

Size Ramsey Number for Matchings Versus Two Small Disconnected Graphs

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Abstract

The concept of the size Ramsey number of a graph was introduced by Erdős, Faudree, Rousseau, and Schelp in 1978. There are some published research papers about the size Ramsey number, (see [1], [2], [5], [6] and [7]), but there are only a few results about the size Ramsey number for a pair of graphs, see [3]. In this paper, we determine the size Ramsey number for three graphs, namely the size Ramsey number for matchings versus two small disconnected graphs.

Keywords: *size Ramsey number, matchings, disconnected graph.*

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1 Introduction

Throughout of this paper, all graphs which we consider are finite, undirected and simple. Let F, G, R and B be finite, undirected and simple graphs. The notation $F \rightarrow (G, R, B)$ means that if all edges of F are arbitrary colored

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by green or red or blue (called *green-red-blue coloring*), then there always exists either a green copy of G , a red copy of R , or a blue copy of B . The size Ramsey number of graphs G, R and B , denoted by $\hat{r}(G, R, B)$, is the smallest positive integer k such that there is a graph F with k edges satisfying $F \rightarrow (G, R, B)$. In this paper, we determine the size Ramsey number for three graphs, namely the size Ramsey number for matchings versus two small disconnected graphs including $\hat{r}(nK_2, 2K_2, 2K_2)$, $\hat{r}(nK_2, 2K_2, 2K_3)$, $\hat{r}(nK_2, 2K_2, 2P_3)$ and $\hat{r}(nK_2, 2K_2, 2K_{1,3})$ for any natural number $n \geq 2$.

2 Preliminaries

We first recall some basic concepts of graph theory, referring to [4]. A graph G consists of a finite nonempty set $V(G)$ of elements called *vertices* and a set $E(G)$ of two elements subsets of $V(G)$ called *edges*. We denote the number of vertices and edges of G by $|V(G)|$ and $|E(G)|$ and say that G has order $|V(G)|$ and size $|E(G)|$, respectively. If $e = \{u, v\}$ is an edge of a graph G , we write simply as uv or vu and say that u and v are *adjacent vertices* in G . We also say u and v are *joined* by e . In this case, the vertex u and the edge e (as well as v and e) are said to be *incident* with each other.

If the vertices of a graph G of order n can be labeled by $v_1, v_2, v_3, \dots, v_n$ and its edges are $v_1v_2, v_2v_3, \dots, v_{n-1}v_n$, then G is called a *path*. A path of order n is denoted by P_n .

A graph is *complete* if every two distinct vertices of G are adjacent. A complete graph of order n is denoted by K_n .

A graph G is a *bipartite* graph if $V(G)$ can be partitioned into two subsets U and W , called *partite sets*, such that every edge of G joins a vertex of U and a vertex of W . In case of G is a bipartite graph with partite sets U and W such that every vertex of U is adjacent to every vertex of W , G is called a *complete bipartite* graph. A complete bipartite graph with partite sets U and W such that $|U| = m$ and $|W| = n$ is denoted by $K_{m,n}$.

Let u and v be two distinct vertices in a graph G . A $u - v$ path is a sequence of vertices in G in which no vertices are repeated, beginning with u and ending at v such that consecutive vertices in the sequence are adjacent. We say the vertices u and v in a graph G are *connected* means that there is some $u - v$ path in G . A graph G is connected if every two vertices of G are connected. A graph G that is not connected is called a *disconnected* graph.

Let G_1, G_2, \dots, G_k , where $k \geq 2$, be graphs with mutually disjoint vertex sets. A graph $G_1 \cup G_2 \cup \dots \cup G_k$ means a graph which every vertex and every edge of it belong to exactly one of these graphs. In case of G_1, G_2, \dots, G_k are the same, we will write kG_1 instead of $G_1 \cup G_2 \cup \dots \cup G_k$. For $n \geq 2$, a *matching* which is denoted by nK_2 , is a graph consisting of $2n$ vertices and n independent edges.

3 Main results

To determine the size Ramsey number for matchings versus two small disconnected graphs, we need the following notations and definition.

Let F, G, R and B be finite, undirected and simple graphs. The notation $F \rightarrow (G, R, B)$ means that if all edges of F are arbitrary colored by either green, red or blue (called *green-red-blue coloring*), then there always exists either a green copy of G , a red copy of R or a blue copy of B in F . The notation $F \rightsquigarrow (G, R, B)$ means that there is some green-red-blue coloring of the edges of F such that F contains neither a green copy of G , a red copy of R nor a blue copy of B .

Example 3.1 $K_3 \rightarrow (P_3, K_2, K_2)$.

We can consider the green-red-blue coloring of K_3 into 7 cases.

Case 1: All edges of K_3 are colored in green. Then K_3 contains a green copy of P_3 .

Case 2: All edges of K_3 are colored in red. Then K_3 contains a red copy of K_2 .

Case 3: All edges of K_3 are colored in blue. Then K_3 contains a blue copy of K_2 .

Case 4: One edge of K_3 is colored in either red or blue and the remaining edges are colored in green. Then K_3 contains a green copy of P_3 .

Case 5: One edge of K_3 is colored in either green or blue and the remaining edges are colored in red. Then K_3 contains a red copy of K_2 .

Case 6: One edge of K_3 is colored in either green or red and the remaining edges are colored in blue. Then K_3 contains a blue copy of K_2 .

Case 7: Each edge of K_3 is colored with a distinct color (green, red, blue). Then K_3 contains a red copy of K_2 and a blue copy of K_2 .

In each case, we conclude that if all edges of K_3 are arbitrary colored by green-red-blue coloring, then there always exists either a green copy of P_3 , a red copy of K_2 or a blue copy of K_2 .

Example 3.2 $K_4 \rightsquigarrow (K_3, P_4, K_2)$.

Let us consider a green-red-blue coloring of K_4 . If two edges which have no common vertex of K_4 are colored in red and the remaining edges are colored in green, K_4 contains neither a green copy of K_3 , a red copy of P_4 nor a blue copy of K_2 . This means that $K_4 \rightsquigarrow (K_3, P_4, K_2)$.

Definition 3.1 Let G, R and B be finite, undirected and simple graphs. The size Ramsey number of graphs G, R and B , denoted by $\hat{r}(G, R, B)$, is the smallest positive integer k such that there is a graph F with k edges satisfying $F \rightarrow (G, R, B)$.

The following theorem is referring to [8].

Theorem 3.2 ([8]) *(The pigeonhole principle) Let n be a positive integer. If $n+1$ or more objects are distributed among n sets, then at least one of the sets must contain at least two objects.*

Example 3.3 $\hat{r}(2K_2, 2K_2, 2K_2) = 4$.

Here, we want to find a graph F with the least number of edges such that $F \rightarrow (2K_2, 2K_2, 2K_2)$. Since $2K_2$ is a disconnected graph, the graph F must be a matching of the form nK_2 where $n \geq 2$. Let F be the graph $4K_2$. Then $|E(F)| = 4$. Since each edge of F is arbitrary colored in either green, red or blue, at least two edges of F must be colored in the same color (by the pigeonhole principle). This means that there always exists either a green copy of $2K_2$, a red copy of $2K_2$ or a blue copy of $2K_2$ for arbitrary green-red-blue coloring of F . So $4K_2 \rightarrow (2K_2, 2K_2, 2K_2)$. Thus $\hat{r}(2K_2, 2K_2, 2K_2) \leq 4$. Next, we will show that $\hat{r}(2K_2, 2K_2, 2K_2) \geq 4$. Let G be a disconnected graph with $|E(G)| < 3$. It is easy to see that if each edge of F is colored by a distinct color (green, red, blue) then F contains neither a green copy of $2K_2$, a red copy of $2K_2$ nor a blue copy of $2K_2$. These mean $G \not\rightarrow (2K_2, 2K_2, 2K_2)$. Hence $\hat{r}(2K_2, 2K_2, 2K_2) = 4$.

In this paper, we determine the size Ramsey numbers involving matchings and two small disconnected graphs including $\hat{r}(nK_2, 2K_2, 2K_2)$, $\hat{r}(nK_2, 2K_2, 2K_3)$, $\hat{r}(nK_2, 2K_2, 2P_3)$ and $\hat{r}(nK_2, 2K_2, 2K_{1,3})$ for any natural number $n \geq 2$.

Theorem 3.3 *For any natural number $n \geq 2$, $\hat{r}(nK_2, 2K_2, 2K_2) = n + 2$.*

Proof Let n be a natural number where $n \geq 2$. We want to find a graph F with the least number of edges such that $F \rightarrow (nK_2, 2K_2, 2K_2)$. Since nK_2 and $2K_2$ are disconnected graphs, the graph F must be a matching of the form kK_2 where $k \geq 2$. Let F be the graph $(n+2)K_2$. Then $|E(F)| = n+2$. Let μ be any green-red-blue coloring of $(n+2)K_2$ that maximizes the number of either green edges, red edges or blue edges and contains neither a green copy of nK_2 , a red copy of $2K_2$ nor a blue copy of $2K_2$. We first color $n-1$ edges in green, one edge in red and one edge in blue. Then $(n+2)K_2$ contains at most $n-1$ green edges, one red edge and one blue edge. This means $n+1$ edges of $(n+2)K_2$ were colored and it still contains neither a green copy of nK_2 , a red copy of $2K_2$ nor a blue copy of $2K_2$. Finally, we color a remaining edge of $(n+2)K_2$ with either green, red or blue. Then $(n+2)K_2$ contains either a green copy of nK_2 , a red copy of $2K_2$ or a blue copy of $2K_2$. This means

$(n+2)K_2 \rightarrow (nK_2, 2K_2, 2K_2)$. Thus $\hat{r}(nK_2, 2K_2, 2K_2) \leq n+2$. Next, we will show that $\hat{r}(nK_2, 2K_2, 2K_2) \geq n+2$. Let G be a disconnected graph with $|E(G)| < n+2$. Let $n-1$ edges of G be colored in green, one edge be colored in red and a remaining edge be colored in blue. Then $G \rightsquigarrow (nK_2, 2K_2, 2K_2)$. Hence $\hat{r}(nK_2, 2K_2, 2K_2) = n+2$.

Theorem 3.4 *For any natural number $n \geq 2$, $\hat{r}(nK_2, 2K_2, 2K_3) = 3n+6$.*

Proof Let n be a natural number where $n \geq 2$. We want to find a graph F with the least number of edges such that $F \rightarrow (nK_2, 2K_2, 2K_3)$. Since nK_2 and $2K_3$ are disconnected graphs, the graph F which $F \rightarrow (nK_2, 2K_2, 2K_3)$ must be a graph of the form kK_3 where $k \geq 2$. Let F be the graph $(n+2)K_3$. Since K_3 is a graph with 3 edges (which we will call K_3 a component of $(n+2)K_3$), $|E(F)| = 3n+6$. Let μ be any green-red-blue coloring of $(n+2)K_3$ that maximizes the number of either green edges, red edges or blue edges and contains neither a green copy of nK_2 , a red copy of $2K_2$ nor a blue copy of $2K_3$. We first color $n-1$ components in green, one component in red and one component in blue. Then $(n+2)K_3$ contains at most $3n-3$ green edges, 3 red edges and 3 blue edges. This means $3n+3$ edges of $(n+2)K_3$ were colored and it still contains neither a green copy of nK_2 , a red copy of $2K_2$ nor a blue copy of $2K_3$. Finally, we color the last component of $(n+2)K_3$. We can see that no matter how we color it, $(n+2)K_3 \rightarrow (nK_2, 2K_2, 2K_3)$. Thus $\hat{r}(nK_2, 2K_2, 2K_3) \leq 3n+6$. Next, we will show that $\hat{r}(nK_2, 2K_2, 2K_3) \geq 3n+6$. Let G be a disconnected graph with $|E(G)| < 3n+6$. Let us consider $G = kK_2 \cup (n+1)K_3$ where $k \leq 2$ because it is possible to contains either a green copy of nK_2 , a red copy of $2K_2$ or a blue copy of $2K_3$ and $|E(G)| \leq 3n+5$. We first color $n-1$ components of $(n+1)K_3$ in green, one component of $(n+1)K_3$ in red and one component of $(n+1)K_3$ in blue. Finally, we color each edges of kK_2 in blue. Then $kK_2 \cup (n+1)K_3 \rightsquigarrow (nK_2, 2K_2, 2K_3)$. Hence $\hat{r}(nK_2, 2K_2, 2K_3) = 3n+6$.

Theorem 3.5 *For any natural number $n \geq 2$, $\hat{r}(nK_2, 2K_2, 2P_3) = 2n+4$.*

Proof Let n be a natural number where $n \geq 2$. We want to find a graph F with the least number of edges such that $F \rightarrow (nK_2, 2K_2, 2P_3)$. Since nK_2 and $2P_3$ are disconnected graphs, the graph F which $F \rightarrow (nK_2, 2K_2, 2P_3)$ must be a graph of the form kP_3 where $k \geq 2$. Let F be the graph $(n+2)P_3$. Since P_3 is a graph with 2 edges (which we will call P_3 a component of $(n+2)P_3$), $|E(F)| = 2n+4$. Let μ be any green-red-blue coloring of $(n+2)P_3$ that maximizes the number of either green edges, red edges or blue edges and contains neither a green copy of nK_2 , a red copy of $2K_2$ nor a blue copy of $2P_3$. We first color $n-1$ components in green, one component in red and one component in blue. Then $(n+2)P_3$ contains at most $2n-2$ green edges, 2 red edges and 2 blue edges. This means $2n+2$ edges of $(n+2)P_3$ were colored and it still contains

neither a green copy of nK_2 , a red copy of $2K_2$ nor a blue copy of $2P_3$. Finally, we color the last component of $(n+2)P_3$. We can see that no matter how we color it, $(n+2)P_3 \rightarrow (nK_2, 2K_2, 2P_3)$. Thus $\hat{r}(nK_2, 2K_2, 2P_3) \leq 2n+4$. Next, we will show that $\hat{r}(nK_2, 2K_2, 2P_3) \geq 2n+4$. Let G be a disconnected graph with $|E(G)| < 2n+4$. Let us consider $G = kK_2 \cup (n+1)P_3$ where $k \leq 2$ because it is possible to contains either a green copy of nK_2 , a red copy of $2K_2$ or a blue copy of $2P_3$ and $|E(G)| \leq 2n+3$. We first color $n-1$ component of $(n+1)P_3$ in green, one component of $(n+1)P_3$ in red, one component of $(n+1)P_3$ in blue. Finally, we color each edges of kK_2 in blue. Then $kK_2 \cup (n+1)P_3 \rightsquigarrow (nK_2, 2K_2, 2P_3)$. Hence $\hat{r}(nK_2, 2K_2, 2P_3) = 2n+4$.

Theorem 3.6 *For any natural number $n \geq 2$, $\hat{r}(nK_2, 2K_2, 2K_{1,3}) = 3n+6$.*

Proof Let n be a natural number where $n \geq 2$. We want to find a graph F with the least number of edges such that $F \rightarrow (nK_2, 2K_2, 2K_{1,3})$. Since nK_2 and $2K_{1,3}$ are disconnected graphs, the graph F which $F \rightarrow (nK_2, 2K_2, 2K_{1,3})$ must be a graph of the form $kK_{1,3}$ where $k \geq 2$. Let F be the graph $(n+2)K_{1,3}$. Since $K_{1,3}$ is a graph with 3 edges (which we will call $K_{1,3}$ a component of $(n+2)K_{1,3}$), $|E(F)| = 3n+6$. Let μ be any green-red-blue coloring of $(n+2)K_{1,3}$ that maximizes the number of either green edges, red edges or blue edges and contains neither a green copy of nK_2 , a red copy of $2K_2$ nor a blue copy of $2K_{1,3}$. We first color $n-1$ components in green, one component in red and one component in blue. Then $(n+2)K_{1,3}$ contains at most $3n-3$ green edges, 3 red edges and 3 blue edges. This means $3n+3$ edges of $(n+2)K_{1,3}$ were colored and it still contains neither a green copy of nK_2 , a red copy of $2K_2$ nor a blue copy of $2K_{1,3}$. Finally, we color the last component of $(n+2)K_{1,3}$. We can see that no matter how we color it, $(n+2)K_{1,3} \rightarrow (nK_2, 2K_2, K_{1,3})$. Thus $\hat{r}(nK_2, 2K_2, 2K_{1,3}) \leq 3n+6$. Next, we will show that $\hat{r}(nK_2, 2K_2, 2K_{1,3}) \geq 3n+6$. Let G be a disconnected graph with $|E(G)| < 3n+6$. Let us consider $G = kK_2 \cup (n+1)K_{1,3}$ where $k \leq 2$ because it is possible to contains either a green copy of nK_2 , a red copy of $2K_2$ or a blue copy of $2K_{1,3}$ and $|E(G)| \leq 3n+5$. We first color $n-1$ components of $(n+1)K_{1,3}$ in green, one component of $(n+1)K_{1,3}$ in red, one component of $(n+1)K_{1,3}$ in blue. Finally, we color each edges of kK_2 in blue. Then $kK_2 \cup (n+1)K_{1,3} \rightsquigarrow (nK_2, 2K_2, 2K_{1,3})$. Hence $\hat{r}(nK_2, 2K_2, 2K_{1,3}) = 3n+6$.

4 Open Problem

In this paper, we conclude that for any natural number $n \geq 2$, $\hat{r}(nK_2, 2K_2, 2K_2) = n+2$, $\hat{r}(nK_2, 2K_2, 2K_3) = 3n+6$, $\hat{r}(nK_2, 2K_2, 2P_3) = 2n+4$ and $\hat{r}(nK_2, 2K_2, 2K_{1,3}) = 3n+6$. There is a modified size Ramsey number, namely the connected size

Ramsey number. So our next aim is to determine the exact values of the connected size Ramsey number for matchings and two small disconnected graphs.

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