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Certain Sufficient Conditions for Meromorphic Starlike Functions

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Abstract

In the present paper, we obtain certain sufficient conditions for meromorphic functions $f(z) \in \Sigma$ in terms of certain differential inequalities to be meromorphically starlike of certain order.

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1 Introduction

Let Σ_n be the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=n-1}^{\infty} a_{n-1} z^{n-1},$$

which are analytic in the punctured unit disc $\mathbb{E}_0 = \mathbb{E} \setminus \{0\}$, where $\mathbb{E} = \{z : |z| < 1\}$. Note that for $n = 1, \Sigma_1 = \Sigma$.
A function $f \in \Sigma$ is said to be meromorphic starlike of order α if $f(z) \neq 0$ for $z \in \mathbb{E}_0$ and

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \qquad (\alpha < 1; z \in \mathbb{E}).$$

The class of such functions is denoted by $\mathcal{MS}^*(\alpha)$ and write $\mathcal{MS}^* = \mathcal{MS}^*(0)$ - the class of meromorphic starlike functions.
In the theory of meromorphic functions, many authors have obtained different sufficient conditions for meromorphically starlike functions.
Z-G Wang et al. [3] proved the following results:

Theorem 1.1. *If $f(z) \in \Sigma_p$ satisfies the following inequality*

$$\Re\left(\frac{zf'(z)}{f(z)} + \beta\frac{z^2f''(z)}{f(z)}\right) < \beta\lambda\left(\lambda + \frac{1}{2}\right) + \frac{1}{2}p\beta - \lambda \quad (\beta \geq 0, p - \frac{1}{2} \leq \lambda \leq p),$$

then $f \in \mathcal{MS}_p^*(\lambda)$.

Theorem 1.2. *If $f(z) \in \Sigma_p$ satisfies the following inequality*

$$\Re\left(\frac{f(z)}{f'(z)}\left(\frac{zf'(z)}{f(z)}\right)'\right) < \begin{cases} \dfrac{\gamma}{2(\gamma - p)}, & 0 \leq \gamma \leq \dfrac{p}{2}, \\ \dfrac{\gamma - p}{2\gamma}, & \dfrac{p}{2} \leq \gamma \leq 1. \end{cases}$$

then $f \in \mathcal{MS}_p^*(\gamma)$.

Goyal and Prajapat [1] proved the following results:

Theorem 1.3. *If $f(z) \in \Sigma$ satisfies the following inequality*

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2zf'(z)}{f(z)}\right| < \frac{3}{2},$$

then $f(z) \in \mathcal{M}S^*$.

Theorem 1.4. *If $f(z) \in \Sigma$ satisfies the following inequality*

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2zf'(z)}{f(z)} + 1\right| < \frac{1}{2},$$

then $f(z) \in \mathcal{M}S^*$.

The purpose of the present paper is to obtain certain sufficient conditions for meromorphically starlike functions. We, here, obtain the same in terms of differential inequalities.

2 Preliminaries

To prove our main results, we shall make use of following lemma [2].

Lemma 2.1. *Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that ϕ is a mapping from $\mathbb{C}^2 \times \mathbb{E}$ to $\mathbb{C}$ which satisfies $\phi(ix, y; z) \notin \Omega$ for $z \in \mathbb{E}$, and for real x, y such that $y \leq -n(1 + x^2)/2$. If the function $p(z) = 1 + c_nz^n + \ldots$ is analytic in $\mathbb{E}$ and $\phi(p(z), zp'(z); z) \in \Omega$ for all $z \in \mathbb{E}$, then $\Re p(z) > 0$.*

3 Main results

Theorem 3.1. *If $f(z) \in \Sigma_n$ satisfies*

$$\Re\left\{-\frac{zf'(z)}{f(z)}\left(\alpha\left(\frac{zf''(z)}{f'(z)}-2\frac{zf'(z)}{f(z)}\right)+1\right)\right\} > \alpha\beta(\beta+\frac{n}{2}-1)+\left(\beta-\frac{n\alpha}{2}\right),$$

then $f(z) \in \mathcal{MS}^*(\beta)$ where $z \in \mathbb{E}, \alpha \geq 0, \beta \leq 1$.

Proof. Define $p(z)$ by

$$(1-\beta)p(z)+\beta = -\frac{zf'(z)}{f(z)}.$$

Then $p(z) = 1 + c_n z^n + \ldots$ and is analytic in $\mathbb{E}$. A computation shows that

$$\frac{zf''(z)}{f(z)} - 2\frac{zf'(z)}{f(z)} = \frac{(1-\beta)zp'(z)+[(1-\beta)p(z)+\beta]^2-((1-\beta)p(z)+\beta)}{(1-\beta)p(z)+\beta}$$

and hence

$$-\frac{zf'(z)}{f(z)}\left\{\alpha\left(\frac{zf''(z)}{f'(z)}-2\frac{zf'(z)}{f(z)}\right)+1\right\}$$
$$= \alpha(1-\beta)zp'(z)+\alpha(1-\beta)^2p^2(z)+(1-\beta)(1+2\alpha\beta-\alpha)p(z)+\beta(\alpha\beta+1-\alpha)$$
$$= \phi(p(z), zp'(z); z),$$

where

$$\phi(r,s;t) = \alpha(1-\beta)s+\alpha(1-\beta)^2r^2+(1-\beta)(1+2\alpha\beta-\alpha)r+\beta(\alpha\beta+1-\alpha).$$

For all real x and y satisfying $y \leq -n(1+x^2)/2$, we have

$$\begin{aligned}\Re\phi(ix,y;z) &= \alpha(1-\beta)y-\alpha(1-\beta)^2x^2+\beta(\alpha\beta+1-\alpha)\\ &\leq -\frac{n\alpha(1-\beta)}{2}-\left[\frac{n\alpha}{2}(1-\beta)+\alpha(1-\beta)^2\right]x^2+\beta(\alpha\beta+1-\alpha)\\ &= -\frac{n\alpha(1-\beta)}{2}-\frac{\alpha(1-\beta)}{2}(n+2-2\beta)x^2+\beta(\alpha\beta+1-\alpha)\\ &\leq \beta(\alpha\beta+1-\alpha)-\frac{\alpha}{2}(1-\beta)n\\ &= \alpha\beta\left(\beta+\frac{n}{2}-1\right)+\left(\beta-\frac{n\alpha}{2}\right).\end{aligned}$$

Let $\Omega = \{w;\ \Re\, w > \alpha\beta\left(\beta+\frac{n}{2}-1\right)+\left(\beta-\frac{n\alpha}{2}\right)$. Then $\phi(p(z), zp'(z); z) \in \Omega$ and $\phi(ix,y;z) \notin \Omega$ for all real x and $y \leq -n(1+x^2)/2,\ z \in \mathbb{E}$. By Lemma 2.1, the result follows. □

Theorem 3.2. *Let* $0 \leq \beta < 1, a = (n/2+1-\beta)^2$ *and* $b = (n/2+\beta)^2$ *satisfy* $(a+b)\beta^2 < b(1-2\beta)$. *Let* t_0 *be the positive real root of the equation*

$$2a(1-\beta)^2t^2 + [3a\beta^2 + b(1-\beta)^2]t + [(a+2b)\beta^2 - b(1-\beta)^2] = 0$$

and

$$\rho^2 = \frac{(1-\beta)^3(1+t_0)^2(at_0+b)}{\beta^2 + (1-\beta)^2 t_0}.$$

If $f(z) \in \Sigma_n$ *satisfies*

$$\left|\left(\frac{zf''(z)}{f(z)} - 2\frac{zf'(z)}{f(z)}\right)\left(1 + \frac{zf'(z)}{f(z)}\right)\right| \leq \rho,\ z \in \mathbb{E},$$

then $f(z) \in \mathcal{MS}^*(\beta)$.

Proof. Define $p(z)$ by

$$(1-\beta)p(z) + \beta = -\frac{zf'(z)}{f(z)}.$$

Then $p(z) = 1 + c_n z^n + \ldots$ and is analytic in $\mathbb{E}$. A computation shows that

$$\frac{zf''(z)}{f(z)} - 2\frac{zf'(z)}{f(z)} = \frac{(1-\beta)zp'(z) + [(1-\beta)p(z)+\beta]^2 - ((1-\beta)p(z)+\beta)}{(1-\beta)p(z)+\beta}$$

and hence

$$\left(\frac{zf''(z)}{f(z)} - 2\frac{zf'(z)}{f(z)}\right)\left(1 + \frac{zf'(z)}{f(z)}\right)$$
$$= -\frac{(1-\beta)(p(z)-1)}{(1-\beta)p(z)+\beta}\left[(1-\beta)zp'(z) + ((1-\beta)p(z)+\beta)^2 - ((1-\beta)p(z)+\beta)\right]$$
$$\equiv \phi(p(z), zp'(z); z).$$

Then, for all real x and y satisfying $y \leq -n(1+x^2)/2$, we have

$$|\phi(ix, y; z)|^2$$
$$= \frac{(1-\beta)^2(1+x^2)}{\beta^2 + (1-\beta)^2x^2}\left\{[(1-\beta)y - \beta + \beta^2 - (1-\beta)^2x^2]^2 + [2\beta(1-\beta) - (1-\beta)]^2x^2\right\}$$
$$= \frac{(1-\beta)^2(1+t)}{\beta^2 + (1-\beta)^2t}\left\{[(1-\beta)y - \beta + \beta^2 - (1-\beta)^2t]^2 + [2\beta(1-\beta) - (1-\beta)]^2t\right\}$$
$$\equiv g(t, y),$$

where $t = x^2 > 0$ and $y \leq -n(1+t)/2$. Since

$$\frac{\partial g}{\partial y} = \frac{(1-\beta)^3(1+t)}{\beta^2 + (1-\beta)^2t}[(1-\beta)y - \beta + \beta^2 - (1-\beta)^2t]^2 < 0,$$

we have

$$g(t,y) \geq g\left(t, -\frac{n}{2}(1+t)\right) \equiv h(t).$$

Note that

$$h(t) = \frac{(1-\beta)^3(1+t)^2}{\beta^2+(1-\beta)^2 t}(at+b)$$

Also it is clear that $h'(-1) = 0$ and other two roots of $h'(t) = 0$ are given by

$$2a(1-\beta)^2t^2 + [3a\beta^2 + b(1-\beta)^2]t + [(a+2b)\beta^2 - b(1-\beta)^2] = 0$$

where $a = (n/2+1-\beta)^2$ and $b = (n/2+\beta)^2$. Since t_0 is the positive root of this equation we have $h(t) \geq h(t_0)$ and hence

$$|\phi(ix, y; z)|^2 \geq h(t_0).$$

Define $\Omega = \{w : |w| < \rho\}$.Then $\phi(p(z), zp'(z); z) \in \Omega$ and $\phi(ix, y; z) \notin \Omega$ for all real x and $y \leq -n(1+x^2)/2,\ z \in \mathbb{E}$. By Lemma 2.1, the result follows. $\square$

4 Deductions

Selecting $\beta = 0$ and $n = 1$ in Theorem 3.1, we have the following result.

Corollary 4.1. *If* $f(z) \in \Sigma$ *satisfies*

$$\Re\left\{-\frac{zf'(z)}{f(z)}\left(\alpha\left(\frac{zf''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)}\right) + 1\right)\right\} > -\frac{\alpha}{2},\ z \in \mathbb{E},$$

for some α $(\alpha \geq 0)$ *then* $f(z) \in \mathcal{MS}^*$.

Taking $\beta = \dfrac{\alpha}{2}$ and $n = 1$ in Theorem 3.1, we get

Corollary 4.2. *If* $f(z) \in \Sigma$ *satisfies*

$$\Re\left\{-\frac{zf'(z)}{f(z)}\left(\alpha\left(\frac{zf''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)}\right) + 1\right)\right\} > -\frac{\alpha^2}{4}(1-\alpha),\ z \in \mathbb{E},$$

for some α $(0 \leq \alpha \leq 2)$ *then* $f(z) \in \mathcal{MS}^*\left(\dfrac{\alpha}{2}\right)$.

Taking $n = 1, \beta = 0$ in Theorem 3.2, we have $t_0 = \frac{\sqrt{73}-1}{36}$ and we, therefore, obtain the following result:

Corollary 4.3. *If* $f(z) \in \Sigma$ *satisfies*

$$\left|\left(\frac{zf''(z)}{f(z)} - 2\frac{zf'(z)}{f(z)}\right)\left(1 + \frac{zf'(z)}{f(z)}\right)\right| < \rho,\ z \in \mathbb{E},$$

where $\rho^2 = \dfrac{827 + 73\sqrt{73}}{288}$, *then* $f(z) \in \mathcal{MS}^*$.

5 Open Problem

The sufficient conditions for starlikeness of meromorphic functions have been obtained in terms of differential operator $-\frac{zf'(z)}{f(z)}\left(\alpha\left(\frac{zf''(z)}{f'(z)}-2\frac{zf'(z)}{f(z)}\right)+1\right)$ only in case where $\alpha \geq 0$. The above conditions are still open for $\alpha < 0$.

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