr{P}^{k,n}_{h,\nu}(\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}(.,y))\rangle_{W^s_{k,n}(\mathbb{R})}. \tag{6.17}$$

But from (6.14), we have

$$\forall\, y \in \mathbb{R}, \quad \lim_{r\to 0^+} (\mathscr{P}^{k,n}_{h,\nu})^* \mathscr{P}^{k,n}_{h,\nu}(\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}(.,y)) = K_s(.,y).$$

Thus

$$\lim_{r\to 0^+} f^*_{r,\nu,g}(y) = \langle f, K_s(.,y)\rangle_{W^s_{k,n}(\mathbb{R})} = f(y).$$

(ii) From (6.14) and (6.17), the extremal function $f^*_{r,\nu,g}$ satisfies

$$\forall\, y \in \mathbb{R}, \quad f^*_{r,\nu,g}(y) = f(y) - r\langle f, \mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}(.,y)\rangle_{W^s_{k,n}(\mathbb{R})}.$$

Thus by Proposition 6.4 (i) we obtain

$$\forall\, y \in \mathbb{R}, \quad |f^*_{r,\nu,g}(y) - f(y)| \le r\|f\|_{W^s_{k,n}(\mathbb{R})}\|\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}(.,y)\|_{W^s_{k,n}(\mathbb{R})} \le C(k,n,s)\|f\|_{W^s_{k,n}(\mathbb{R})}.$$

(iii) From (6.17) and Proposition 6.4 (iii), the extremal function $f^*_{r,\nu,g}$ satisfies

$$\forall\, y \in \mathbb{R}, \quad |f^*_{r,\nu,g}(y)| \le \|f\|_{W^s_{k,n}(\mathbb{R})}\|(\mathscr{P}^{k,n}_{h,\nu})^* \mathscr{P}^{k,n}_{h,\nu}(\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}(.,y))\|_{W^s_{k,n}(\mathbb{R})} \le C(k,n,s)\|f\|_{W^s_{k,n}(\mathbb{R})}.$$

□

**Remark 6.7.** *Let $s > \frac{(2k-1)n+2}{2n}$ and $r > 0$, $\nu \in \mathbb{R}$. If $\mathscr{P}^{k,n}_{h,\nu}$ is isometry (i.e. $(\mathscr{P}^{k,n}_{h,\nu})^* \mathscr{P}^{k,n}_{h,\nu} = Id$) then*
*(i) $\langle .,.\rangle_{\mathscr{P}^{k,n}_{h,\nu},r,W^s_{k,n}(\mathbb{R})} = (r+1)\langle .,.\rangle_{W^s_{k,n}(\mathbb{R})}$.*
*(ii) $\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}(x,y) = \frac{1}{r+1}K_s(x,y)$, for all $x, y \in \mathbb{R}$.*
*(iii) For all $y \in \mathbb{R}$, $f^*_{r,\nu,g}(y) = \frac{1}{r+1}(\mathscr{P}^{k,n}_{h,\nu})^* g(y)$, $g \in L^2_{k,n}(\mathbb{R})$.*
*(iv) For all $y \in \mathbb{R}$, $f^*_{r,\nu,\mathscr{P}^{k,n}_{h,\nu}(u)}(y) = \frac{1}{r+1}u(y)$, $u \in W^s_{k,n}(\mathbb{R})$.*

**Proposition 6.5.** *Let $s > \frac{(2k-1)n+2}{2n}$ and $h$ be in $L^2_{k,n,e}(\mathbb{R}) \cap L^\infty_{k,n}(\mathbb{R})$.*
i*) For any $g \in L^2_{k,n}(\mathbb{R})$ and for any $r > 0$, $\nu \in \mathbb{R}$, the best approximate function $f^*_{r,\nu,g}$ is represented by*

$$f^*_{r,\nu,g}(x) = \int_{\mathbb{R}} g(y)\overline{\mathscr{Q}^{k,n}_{r,\nu,h}(x,y)}d\gamma_{k,n}(y),$$

*where*

$$\mathscr{Q}^{k,n}_{r,\nu,h}(x,y) = \int_{\mathbb{R}} \frac{\sqrt{\tau^{k,n}_\nu(|h|^2)((-1)^n\xi)}B_{k,n}((-1)^n\xi,x)B_{k,n}(\xi,y)}{r(1+|\xi|^2)^s + \tau^{k,n}_\nu(|h|^2)((-1)^n\xi)}d\gamma_{k,n}(\xi). \tag{6.18}$$

ii*) If we take $g = \mathscr{P}^{k,n}_{h,\nu}(f)$, then*

$$\lim_{r\to 0^+} \|f^*_{r,\nu,g} - f\|_{W^s_{k,n}(\mathbb{R})} = 0.$$

*Moreover,* $\{f^*_{r,\nu,g}\}_{r>0}$ *converges uniformly to* $f$ *as* $r \to 0^+$.

iii) *Let* $\delta > 0$ *and let* $g$, $g_\delta$ *satisfy* $\|g - g_\delta\|_{L^2_{k,n}(\mathbb{R})} \leq \delta$. *Then*

$$\|f^*_{r,\nu,g} - f^*_{r,\nu,g_\delta}\|_{W^s_{k,n}(\mathbb{R})} \leq \frac{\delta}{2\sqrt{r}}.$$

*Proof.* i) By Remark 6.6 and Theorem 6.3 i), the infimum given by (6.16) is attained by a unique function $f^*_{r,\nu,g}$, and the extremal function $f^*_{r,\nu,g}$ is represented by

$$f^*_{r,\nu,g}(y) = \langle g, \mathscr{G}^{k,n}_h(\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}(.,y))(.,\nu)\rangle_{L^2_{k,n}(\mathbb{R})}, \quad y \in \mathbb{R},$$

where $\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}$ is the kernel given by Remark 6.6.
On the other hand we have

$$\mathscr{G}^{k,n}_h(f)(x,\nu) = \int_{\mathbb{R}} \sqrt{\tau^{k,n}_\nu(|h|^2)((-1)^n\xi)}\mathscr{F}_{k,n}(f)(\xi)B_{k,n}((-1)^n\xi, x)d\gamma_{k,n}(\xi), \quad \text{for all } x, \nu \in \mathbb{R}.$$

Using the properties of the kernel $\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}$ and the definition of the deformed Gabor transform, we get

$$\mathscr{G}^{k,n}_h\Big(\mathscr{K}_{\mathscr{P}^{k,n}_{h,\nu},r}(.,y)\Big)(.,\nu)(x) = \int_{\mathbb{R}} \frac{\sqrt{\tau^{k,n}_\nu(|h|^2)((-1)^n\xi)}B_k((-1)^n\xi, x)B_k(\xi, y)}{r(1+|\xi|^2)^s + \tau^{k,n}_\nu(|h|^2)((-1)^n\xi)}d\gamma_{k,n}(\xi) = \mathscr{Q}^{k,n}_{r,\nu,h}(x,y).$$

This gives (6.18).

ii) From (6.18) and Fubini's theorem we have

$$\mathscr{F}_{k,n}(f^*_{r,\nu,g})(\xi) = \frac{\sqrt{\tau^{k,n}_\nu(|h|^2)((-1)^n\xi)}\mathscr{F}_{k,n}(g)(\xi)}{r(1+|\xi|^2)^s + \tau^{k,n}_\nu(|h|^2)((-1)^n\xi)}.$$

Hence

$$\mathscr{F}_{k,n}(f^*_{r,\nu,g} - f)(\xi) = \frac{-r(1+|\xi|^2)^s\mathscr{F}_{k,n}(f)(\xi)}{r(1+|\xi|^2)^s + \tau^{k,n}_\nu(|h|^2)((-1)^n\xi)}.$$

Then we obtain

$$\|f^*_{r,\nu,g} - f\|^2_{W^s_{k,n}(\mathbb{R})} = \int_{\mathbb{R}} h_{r,\nu,s}(\xi)|\mathscr{F}_{k,n}(f)(\xi)|^2 d\gamma_{k,n}(\xi),$$

with

$$h_{r,\nu,s}(\xi) = \frac{r^2(1+|\xi|^2)^{3s}}{\Big(r(1+|\xi|^2)^s + \tau^{k,n}_\nu(|h|^2)((-1)^n\xi)\Big)^2}.$$

Since

$$\lim_{r\to 0} h_{r,\nu,s}(\xi) = 0$$

and

$$|h_{r,\nu,s}(\xi)| \leq (1+|\xi|^2)^s,$$

we obtain the result from the dominated convergence theorem.

iii) From (6.18) and Fubini's theorem we have

$$\mathscr{F}_{k,n}(f^*_{r,\nu,g})(\xi) = \frac{\sqrt{\tau^{k,n}_\nu(|h|^2)((-1)^n\xi)}\mathscr{F}_{k,n}(g)(\xi)}{r(1+|\xi|^2)^s + \tau^{k,n}_\nu(|h|^2)((-1)^n\xi)}. \tag{6.19}$$

Hence

$$\mathscr{F}_{k,n}(f^*_{r,\nu,g} - f^*_{r,\nu,g_\delta})(\xi) = \frac{\sqrt{\tau^{k,n}_\nu(|h|^2)((-1)^n\xi)}\mathscr{F}_{k,n}(g-g_\delta)(\xi)}{r(1+|\xi|^2)^s + \tau^{k,n}_\nu(|h|^2)((-1)^n\xi)}.$$

Using the inequality $(x+y)^2 \geq 4xy$, we obtain

$$(1+|\xi|^2)^s\Big|\mathscr{F}_{k,n}(f^*_{r,\nu,g} - f^*_{r,\nu,g_\delta})(\xi)\Big|^2 \leq \frac{1}{4r}|\mathscr{F}_{k,n}(g-g_\delta)(\xi)|^2.$$

Thus from Plancherel's formula (2.7) we obtain

$$\|f^*_{r,\nu,g} - f^*_{r,\nu,g_\delta}\|^2_{W^s_{k,n}(\mathbb{R})} \leq \frac{1}{4r}\|\mathscr{F}_{k,n}(g-g_\delta)\|^2_{L^2_{k,n}(\mathbb{R})} = \frac{1}{4r}\|g-g_\delta\|^2_{L^2_{k,n}(\mathbb{R})},$$

which gives the desired result. □

**Remark 6.8.** *(i) One of our motivations for introducing the theory of reproducing kernels for the best approximation problems involving the deformed Gabor transform is to push forward the connection between Gabor analysis and numerical analysis. We think of the results presented in this Section as opening potentially interesting studies by using computers and graphs, to illustrate numerical experiments approximation formulas for the limit case* $r \uparrow 0$.

*(ii) We note that we have studied these types of time-frequency analysis problems presented in the current paper and others for some integral transforms as the Dunkl Gabor transform on* $\mathbb{R}^d$*, the* $(k,a)$*-generalized wavelet transform on* $\mathbb{R}^d$*, the deformed Hankel Gabor transform on* $\mathbb{R}$*, the generalized Stockwell transforms and others integral transforms. (See as examples [42, 43, 44]).*

*(iii) We mention that we have studied the localization operators, the spectrograms and the scalograms respectively to the deformed Gabor, wavelet and Stockwell transforms. These studies have given some papers. We cite as examples [46, 47, 48].*

*(iv) Let h be in* $L^2_{k,n,e}(\mathbb{R})$*. We proceed as in [10], we define the modulation of h by ν otherwise, as follow:*

$$\mathscr{M}_\nu(h) := \mathscr{F}_{k,n}\left(\sqrt{\tau^{k,n}_\nu(|\mathscr{F}_{k,n}(h)|^2)}\right). \tag{6.20}$$

*Subsequently, we define the generalized Gabor transform* $\mathscr{V}^{k,n}_h$ *as follow:*

$$\forall\,(y,\nu)\in\mathbb{R}^2,\ \mathscr{V}^{k,n}_h(f)(y,\nu) := \int_{\mathbb{R}} f(x)\overline{\tau^{k,n}_{(-1)^n y}(\mathscr{M}_\nu(h))(y)}d\gamma_k(x) = f *_{k,n} \overline{r_n(\mathscr{M}_\nu(h))}(y). \tag{6.21}$$

*It is clear that*

$$\mathscr{V}^{k,n}_h = \mathscr{G}^{k,n}_{\mathscr{F}_{k,n}(h)}. \tag{6.22}$$

*Thus, by involving Plancherel's formula* (2.7)*, we derive that the two integral transforms are equivalent and then all results proved for one are valuables for the second. So, I reclame that all results proved in [45] and in this paper for the deformed Gabor transform* $\mathscr{G}^{k,n}_h$ *are valuables for the integral transform* $\mathscr{V}^{k,n}_h$ *and it is suffice to replace h by* $\mathscr{F}_{k,n}(h)$ *to derive the analogues results. Finally, I note and I insist that any adaptation of results proved for the deformed Gabor transform* $\mathscr{G}^{k,n}_h$ *in the context of the transformation* $\mathscr{V}^{k,n}_h$ *is a plagiarism (in particular results proved in [45, 47] and in the current paper), since I mentioned that the two transformations coincide modulo the formulas (6.22) and* (2.7).

# 7 Open Problem

In the present paper, we have successfully studied some problems of time-frequency analysis and reproducing kernel theory for the deformed Gabor transforms. The obtained results have a novelty and contribution to the literature. It is our hope that this work motivate the researchers to study the generalized translation operators associated with the $(k,a)$-generalized Fourier transform in the multi-dimensional case and for any positive real $a$. When we solve this open problem it is easy for extending the time-frequency analysis presented in the current paper to multi-dimensional signals.

# References

[1] M. Benedicks, *On Fourier transforms of functions supported on sets of finite Lebesgue measure.* J. Math. Anal. Appl., 1985; **106**: 180-183.

[2] A. Bansal, A. Kumar, *Heisenberg uncertainty inequality for Gabor transform.* J. Math. Inequal., 2016; **10**(3): 737-749.

[3] A. Bansal and A. Kumar, *Qualitative uncertainty principle for Gabor transform.* Bull. Korean Math. Soc., 2017; **54**(1): 71–84.

[4] S. Ben Saïd, T. Kobayashi and B. Ørsted, Laguerre semigroup and Dunkl operators. Compos. Math., 2012; **148**(04): 1265-1336.

[5] N. Ben Salem and A.R. Nasr, *Heisenberg-type inequalities for the Weinstein operator.* Integ. Transf. Special Funct., 2015; **26**(9): 700-718.

[6] MA. Boubatra, S. Negzaoui and M. Sifi, A New Product Formula Involving Bessel Functions. arXiv:2011.08104.

[7] P. Ciatti, F. Ricci and M. Sundari, Heisenberg-Pauli-Weyl uncertainty inequalities and polynomial volume growth. Adv. Math., 2007; **215**: 616-625.

[8] L. Cohen, Time-frequency distributions-a review. Proc IEEE, 1989; **77**: 941-981.

[9] D. Constales, H. De Bie and P. Lian, Explicit formulas for the Dunkl dihedral kernel and the $(\kappa,a)$-generalized Fourier kernel. J. Math. Anal. Appl., 2018; **460**(2): 900-926.

[10] W. Czaja and G. Gigante, Continuous Gabor transform for strong hypergroups. J Fourier Anal Appl., 2003; **9**: 321-339.

[11] H. De Bie, B. Ørsted, P. Somberg and V. Souček, Dunkl operators and a family of realizations of osp(1—2). Trans. Amer. Math. Soc., 2012; **364**: 3875-3902.

[12] D.L. Donoho, P.B. Stark, *Uncertainty principles and signal recovery.* SIAM J. Appl. Math., 1989; **49**: 906-931.

[13] C.F. Dunkl, *Differential-difference operators associated to reflection groups.* Trans. Am. Math. Soc., 1989; **311**: 167-183.

[14] C.F. Dunkl, *Hankel transforms associated to finite reflection groups.* in Proceedings of the special session on hypergeometric functions on domains of positivity, Jack polynomials and applications. (Tampa, FL, 1991), Contemp. Math., 1992; **138**: 123–138.

[15] W.G. Faris, Inequalities and uncertainty inequalities, Math. Phys., 1978; **19**:461-466.

[16] C. Fernandez and A. Galbis, *Annihilating sets for the short time Fourier transform.* Adv. Math., 2010; **224**(5): 1904–1926.

[17] C. Fernandez, A. Galbis, and J. Martinez, *Localization operators and an uncertainty principle for the discrete short time Fourier transform.* Abstr. Appl. Anal., 2014; Art. ID 131459, 6 pp.

[18] G.B. Folland and A. Sitaram, The uncertainty principle: a mathematical survey. J Fourier Anal Appl., 1997; **3**(3): 207-238.

[19] D. Gabor, Theory of communication. part 1: The analysis of information. Journal of the Institution of Electrical Engineers-Part III: Radio and Communication Engineering, 1946; **93**(26): 429-441.

[20] S. Ghobber and P. Jaming, *Uncertainty principles for integral orperators.* Studia Math., 2014; **220**: 197-220.

[21] D. Gorbachev, V. Ivanov and S. Tikhonov, *Pitt's inequalities and uncertainty principle for generalized Fourier transform.* Int. Math. Res. Notices, 2016; **23**: 7179-7200.

[22] K. Gröchenig, Aspects of Gabor analysis on locally compact abelian groups. Gabor analysis and algorithms, 211-231, Appl. Numer. Harmon. Anal., Birkhuser Boston, Boston, MA, 1998.

[23] K. Gröchenig, Foundations of time-frequency analysis. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Inc., Boston, MA, 2001.

[24] V. Havin and B. Jöricke, *The uncertainty principle in harmonic analysis. Volume 24. Berlin: Springer Verlag, 1994.*

[25] R. Howe, The oscillator semigroup. The mathematical heritage of Hermann Weyl (Durham, NC, 1987), 61-132, Proc. Sympos. Pure Math., 48, Amer. Math. Soc., Providence, RI, 1988.

[26] T.R. Johansen, Weighted inequalities and uncertainty principles for the $(k,a)$-generalized Fourier transform. Int. J. Math., 2016; **27**(3): 1650019.

[27] T. Kawazoe and H. Mejjaoli, *Uncertainty principles for the Dunkl transform.* Hiroshima Math. J., 2010; **40**(2): 241-268.

[28] T. Kobayashi and G. Mano, The Schrödinger model for the minimal representation of the indefinite orthogonal group $O(p,q)$. vi + 132pp. the Mem. Amer. Math. Soc. vol., **212**(1000), (2011).

[29] E. Malinnikova, Orthonormal sequences in $L^2(\mathbb{R})$ and time frequency localization. J. Fourier Anal. Appl., 2010; **16**(6): 983-1006.

[30] H. Mejjaoli and N. Sraieb, *Uncertainty principles for the continuous Dunkl wavelet transform and the Dunkl continuous Gabor transform.* Mediterr. J. Math., 2008; **5**: 443-466.

[31] H. Mejjaoli and N. Sraieb, *Gabor transform in quantum calculus and applications.* Fractional Calculus Appl. Anal., 2009; **12**(3): 319-336.

[32] H. Mejjaoli, Practical inversion formulas for the Dunkl-Gabor transform on $\mathbb{R}^d$. Integr. Transforms Spec. Funct., 2012; **23**(12): 875-890.

[33] H. Mejjaoli, *Uncertainty principles for the generalized Fourier transform associated to a Dunkl-type operator.* Appl Anal., 2017; **95**(9): 1930-1956.

[34] H. Mejjaoli, *Harmonic analysis associated with the generalized differential-difference operator on the real line and quantitative uncertainty principles for its Hartley transform.* Appl Anal., 2017; **96**(7): 1146-1169.

[35] H. Mejjaoli, Wavelet-multipliers analysis in the framework of the $k$-Laguerre theory. Linear Multilinear Algebra, 2017; **67**(1): 1-24.

[36] H. Mejjaoli, Spectral theorems associated with the $(k,a)$-generalized wavelet multipliers. J. Pseudo-Differ. Oper. Appl., 2018; **9**:735-762.

[37] H. Mejjaoli, $(k,a)$-generalized wavelet transform and applications. J. Pseudo-Differ. Oper. Appl., 2020; **11**: 55-92.

[38] H. Mejjaoli and K. Trimèche, $k$-Hankel two-wavelet theory and localization operators. Integr. Transforms Spec. Funct., 2020; **31**(8): 620-644.

[39] H. Mejjaoli, $k$-Hankel Gabor transform on $\mathbb{R}^d$ and its applications to the reproducing kernel theory. Complex Anal. Oper. Theory, 2021; **15**(14): 1-54.

[40] H. Mejjaoli, New uncertainty principles for the $(k,a)$-generalized wavelet transform. To appear in Revista de la Unin Matemtica Argentina.

[41] H. Mejjaoli, Time-frequency analysis associated with $k$-Hankel Gabor transform on $\mathbb{R}^d$. J. Pseudo-Differ. Oper. Appl. https://doi.org/10.1007/s11868-021-00399-7.

[42] H. Mejjaoli, Dunkl-Stockwell transform and its applications to the time-frequency analysis. J. Pseudo-Differ. Oper. Appl. https://doi.org/10.1007/s11868-021-00378-y

[43] H. Mejjaoli and K. Trimèche, Quantitative uncertainty principles associated with the $k$-generalized Stockwell transform. To appear in Mediterr. J. Math.

[44] H. Mejjaoli, Deformed Stockwell transform and applications on the reproducing kernel theory. To appear in International Journal of Reproducing Kernels.

[45] H. Mejjaoli, Generalized translation operator and quantitative uncertainty principles associated with the deformed Gabor transform. In Revision.

[46] H. Mejjaoli, Time-frequency analysis associated with the deformed Hankel wavelet transform. To appear in International Journal of Reproducing Kernels.

[47] H. Mejjaoli, Time-frequency concentration associated with the deformed Gabor transform on $\mathbb{R}$ and its applications to the approximation theory. In Revision.

[48] H. Mejjaoli, Localization operators associated with the deformed Stockwell transform and applications. In Revision.

[49] J.F. Price, Sharp local uncertainty principles, Studia Math., 1987; **85**:37-45.

[50] J.F. Price, A. Sitaram, Local uncertainty inequalities for locally compact groups, Trans. Amer. Math. Soc., 1988; **308**: 105-114.

[51] S. Saitoh, Theory of reproducing kernels and its applications. Longman Scientific Technical, Harlow, 1988.

[52] S. Saitoh, Reproducing Kernels and their Applications. Pitman Research Notes in Math. Series 369, Addison Wesley Longman, Harlow, 1997.

[53] D. Slepian, *Prolate spheroidal wave functions, Fourier analysis and uncertainty IV: Extensions to many dimensions, generalized prolate spheroidal functions.* Bell. System Tech. J., 1964; **43**: 3009-3057.

[54] D. Slepian, Some comments on Fourier analysis, uncertainty and modeling. SIAM Review. 1983; **25**(3): 379-393.

[55] K. Trimèche, Generalized Wavelets and Hypergroups. Gordon and Breach Science Publishers, 1997.

[56] S.B. Yakubovich, *Uncertainty principles for the Kontorovich-Lebedev transform.* Math. Model. Anal., 2008; **13**(2): 289-302.

[57] E. Wilczok, *New uncertainty principles for the continuous Gabor transform and the continuous Gabor transform.* Doc. Math., J., DMV (electronic) 2000; **5**:201-226.