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L-Fuzzy reflexive Banach Spaces of LG-Tangent Vectors of C^∞ L-Fuzzy Manifolds with L-Gradation of Openness

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Abstract

In this manuscript, first we introduce a new L-fuzzy topological space with L-gradation of openness and compare three kind of categories of all L-fuzzy topological spaces. Then we define C^∞ LG-fuzzy local one parameter groups acting on an LG-fuzzy manifold $(X, \mathfrak{T})$ which L denotes a complete distributive lattice, X is an L-fuzzy subset of a crisp set M and $\mathfrak{T}$ is an L-gradation of openness on X. We prove Existence theorem for LG-differential equations on $(1_{\mathbb{R}^n}, \mathfrak{T}_{Ln})$ and we show that each LG-fuzzy vector field Z on X determines an LG-fuzzy local one-parameter group (θ, W) of which, Z is the LG-infinitesimal generator and conversely. Next we introduce the consept of C^∞ LG-fuzzy covector fields on X and L-fuzzy tensors of order (k, l) on all points of M. In regards to the notion of an L-fuzzy vector, we define an LG^P norm on the LG-fuzzy tangent space $LGT_p(X)$ and prove that this space is an LG-fuzzy Banach space.

Keywords: *C^∞ LG-fuzzy local one parameter groups acting on an C^∞ LG-fuzzy n-manifold; LG-integral curve of an LG-fuzzy vector field on X; L-tensors on $LGT_p(X)$ of order (k, l); LG-fuzzy Banach space.*

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1 Introduction

The concept of fuzzy topological spaces was introduced in 1968 by C. L. Chang [3]. In 1985, A. P. Shostak [24], redefined in a somewhat different way a fuzzy topology. Later K. C. Chattopadhyay et. al. [4] introduced a concept of gradation of openness of a fuzzy set of X and V. Gregori and A. Vidal [8], defined fuzziness in Chang's fuzzy topological spaces. In consequence to the development of this kind of fuzzy topology, we considered in [14], a complete distributive lattice set $L =< L, \leq, \bigwedge, \bigvee, ' >$ with at least 2 elements, zero 0 and infinity 1 and we assumed that X is an L-fuzzy subset of of the crisp set M, in Goguen's sense [7], Then any L-fuzzy subset of M which is less than or equal to X is called an L-fuzzy subset of X and the set of such subsets is denoted by L_X^M and we defined an LG-fuzzy topological space $(X,\ \mathfrak{T})$, which $\mathfrak{T} : L_X^M \to L$, is an L-gradation of openness on X. This approach resulted into our definition of C^∞ LG-fuzzy manifolds [14], different from the concept of C^1 fuzzy manifolds introduced by M. Ferraro, D. H. Foster [6] and differential topology presented by J. W. Robbin, D. A. Salamon [18].

Since differential equations naturaly occure in many fields of mechanics and mathematical physics, there are many studies related to them. Differential equations in fuzzy setting are a natural way to model uncertainty of dynamical systems. As it's history, in 1984, E. Sanchez [22], concerned to the solution of fuzzy equations with extended operations. Then S. Seikkala [23], investigated the fuzzy initial value problems and P.E. Kloeden [12], developed Peano theorem for fuzzy differential equations. E. Hullermeier [9], approached to modelling and simulation of uncertain dynamical systems. Later S. Song et. al. investigated approximate solutions of the Cauchy problem of fuzzy differential equations in [27] and discussed the existence and uniqueness solutions of them in [26]. In 2000, P. Diamond [5], presented stability and periodicity in fuzzy differential equations. Fuzzy initial value problem for Nth-order linear differential equations was investigated by J.J. Buckley, T. Feuring [2], and more recently by S. Salahshour [21]. A. Jafarian, S. Measoomy Nia [10], gained new iterative method for solving linear Fredholm fuzzy integral equations of the second kind. In 2017, Y. M. Soliman [25] concerned the theory and applications to machine learning and manifold estimation

In the present manuscript, we base our work upon the already well-established L-fuzzy and LG-fuzzy structures in [14] and [20]. First we discuss three kinds of categories: LF-Top denote the category of all L-fuzzy topological spaces and continuous L-related functions and LGF-Top denote the category of all LG-fuzzy topological spaces and continuous LG-related functions in our sence. For each $r \in (0, 1]$, IGF_r-Top denote the category of all r-th graded I-fuzzy topological spaces and continuous I-related functions. We prove that for each $r \in (0, 1]$, IGF_r-Top is a full subcategory of IGF-Top and IF-Top and IGF_r-Top are isomorphic.

Next we define C^∞ LG-fuzzy local one parameter groups acting on an C^∞ LG-fuzzy n-manifold. We define LG-integral curve of an C^∞ LG-fuzzy vector field on X and obtain the existence and uniqueness theorems for the solution of an *LG*-differential equation. The fifth section is devoted to the C^∞ *LG*-fuzzy covector fields on X. Then we introduce $L\mathcal{T}^k_{lp}(X)$, the set of L-tensors on $LGT_p(X)$ of order (k, l). Finally in regards to the notion of an *LG*-fuzzy vector, we define locally the LG^P norm on $LGT_p(X)$, that reduces this space to *LG*-fuzzy Banach space.

2 Preliminaries

For the benefit of the reader we summarize some of the relevant background material and bring out requirement definitions and theorems from our previous works [13] and [14]:

Definition 2.1 *Let X be an L-fuzzy subset of M. If τ as a collection of L-fuzzy subsets of X, satisfies the following conditions, then (X, τ) is called an L-fuzzy topological space (L-fts):*

1) $X,\ \phi \in \tau$,

2) $A, B \in \tau \ \Rightarrow\ A \cap B \in \tau$.

3) $\{A_i\}_{i\in I} \subseteq \tau \ \Rightarrow\ \bigcup\limits_{i\in I} A_i \in \tau$,

Definition 2.2 *If* $\mathfrak{T} : L^M_X \to L$, *be a mapping satisfying:*

i) $\mathfrak{T}(X) = \mathfrak{T}(\tilde{0}) = 1$.

ii) $\mathfrak{T}(A \cap B) \geq \mathfrak{T}(A) \wedge \mathfrak{T}(B)$.

iii) $\mathfrak{T}(\bigcup_{j\in J} A_j) \geq \bigwedge_{j\in J} \mathfrak{T}(A_j)$

Then $\mathfrak{T}$ is called an L-gradation of openness on X and $(X,\ \mathfrak{T})$ is called an LG-fuzzy topological space (L-gfts).

Set $supp\mathfrak{T} = \{A \in L^M_X :\ \mathfrak{T}(A) > 0\}$, then A is called an *LG*-fuzzy open subset of X if $A \in supp\mathfrak{T}$.

Example 2.3 *Let $M = \mathbb{R}^n$ and $X = \tilde{1}$ be a constant fuzzy subset of M. Let $B(a,\ r,\ b)$ be an L-fuzzy subset of X, that is equal to zero outside or on the sphere $B(a,\ r)$ and equal to the function b with values in L, inside $B(a,\ r)$. We call the fuzzy topology induced by*

$$\beta_{Ln} = \{B(a,\ r,\ b),\ a \in \mathbb{R}^n,\ r \in \mathbb{R}^+,\ b : B(a, r) \to L\ is\ a\ function\}$$

the L-fuzzy Euclidean topology of dimension n and denote it by $\tau_{_{Ln}}$. Therefore we have the L-fuzzy Euclidean topological space $(1_{\mathbb{R}^n},\ \tau_{_{Ln}})$.

Example 2.4 *Let $M = \mathbb{R}^n$ and X be a constant fuzzy subset of M, equal to 1. As two useful examples we define*

$$\mathfrak{T}_{Ln} : I_X^M \to L \qquad \mathfrak{T}_{Ln}(B) = \begin{cases} 1 & B \in \tau_{_{Ln}}, \\ 0 & elsewhere. \end{cases} \tag{1}$$

and

$$\mathfrak{T}_{Linf} : L_X^M \to L, \quad \mathfrak{T}_{Linf}(B) = \begin{cases} 1 & B = \tilde{0} \\ inf\{B(x) : x \in M\} & \tilde{0} \neq B \in \tau_{_{Ln}} \\ 0 & elsewhere, \end{cases} \tag{2}$$

Let $\mathfrak{T}_{Ln}$ be any L-gradation of openness on $1_{\mathbb{R}^n}$, such that $supp\mathfrak{T} = \tau_{_{Ln}}$, then we call $(1_{\mathbb{R}^n}, \mathfrak{T}_{Ln})$ the LG-fuzzy Euclidean topological space.

Proposition 2.5 *Let $(X, \mathfrak{T})$ be an L-gfts. For any $r \in L$, we define $\mathfrak{T}_r = \{A \in L_X^M : \mathfrak{T}(A) \geq r\}$. Then $(X, \mathfrak{T}_r)$ is an L-fuzzy topologiacal space. For each $r \in L$, $\mathfrak{T}_r$ is called the r-level L-fuzzy topology on X with respect to the L-gradation of openness $\mathfrak{T}$.*

Corollary 2.6 *Two L-gradations of openness $\mathfrak{T}$ and $\mathfrak{T}'$ on X are equal iff $\mathfrak{T}_r = \mathfrak{T}'_r, \ \forall r \in L$.*

Definition 2.7 *If $\mathfrak{C} : L_X^M \to L$, satisfies the following conditions:*

i) $\mathfrak{C}(X) = \mathfrak{C}(\tilde{0}) = 1$.

ii) $\mathfrak{C}(A \cup B) \geq \mathfrak{C}(A) \wedge \mathfrak{C}(B)$.

iii) $\mathfrak{C}(\bigcap_{j \in J} A_j) \geq \bigwedge_{j \in J} \mathfrak{C}(A_j)$

Then $\mathfrak{C}$ is called an L-gradation of closedness on X.

Proposition 2.8 *Let $\mathfrak{C}$ and $\mathfrak{T}$ be L-gradations of closedness and openness respectively on X. Then*

i) The mapping $\mathfrak{T}_{\mathfrak{C}} : L_X^M \to L$, defined by $\mathfrak{T}_{\mathfrak{C}}(A) = \mathfrak{C}(X - A)$, is an L-gradation of openness on X, where $(X - A)$ is an L-fuzzy subset of M defined by $(X - A)(p) = X(p) - A(p)$.

ii) The mapping $\mathfrak{C}_{\mathfrak{T}} : L_X^M \to L$, defined by $\mathfrak{C}_{\mathfrak{T}}(A) = \mathfrak{T}(X - A)$, is an L-gradation of closedness on X.

iii) We have $\mathfrak{C}_{\mathfrak{T}_{\mathfrak{C}}} = \mathfrak{C}, \ \mathfrak{T}_{\mathfrak{C}_{\mathfrak{T}}} = \mathfrak{T}$.

Proposition 2.9 *Let $\mathfrak{M}_{\mathfrak{T}}(X)$ be the set of all L-gradations of openness on X. We write $\mathfrak{T}_1 \leq \mathfrak{T}_2$, if we have $\mathfrak{T}_1(A) \leq \mathfrak{T}_2(A), \ \ \forall A \in L_X^M$. Then $(\mathfrak{M}_{\mathfrak{T}}(X), \leq)$ is a complete lattice.*

From no on we assume that $M_1,\ M_2,\ M_3$ are three crisp sets and $X \in L^{M_1},\ Y \in L^{M_2},\ Z \in L^{M_3}$ such that $(X, \mathfrak{T}),\ (Y, \mathfrak{R}),\ (Z, \mathfrak{S})$ are LG-fuzzy topological spaces.

Definition 2.10 *Let $f : M_1 \to M_2$ be a function and $f[X]$ be an L-fuzzy subset of M_2, defined by $f[X](y) = \bigvee\{X(x) \mid x \in f^{-1}(y)\}$.*
If we have $f[X] \le Y$, then f is called an LG-related function from X to Y and the set of all these functions is denoted by $LGRf(X,\ Y)$.
Further more if we have $\mathfrak{R}(H) \le \mathfrak{T}(f^{-1}[H])$, for all LG-fuzzy subset H of Y, then f is an L-gradation preserving LG-related function so it is called an LGP-related function or LGP-related function from X to Y, or $f \in LGPRf(X,\ Y)$.

i) *f is called a one-to-one LG-related (LGP-related) function if $f|_{suppX} :\ suppX \to suppY$ is one-to-one.*

ii) *f is called a onto LG-related (LGP-related) function if $f[X] = Y$.*

Definition 2.11 *Let $f \in LGRf(X,\ Y)$ then*

i) *f is called LG-open if $f[A] \in supp\mathfrak{R} - \{\tilde{0}, Y\},\ \forall A \in supp\mathfrak{T} - \{\tilde{0}, X\}$ and $f[X] \in supp\mathfrak{R}$.*

ii) *f is called LG-continuous if $f^{-1}[H] \in supp\mathfrak{T} - \{\tilde{0}, X\},\ \forall H \in supp\mathfrak{R} - \{\tilde{0}, Y\}$ and $f^{-1}[Y] \in supp\mathfrak{T}$.*

iii) *f is called an LG-homeomorphism if is one -to -one, onto, LG-continuous, LG-open and $f^{-1} \in LGRf(Y,\ X)$.*

iv) *f is called an LGP-homeomorphism if is a bijective and $f,\ f^{-1}$ are LGP-related function.*

Definition 2.12 *An LG-fuzzy topological space $(X, \mathfrak{T})$ is an LG-fuzzy topological space of dimension n, if for any $x \in X$, there exists an LG-fuzzy open subset A of X such that $x \in A$ and $B \in \mathfrak{T}_{Ln}$ along with an LGP-homeomorphism $\psi \in LGPRf(A, B)$.*

Definition 2.13 *Let $\mathfrak{A} = \{(A_i,\ \psi_i)\ i \in K\}$ be a collection of LG-local coordinate neighborhoods. Since ψ_i is an LGP-homeomorphism for all $i \in K$, then for every $i, j \in K$, whenever $A_i \cap A_j \ne \phi$,*

$$\psi_j \circ\ \psi_i^{-1} : \psi_i(supp(A_i \cap A_j)) \to \psi_j(supp(A_i \cap A_j))$$

is a homeomorphism, called an LG-transition function.

$$\psi_j \circ \psi_i^{-1}(x_1^i,\ x_2^i,\ ...,\ x_n^i) = (x_1^j,\ x_2^j,\ ...,\ x_n^j)$$

If $\psi_i \circ \psi_j^{-1}$ and $\psi_j \circ \psi_i^{-1}$, changing the LG-local coordinates, are infinitely differentiable or C^∞, we shall say that $(A_i,\ \psi_i)$ is C^∞ compatible with $(A_j,\ \psi_j)$ when $A_i \cap A_j \ne \phi$.

Definition 2.14 *An LG-fuzzy topological space* $(X, \mathfrak{T})$ *is called an LG-fuzzy topological manifold of dimension* n*, if it satisfies the following conditions:*

i) X *is an LG-fuzzy topological space of dimension* n

ii) X *is a Hausdorff L-gfts.*

An C^∞ LG-fuzzy manifold is an LG-fuzzy topological manifold with an C^∞ LG-structure on it. For convenience, "LG-fuzzy manifold" will mean C^∞ LG-fuzzy manifold.
From no on we assume that $(X, \mathfrak{T})$, $(Y, \mathfrak{R})$ are two LG-fuzzy manifolds of dimensions m, n respectively.

Theorem 2.15 $(X\times Y,\ \mathfrak{T}\times\mathfrak{R})$ *is an LG-fuzzy manifold of dimension* $m+n$ *where For all* $A \in \mathfrak{T},\ D \in \mathfrak{R}$ *, we define*

$$(A \times D)(x, y) = A(x) \bigwedge D(y),\ \forall x \in M_1,\ y \in M_2$$

and

$$\mathfrak{T}\times\mathfrak{R}\ :\ I^{M\times N}_{X\times Y} \to L, \quad (\mathfrak{T}\times\mathfrak{R})(A\times D) = \begin{cases} \mathfrak{T}(A) \wedge \mathfrak{R}(D) & A \in \mathfrak{T},\ D \in \mathfrak{R} \\ 0 & elsewhere. \end{cases}$$

Definition 2.16 *Let* M *be a crisp set and* $\sim$ *be an equivalence relation on it. If* A *is an L-fuzzy subset of* M *such that* $A(y) = A(x)$ *whenever* $y \sim x$*, then we define the L-fuzzy subset:*

$$\frac{A}{\sim} : \frac{M}{\sim} \to L, \quad \frac{A}{\sim}([x]) = A(x),\ \forall x \in M.$$

where $[x] = \{y \mid x \sim y\}$*. Since* $A \leq X$*, thus* $\frac{A}{\sim} \leq \frac{X}{\sim}$ *and hence* $\frac{A}{\sim} \in L^{\frac{M}{\sim}}_{\frac{X}{\sim}}$*.*

Theorem 2.17 *Let* $(X, \mathfrak{T})$ *be an LG-fuzzy topological space, such that* $X(y) = X(x)$ *whenever* $y \sim x$*, then* $(\frac{X}{\sim}, \frac{\mathfrak{T}}{\sim})$ *is an LG-fuzzy topological space, called the LG-fuzzy quotient space, where*

$$\frac{\mathfrak{T}}{\sim} :\ L^{\frac{M}{\sim}}_{\frac{X}{\sim}} \to L, \quad \frac{\mathfrak{T}}{\sim}(\frac{A}{\sim}) = \mathfrak{T}(A)$$

Definition 2.18 *Consider an LG-fuzzy quotient space* $(\frac{X}{\sim}, \frac{\mathfrak{T}}{\sim})$*. An equivalence relation* $\sim$ *is called an LG-open relation if for each fuzzy subset* $A \in supp\mathfrak{T}$ *we have* $\frac{A}{\sim} \in supp\frac{\mathfrak{T}}{\sim}$*.*

Theorem 2.19 *Let $(X, \mathfrak{T})$ be an LG-fuzzy manifold and $\sim$ be an LG-open relation. Then $(\frac{X}{\sim}, \frac{\mathfrak{T}}{\sim})$ is an LG-fuzzy topological space of dimension n called LG-fuzzy quotient topological space of dimension n.*

Example 2.20 *Let $M = \mathbb{R} \times \mathbb{R}^2$ and $L = \mathbb{N} \cup \{\infty\}$. Consider the L-fuzzy subset $X : M \to L$ defined by $X(t, x, y) = \lfloor t + xy \rfloor$ and L-gradation of openness $\mathfrak{T} : L_X^M \to L$ defined by $\mathfrak{T}(A) = sup\{A(t, x, y) : (t, x, y) \in M\}$. We define $(t_1, x_1, y_1) \sim (t_2, x_2, y_2)$ iff*

$$\textit{either} \quad y_1 = y_2 \neq 0; \;\; t_1 + x_1y_1 = t_2 + x_2y_2$$

$$\textit{or} \quad y_1 = y_2 = 0; \;\; t_1 = t_2; \;\; x_1 = x_2.$$

Then $\sim$ is an equivalence relation.
For $y \neq 0$ we have $(0, x, y) \sim (t, x - \frac{t}{y}, y)$, however, each point $(x, 0)$ on the x-axis gets replaced by the uncountable set $\mathbb{R} \times \{(x, 0)\}$.
Since we have $X\big((t_1, x_1, y_1)\big) = X\big((t_2, x_2, y_2)\big)$ whenever $(t_1, x_1, y_1) \sim (t_2, x_2, y_2)$, using Theorem 2.17, we have the LG-fuzzy quotient space $(\frac{X}{\sim}, \frac{\mathfrak{T}}{\sim})$. Also it is an LG-fuzzy manifold with LG-local coordinate neighborhoods (A_t, φ_t) where $A_t\big([(r, x, y)]\big) = \begin{cases} \infty & r = t \\ 1 & \textit{elsewhere}, \end{cases}$

$$suppA_t = \{\ [(t, x, y)] \mid (x, y) \in \mathbb{R}^2\ \} \quad \varphi_t\big([(t, x, y)]\big) = (x, y)$$

Hence $(\frac{X}{\sim}, \frac{\mathfrak{T}}{\sim})$ has an uncountable LG-atlas.

Definition 2.21 *An LG-related function $F \in LGRf(U, Y)$ is an C^∞ LG-related function if for every $p \in U$,*

$$\hat{F} = \varphi \circ F \circ \psi^{-1} : \psi(supp(A \cap U)) \;\to\; \varphi(suppB)$$

is C^∞ where (A, ψ), (B, φ) are LG-local coordinate neighborhoods of p and $F(p)$ respectively.
$F \in LGRf(X, Y)$ is called an LG-diffeomorphism if it is an LG-homeomorphism and F and F^{-1} are C^∞.

More precisely, if $\psi(q) = (x_1, \ldots, x_n)$, $\forall q \in supp(A \cap U)$ and $\varphi(w) = (y_1, \ldots, y_m)$, $\forall w \in B$, then

$$\hat{F}(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n))$$

and each $y_i = f_i(x_1, \ldots, x_n)$ is C^∞ on $\psi(A)$.

Definition 2.22 *An LG-related function* $F = LGRF(U,Y)$ *is an LG-related function if for every* $p \in U$*,*

$$\hat{F} = \varphi \circ F \circ \psi^{-1} : \psi(supp(A \cap U)) \ \rightarrow \ \varphi(suppB)$$

is C^∞ *where* (A,ψ)*,* (B,φ) *are LG-local coordinate neighborhoods of* p *and* $F(p)$ *respectively.* $F \in LGRf(X,Y)$ *is called a diffeomorphism if it is a homeomorphism and* F^{-1} *is* C^∞*.*

More precisely, if $\psi(q) = (x_1, \ ..., \ x_n)$, $\forall q \in supp(A \cap U)$ and $\varphi(w) = (y_1, \ ..., \ y_m)$, $\forall w \in B$, then

$$\hat{F}(x_1, \ ..., \ x_n) = (f_1(x_1, \ ..., \ x_n), \ ..., \ f_n(x_1, \ ..., \ x_n))$$

and each $y_i = f_i(x_1, \ ..., \ x_n)$ is C^∞ on $\psi(A)$.

Theorem 2.23 *(LG-rank theorem) Let* $F \in LGRf(U,Y)$ *be an* C^∞ *fuzzy mapping and* $\operatorname{rank} F = k$ *at every point of* X*. If* $p \in X$*, then there exist LG-local coordinate neighborhoods* (A,ψ)*,* (B,φ) *such that*

$$\psi(p) = (0, \ldots, 0) \in \mathbb{R}^n, \ \ \varphi(F(p)) = (0, \ldots, 0) \in \mathbb{R}^m$$

and $\hat{F} = \varphi \circ F \circ \psi^{-1}$ *is given by:*

$$\hat{F}(x_1, \ \ldots, \ x_n) = (x_1, \ \ldots, \ x_k, 0, \ldots, 0) \qquad (*)$$

Remark 2.24 *We can cover* X *and* $\tilde{X} = F[X]$ *by these LG-local coordinate neighborhoods* $\mathfrak{A} = \{(A_s, \psi_s)| \ s \in S\}$*, and* $\mathfrak{D} = \{(D_s, \varphi_s)| \ s \in S\}$ *respectively where* $S \subseteq K$*. Since* $\mathfrak{A}$ *is an LG-structure of* X*, one can show that* $\mathfrak{D}$ *is an LG-structure of* $F[X]$*. If* F *is an LG-diffeomorphism, then we have* $\operatorname{rank} F = \dim X = \dim Y$*.*

Definition 2.25 *The* C^∞ *L-related function* $F \in LGRf(X,Y)$ *is an LG-immersion(LG-submersion), if* $\operatorname{rank} F = dimX (= dimY)$ *at every point of*X*.*

Theorem 2.26 *Let* $F \in LGRf(X,Y)$ *be an* C^∞ *LG-related function. If* F *is an injective LG-immersion, then* $(\tilde{X}, \tilde{\tau})$ *is an LG-fuzzy submanifold of dimension* n*, called an LG-immersed fuzzy submanifold and* $F \in LGRf(X, \tilde{X})$ *is an LG-diffeomorphism.*

3 Main results

Category of LG-topological spaces

Let LF-Top denote the category of all L-fuzzy topological spaces and continuous L-related functions and LGF-Top denote the category of all LG-fuzzy topological spaces and LGP-related functions in our sence and for each $r \in (0, 1]$, LGF_r-Top denote the category of all r-th graded L-fuzzy topological spaces and continuous L-related functions.

Proposition 3.1 *Let (X, τ) be an L-fuzzy topological space. Define for each $r \in L$, a mapping $\tau^r : L_X^M \to L$, by the rule:* $\tau^r(A) = \begin{cases} 1 & A = X \ or \ \tilde{0} \\ r & A \in \tau - \{X, \tilde{0}\} \\ 0 & otherwise. \end{cases}$
Then τ^r is an L-gradation of openness on X, such $(\tau^r)_r = \tau$.
τ^r is called an r-th L-gradation on X and (X, τ^r) is called an r-th LG-fuzzy topological space.

The proof is straightforward.

Proposition 3.2 *Let $(X, \mathfrak{T})$ be an I-fuzzy topological space and $\{\mathfrak{T}_r : r \in I\}$ be the family of all r-level I-fuzzy topologies with respect to $(X, \mathfrak{T})$. Then this family is a descending family and for each $r \in I$, $\mathfrak{T}_r = \bigcap_{s<r} \mathfrak{T}_s$.*

Proof If $r > s$, then obviously, $\mathfrak{T}_r \subset \mathfrak{T}_s$. Hence $\{\mathfrak{T}_r : r \in I\}$ is a descending family of I-fuzzy topologies. Clearly
$0 < r \in I$, $\mathfrak{T}_r \subseteq \bigcap_{s<r} \mathfrak{T}_s$. Also, if $A \notin \mathfrak{T}_r$ then $\mathfrak{T}(A) < r$. So $\exists s \in (0, 1]$ such that $\mathfrak{T}(A) < s < r$. So $A \notin \mathfrak{T}_s$ for some $s < r$. Hence $A \notin \bigcap_{s<r} \mathfrak{T}_s$. Hence $\bigcap_{s<r} \mathfrak{T}_s \subseteq \mathfrak{T}_r$. This completes the proof.

Proposition 3.3 *Let $\{\ T_r : r \in (0, 1]\ \}$ be a nonempty descending family of I-fuzzy topologies on X, and let $\mathfrak{T} : I_X^M \to I$ be a mapping defined by $\mathfrak{T}(A) = \bigvee\{\ r \mid r \in (0, 1],\ A \in T_r\ \}$. Then $\mathfrak{T}$ is an I-gradation of openness on X. If for any $r \in (0, 1]$, $T_r = \bigcap_{s<r} T_s$ then $\mathfrak{T}_r = T_r$, holds for all $r \in (0, 1]$.*

Proof Since $\forall r \in (0, 1]$, T_r is an I-fuzzy topology on X, then $\tilde{0}$, $X \in T_r$, , $\forall r \in (0, 1]$. Hence $\mathfrak{T}(\tilde{0}) = \mathfrak{T}(X) = 1$. Let $\mathfrak{T}(A_i) = n_i$. If $n_i = 0$ for some $i = 1, 2$, then obviously $\mathfrak{T}(A_1 \cap A_2) \geq \mathfrak{T}(A_1) \wedge \mathfrak{T}(A_2)$.
Suppose $n_i > 0,\ for\ i = 1, 2$. Let $s < n_i,\ for\ i = 1, 2$. Then $\forall \varepsilon > 0$, $\exists\ r_1,\ r_2 \in (0, 1]$ such that $n_i - \varepsilon < r_i < n_i$ and $A_i \in T_{r_i}$, $i = 1, 2$. Let $r = min\{r_l, r_2\}\ and\ n = min\{n_1, n_2\}$. Then $A_1, A_2 \in T_r$, hence
$\mathfrak{T}(A_1 \cap A_2) \geq r > n - \varepsilon > s - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, therefore
$\mathfrak{T}(A_1 \cap A_2) \geq s$. This follows that $\mathfrak{T}(A_1 \cap A_2) \geq \mathfrak{T}(A_1) \wedge \mathfrak{T}(A_2)$

Now suppose $\mathfrak{T}(A_j) = k_j,\ j \in J$ and $k = \bigwedge_{j\in J} k_j$. If $k = 0$, then clearly $\mathfrak{T}(\ \bigcup_{j\in J} A_j\ ) \geq \bigwedge_{j\in J} \mathfrak{T}(A_j)$.
If $l > 0$, choose $l > \varepsilon > 0$. Then $l_j > l-\varepsilon > 0$, $\forall j \in J$. So, $A_j \in T_{k-\varepsilon}$, $\forall j \in J$. Since $T_{k-\varepsilon}$ is an I-fuzzy topology, $\bigcup_{j\in J} A_j \in T_{k-\varepsilon}$. Thus $\mathfrak{T}(\ \bigcup_{j\in J} A_j\ ) \geq k-\varepsilon$. Since $\varepsilon > 0$ is arbitrary, then

$$\mathfrak{T}(\ \bigcup_{j\in J} A_j\ ) \geq k = \bigwedge_{j\in J} \mathfrak{T}(A_j).$$

Hence it follows that $\mathfrak{T}$ is an I-gradation of openness on X.
Next assume that for any $r \in (0,1]$, $T_r = \bigcap_{s<r} T_s$. Let $A \in T_r$. Then, $\mathfrak{T}(A) \geq r$. Hence $A \in \mathfrak{T}_r$.
Conversely, if $A \in \mathfrak{T}_r$. then $\mathfrak{T}(A) > r$ which implies

$$\bigvee \{t \mid t \in (0,1],\ A \in T_t\} = s > r.$$

Therefore, for any $\varepsilon > 0$, there exists $l \in (0,1]$ such that $s-\varepsilon < l$ and $A \in T_l$. Hence, we have $r-\varepsilon < s-\varepsilon < l$ and $A \in T_l$. Thus $A \in T_{r-\varepsilon}$. Since $\varepsilon > 0$ is arbitrary, then $A \in \bigcap_{\varepsilon>0} \mathfrak{T}_{r-\varepsilon}$. So $A \in T_r$. This completes the proof.

Proposition 3.4 *Let $F : M_1 \to M_2$ and $G : M_2 \to M_3$ be two functions such that F is an LGP-related function from X to Y and G is an LGP-related function from Y to Z. Then $G \circ F$ is an LGP-related function from X to Z.*

Proof First we show that $(G \circ F)[X] \leq Z$:

$$\begin{aligned}(G\circ F)[X](z) &= \bigvee \{X(x) \mid (G\circ F)(x) = z\ \}\\ &= \bigvee \{X(x) \mid y = F(x)\ for\ some\ y \in G^{-1}(z)\}\\ &\leq \bigvee \{Y(y) \mid G(y) = z\}\\ &= G[Y](z) \leq Z(z)\end{aligned}$$

Next we prove that $\mathfrak{S}(K) \leq \mathfrak{T}((G\circ F)^{-1}[K])$, for all LG-fuzzy open subset K of Z: Let K be an LG-fuzzy open subset of Z. Since G is an LGP-mapping, hence $\mathfrak{S}(K) \leq \mathfrak{R}(G^{-1}[K])$. Since $G^{-1}[K]$ is an LG-fuzzy open subset of Y and F is an LGP-mapping, then

$$\mathfrak{S}(K) \leq \mathfrak{R}(G^{-1}[K]) \leq \mathfrak{T}\big(F^{-1}(G^{-1}[K])\big) = \mathfrak{T}((G \circ F)^{-1}[K]).$$

This completes the proof.

Proposition 3.5 *The function $F : M_1 \to M_2$ is an LGP-related function from X to Y iff $F : (X, \mathfrak{T}_r) \to (Y, \mathfrak{R}_r)$ is an L-continuous for all $r \in L$.*

Proof Suppose $F \in LGPRf(X, Y)$. Take $G \in \mathfrak{R}_r$. Then $r \leq \mathfrak{R}(G) \leq \mathfrak{T}(F^{-1}[G])$. Hence $F^{-1}[G] \in \mathfrak{T}_r$. Therefore F is an L-continuous. Conversely, suppose $F : (X, \mathfrak{T}_r) \to (Y, \mathfrak{R}_r)$ is an L-continuous for all $r \in L$. Let G be an L-fuzzy subset of Y, $\mathfrak{R}(G) = r$. If $r = 0$ then clearly $0 = \mathfrak{R}(G) \leq \mathfrak{T}(f^{-1}[G])$. If $r \neq 0$ then $G \in \mathfrak{R}_r$ by continuity of F, implies that $F^{-1}[G] \in \mathfrak{T}_r$. Hence $\mathfrak{T}(F^{-1}[G]) \geq r = \mathfrak{R}(G)$. This completes the proof.

Proposition 3.6 *Let (X, τ) and (Y, ϱ) be two L-fuzzy topological spaces and $F : M_1 \to M_2$ be a function which $F[X] \leq Y$. Then $F : (X, \tau) \to (Y, \varrho)$ is an L-continuous iff $F : (X, \tau^r) \to (Y, \varrho^r)$ is an LGP-related function for all $r \in L$.*

Proof Suppose $F : (X, \tau) \to (Y, \varrho)$ is an L-continuous. Take $H \in L_Y^{M_2}$. Then we have four possibilities:

i) $H = \tilde{0}$. Since $1 = \varrho^r(\tilde{0})$ and $\tau^r(F^{-1}[\tilde{0}]) = \tau^r(\tilde{0}) = 1$, then $\varrho^r(\tilde{0}) \leq \tau^r(F^{-1}[\tilde{0}])$.

ii) $H = Y$. We prove that $F^{-1}[Y] = X$:
Since $Y(y) \geq F[X](y) = \bigvee\{X(t) \mid F(t) = y\}$, hence

$$F^{-1}[Y](x) = Y(F(x)) \geq F[X](F(x)) = \bigvee\{X(t) \mid F(t) = F(x)\} \geq X(x).$$

Since $Y \in \varrho$ and F is an L-continuous, we have $F^{-1}[Y] \in \tau$. Hence $F^{-1}[Y] \leq X$. Therefore $F^{-1}[Y] = X$. Then $1 = \varrho^r(Y)$ and $\tau^r(F^{-1}[Y]) = \tau^r(X) = 1$ implies that $\varrho^r(Y) \leq \tau^r(F^{-1}[Y])$.

iii) $H \in \varrho - \{\tilde{0}, Y\}$. Since F is an L-continuous, hence $F^{-1}(H) \in \tau - \{\tilde{0}, X\}$. Therefore by Proposition 3.1, we have $\varrho^r(H) = r$ and $\tau^r(F^{-1}[H]) = r$. Thus $\varrho^r(H) \leq \tau^r(F^{-1}[H])$.

iv) $H \notin \varrho$. Hence $\varrho^r(H) = 0$. Therefore $0 = \varrho^r(H) \leq \tau^r(F^{-1}[H])$.

Consequently $F : (X, \tau^r) \to (Y, \varrho^r)$ is an LGP-related function for all $r \in L$. The converse part follows from Propositions 3.1, 3.2, 3.3 and 3.5.

Proposition 3.7 *Let $(X, \mathfrak{T})$ be an IG-fuzzy topological space and $F : M_1 \to M_2$ be a mapping such that $F[X] \leq Y$. Let $\{R_r : r \in (0,1]\}$ be a nonempty descending family of I-fuzzy topologies on Y and $\mathfrak{R}$ be the I-gradation of openness on Y generated by this family. Further more suppose, for each $r \in (0,1]$, $\beta_r = \{B_{j_r}, j_r \in J_r\}$ is a basis and $\zeta_r = \{S_{k_r}, j_r \in K_r\}$, is a subbasis of R_r.*

i) If $F : (X, \mathfrak{T}) \to (Y, \mathfrak{R})$ is an IGP-related function then $\mathfrak{T}(F^{-1}[H]) \geq r$, $\forall H \in R_r$, $\forall r \in (0,1]$, *ii) If* $F : (X, \mathfrak{T}) \to (Y, \mathfrak{R})$ *is an IGP-related function then* $\mathfrak{T}(F^{-1}[H]) \geq r, \;\; \forall H \in \beta_r, \; \forall r \in (0,1],$

iii) If $F : (X, \mathfrak{T}) \to (Y, \mathfrak{R})$ *is an IGP-related function then* $\mathfrak{T}(F^{-1}[H]) \geq r, \;\; \forall H \in \zeta_r, \; \forall r \in (0,1].$

Proof Let $r \in (0,1]$. Using Proposition 3.1, $\forall H \in R_r$ we have $\mathfrak{R}(H) = \bigvee\{t \mid t \in (0,1], \; A \in T_t\} \geq r.$

i) Assume that $F : (X, \mathfrak{T}) \to (Y, \mathfrak{R})$ is an IGP-related function. Then $\forall r \in (0,1]$, $\forall H \in R_r$, we have $\mathfrak{T}(F^{-1}[H]) \geq \mathfrak{R}(H) \geq r$.

ii), iii) Since each $H \in R_r$ is an union of elements of β_r we have

$$\mathfrak{R}(H) = \mathfrak{R}\big(\bigcup_{j_r \in J_r} B_{j_r}\big) \geq \bigwedge_{j_r \in J_r} \mathfrak{R}(B_{j_r}) = r.$$

Also H is an union of finitely intersection of elements of ζ_r,

$$\mathfrak{R}(H) = \mathfrak{R}\Big(\bigcup_{j_r \in J_r} \big(\bigcap_{k_{jr}=1}^{n_{jr}} S_{k_{jr}}\big)\Big) \geq \bigwedge_{j_r \in J_r} \bigwedge_{k_{jr}=1}^{n_{jr}} \mathfrak{R}(S_{k_{jr}}) = r$$

Using this and arguing as (i), we get (ii) and (iii).

Proposition 3.8 *Let Z be an LG-fuzzy open subset of LG-fuzzy topological space* $(X, \mathfrak{T})$. *Define* $\mathfrak{T}_Z : L_Z^M \to L$, *by* $\mathfrak{T}_Z(A) = \mathfrak{T}(A)$. *Then* $(Z, \mathfrak{T}_Z)$ *is an LG-fuzzy topological space, called LG-fuzzy topological subspace of X (L-gtfss). Further more if W be an LG-fuzzy open subset of X such that* $W \subset Z \subset X$, *then* $\mathfrak{T}_W = (\mathfrak{T}_Z)_W$

The proof is straightforward.

Proposition 3.9 *For each* $r \in (0,1]$, *we have the following axioms:*

i) IGF_r*-Top is a full subcategory of IGF-Top.*

ii) IF-Top and IGF_r*-Top are isomorphic.*

iii) IGF_r*-Top a bireflexive full subcategory of IGF-Top.*

Proof The results (i) and (ii) follow from the following facts:
If $\mathfrak{T}$ is an I-gradation openness on X, then we have $(\mathfrak{T}_r)^r = \mathfrak{T}$ and if τ is an I-fuzzy topology, then by Proposition 3.1, we have $(\tau^r)_r = \tau$. Also using Proposition 3.6 we see the function $F : (X, \tau) \to (Y, \varrho)$ is an L-continuous iff $F : (X, \tau^r) \to (Y, \varrho^r)$ is an IGP-related function $\forall r \in (0,1]$.

In order to prove (iii), let us take a member $(X, \mathfrak{T})$ of IGF-Top. Then for $r \in (0,1]$, $\big(X, (\mathfrak{T}_r)^r\big)$ is a member of IGF_r-Top and also $I_X : (X, \mathfrak{T}) \to (X, (\mathfrak{T}_r)^r$ is an IGP-related function. Let $(Y, \mathfrak{R})$ be a member of IGF_r-Top and $F : (X, \mathfrak{T}) \to (Y, \mathfrak{R})$ be an IGP-related function. We need to check only that $F : (X, (\mathfrak{T}_r)^r) \to (Y, \mathfrak{R})$ is an IGP-related function. Since $F \in IGPRf(X,Y)$, hence $F[X] \leq Y$ and $\mathfrak{R}(H) \leq \mathfrak{T}(F^{-1}[H])$ $\forall H \in supp\mathfrak{R}$.

$$\mathfrak{R}(\tilde{0}) = \mathfrak{T}\big(F^{-1}[\tilde{0}]\big) = \mathfrak{T}(\tilde{0}) = (\mathfrak{T}_r)^r(\tilde{0}) = (\mathfrak{T}_r)^r\big(F^{-1}[\tilde{0}]\big) = 1$$

Similarly $\mathfrak{R}(Y) = (\mathfrak{T}_r)^r\big(F^{-1}[Y]\big) = 1$.
Let H be an LG-fuzzy subset of Y. If $\mathfrak{R}(H) = 0$, then obviously $\mathfrak{R}(H) \leq (\mathfrak{T}_r)^r(F^{-1}[H])$. If $\mathfrak{R}(H) = r$, then $\mathfrak{R}(H) \leq \mathfrak{T}(F^{-1}[H])$ implies that $F^{-1}[H] \in \mathfrak{T}_r$. Thus $(\mathfrak{T}_r)^r(F^{-1}[H]) \geq r$. So we have $\mathfrak{R}(H) \leq (\mathfrak{T}_r)^r(F^{-1}[H])$. Therefore F is an IGP-related function.

Proposition 3.10 *Let M, $\{M_j,\ j \in J\}$ be a family of crisp sets and $X \in I^M$, $\{\ Y_j \in I^{M_j},\ j \in J\ \}$ be a family of I-fuzzy subsets of them such that for each $j \in J$, $(Y_j, \mathfrak{R}_j)$ is an IG-fuzzy topological space and $f_j : M \to M_j$ is a function. Then there exists an I-gradation of openness $\mathfrak{T}$ on X, such that the following conditions hold:*

- *i) For each $j \in J$, $f_j : (X, \mathfrak{T}) \to (Y_j, \mathfrak{R}_j)$ is an IGP-related function,*
- *ii) If $(Z, \mathfrak{C})$ is an IG-fuzzy topological space, then $g : (Z, \mathfrak{C}) \to (X, \mathfrak{T})$ is an IGP-related function iff $f_j \circ g$ is an IGP-fuzzy map, for all $j \in J$.*

Proof For each $r \in (0,1]$ and for each $k \in J$, define

$$\tau_{k,r} = \{(f_k)^{-1}[H] : H \in (\mathfrak{R}_j)_r\}.$$

It can be proved that each $\tau_{k,r}$ is an I-fuzzy topology on X. Clearly $\{\ \tau_{k,r}\ :\ r \in (0,1]\ \}$ is a descending chain of I-fuzzy topologies on X. For each $r \in (0,1]$, define

$$S_r = \bigcup_{k \in J} \tau_{k,r}.$$

and let T_r be the I-fuzzy topology on X generated by S_r as a subbasis. We can prove that $\{\ T_r\ :\ r \in (0,1]\ \}$ is a descending chain of I-fuzzy topologies on X. Now, by Proposition 3.3 we have an I-gradation of openness $\mathfrak{T}$ on X associated to $\{\ T_r\ :\ r \in (0,1]\ \}$, where $\mathfrak{T}(A) = \bigvee\{\ r \in (0,1] : A \in T_r\}$:
First we show that for each $j \in J$, $f_j : (X, \mathfrak{T}) \to (Y_j, \mathfrak{R}_j)$ is an IGP-related function. For this let $H \in I^{M_j}_{Y_j}$ and $\mathfrak{R}_j(H) = r$, $r > 0$. Then

$$(f_j)^{-1}[H] \in \tau_{j,r} \subseteq S_j \subseteq T_j$$

Therefore

$$\mathfrak{T}((f_j)^{-1}[H] \geq r = \mathfrak{R}_j(H).$$

Next suppose $g : (Z, \mathfrak{E}) \to (X, \mathfrak{T})$ is an IGP-related function.
Since $f_j : (X, \mathfrak{T}) \to (Y_j, \mathfrak{R}_j)$ is an IGP-related function, hence by Proposition 3.4, $f_j \circ g$ is an LGP-related function, for all $j \in J$.
Conversely assume that for all $j \in J$ the function $f_j \circ g$ is an IGP-related function. We show that $g : (Z, \mathfrak{E}) \to (X, \mathfrak{T})$ is an IGP-related function:
Using Proposition 3.7, it is sufficient to prove that $\mathfrak{E}(g^{-1}[A]) \geq r$, for all $A \in S_r,\ r \in (0, 1]$. Let $r \in (0, 1],\ A \in S_r$. Then $A \in \tau_{j,r}$, for some $j \in J$. Hence there is an $H \in \tau_{j,r}$ such that $A = (f_j)^{-1}[H]$.
Since for all $j \in J$, $\{(\mathfrak{R}_j)_r : r \in (0, 1]\}$ is a family of associated I-fuzzy topologies on Y_j and since $f_j \circ g : (Z, \mathfrak{E}) \to (Y_j, \mathfrak{R}_j)$ is an IGP-related function, by Proposition 3.7, we have

$$\mathfrak{E}\big((f_j \circ g)^{-1}[H]\big) \geq r \quad \Longrightarrow \quad \mathfrak{E}\big(g^{-1}[(f_j)^{-1}[H]]\big) \geq r \quad \Longrightarrow \quad \mathfrak{E}\big(g^{-1}[A]\big) \geq r.$$

This completes the proof (ii).

C^{∞} LG-fuzzy local one parameter groups acting on an LG-fuzzy manifold

The concept of fuzzy vector space and fuzzy topological vector space was introduced by A. K. Katsaras and D. B. Liu [11] in 1977. We extend these concepts by LG-fuzzification:

Definition 3.11 *An L-fuzzy vector space (V, η) or ηV over a field F is an ordinary vector space V over F, with a map $\eta : V \to L$ satisfying the following conditions for all $a, b \in V$ and $r \in F$*

1) $\eta(a + b) \geq min\{\eta(a),\ \eta(b)\}$,

2) $\eta(-a) = \eta(a)$,

3) $\eta(0) = 1$,

4) $\eta(ra) \geq \eta(a)$,

Definition 3.12 *An L-fuzzy vector space (V, η) that $(V, \mathfrak{T})$ is at the same time an LG-fuzzy topological space, is called an LG-fuzzy topological vector space (L-gftvs) if two fuzzy functions $(v, w) \to v + w$ and $(\alpha, v) \to \alpha v$ are LG-continuous.*

We bring up two Definitions from [20] to establish concept of LG-fuzzy one parameter subgroups of an Lie groups:

Definition 3.13 *Let* $(X, \mathfrak{T})$ *be an* C^∞ *LG-fuzzy manifold of dimension* n *and* $p \in X$. *We define the LG-fuzzy tangent space* $LGT_p(X)$ *of* X *at* p *to be the set of all mappings* $Z_p : C_L^\infty(p) \to \mathbb{R}$ *satisfying two conditions for all* $\alpha, \beta \in \mathbb{R}$ *and* $f, g \in C_L^\infty(p)$*:*

i) $Z_p(\alpha f + \beta g) = \alpha(Z_p f) + \beta(Z_p g)$ *(linearity),*

ii) $Z_p(fg) = (Z_p f)g(p) + f(p)(Z_p g)$ *(leibniz rule),*

with the LG-fuzzy vectore space operations in $LGT_p(X)$ *defined by:*
$(Z_p + W_p)f = Z_p f + W_p f$
$(\alpha Z_p)f = \alpha(Z_p f)$.
We define $\eta : LGT_p(X) \to L, \quad \eta(f) = \begin{cases} A(p) & f \neq 0 \\ 0 & f = 0. \end{cases}$
where (A, ψ) *is an LG-local coordinate neighborhood of* p. *Then* $(LGT_p(X),\ \eta)$ *is an LG-fuzzy vector space.*
Each $Z_p \in LGT_p(X)$ *is called an LG-fuzzy tangent vector of* X *at* p.

Definition 3.14 *An* C^∞ *LG-fuzzy vector field* Z *on* X *is a function assigning to each point* p *of* X *a vector* $Z \in LGT(X)$ *whose components in the LG-frames of any LG-local coordinates* (A, ψ)*, are* C^∞ *LG-related function on the domain suppA. We denote the set of all LG-fuzzy vector fields on* X *by* $LG\mathfrak{X}(X)$.

We will use *LG*- fuzzy vector field to mean C^∞ *LG*-fuzzy vector field hereafter. For $p \in X$, let (A, ψ) be any *LG*-local coordinate neighborhood of p, and let $H_{1p}, ..., H_{np}$, be the corresponding basis of $LGT_p(X)$ (*LG*-local coordinate frames) defined by $H_{ip} = \psi_*^{-1}(\dfrac{\partial}{\partial x^i})|_{\psi(p)}$.
Then Z_p, the value of Z at p, equals to $\sum_{i=1}^n \alpha^i(p) H_{ip}$. If p is varied in *suppA*, the components $\alpha^1, ..., \alpha^n$ are well-defined functions of p.

Definition 3.15 *Let* Q *be a group. An L-fuzzy subset* G *of* Q *is called an L-fuzzy subgroup of* Q *if*

$$G(xy^{-1}) \geq \ min\{G(x), G(y)\} \ \textit{and} \ G(e) = 1.$$

Definition 3.16 *Let* Q *be a group and* G *be an L-fuzzy subgroup of* Q *which is at the same time an* C^∞ *L-fuzzy manifold with gradation of opennes* $\mathfrak{G}$. *Then* $(G, \mathfrak{G})$ *is an LG-fuzzy Lie group provided that the mapping of* $G \times G \to G$ *defined by* $(x, y) \to xy$ *and the mapping of* $G \to G$ *defined by* $x \to x^{-1}$ *are both* C^∞ *LG-related functions.*

Example 3.17 *$(1_{\mathbb{R}^n},\ \mathfrak{T}_{Ln})$ is an LG-fuzzy manifold. Since $1_{\mathbb{R}^n}$ satisfies two conditions of Definition* 3.15*, therefore $\mathbb{R}^n$ is an LG-fuzzy subgroup of additive group $\mathbb{R}^n$. On the othere hand, the mapping of $1_{\mathbb{R}^n} \times 1_{\mathbb{R}^n} \to 1_{\mathbb{R}^n}$ defined by $(x, y) \to x + y$ and the mapping of $1_{\mathbb{R}^n} \to 1_{\mathbb{R}^n}$ defined by $x \to -x$ are both LG-continuous, so $(1_{\mathbb{R}^n},\ \mathfrak{T}_{Ln})$ is an LG-fuzzy Lie group.*

Definition 3.18 *Let $(G, \mathfrak{G})$ be an LG-fuzzy Lie group and $(X, \mathfrak{T})$ an C^∞ LG-fuzzy manifold. Then θ is called an LG-action of G on X (on the left) if there is an C^∞ LG-related function θ: $G \times X \to X$ satisfying two conditions:*

1) If e is the identity element of G, then $\theta(e, x) = x$ for all $x \in X$

2) If $g_1, g_2 \in G$, then $\theta(g_1, \theta(g_2, x)) = \theta(g_1\ g_2, x)$ for all $x \in X$.

Definition 3.19 *Let $(G, \mathfrak{G})$ be an arbitrary LG-fuzzy Lie group. An LG-fuzzy one-parameter subgroup H of G is the image of an LG-homomorphism $F : 1_{\mathbb{R}} \to G$, i.e. $H = F[1_{\mathbb{R}}]$.*

Theorem 3.20 *Let $F : 1_{\mathbb{R}} \to G$ be an LG-fuzzy one-parameter subgroup of LG-fuzzy Lie group G and Z the left-invariant LG-fuzzy vector field on G defined by $Z = F(0)$. Then $\theta(t, g) = Z_{F(t)}(g)$ defines an LG-action $\theta : 1_{\mathbb{R}} \times G \to G$, having Z as LG-infinitesimal generator. Conversely, let Z be a left-invariant LG-fuzzy vector field and $\theta : 1_{\mathbb{R}} \times G \to G$ the corresponding action. Then $F(t) = \theta(t, e)$ is an LG-fuzzy one-parameter subgroup $\theta : 1_{\mathbb{R}} \times G \to G$ and $\theta(t, g) = Z_{F(t)}(g)$.*

Definition 3.21 *Given an LG-fuzzy vector field Z on an LG-fuzzy manifold $(X, \mathfrak{T})$. We say that an L-curve $F : \chi_J \to X$ where J is an open interval of $\mathbb{R}$ is an LG-integral curue of Z if $dF/dt = Z_{F(t)}$ on J.*

Theorem 3.22 *(Existence Theorem for ordinary LG-differential equations on $(1_{\mathbb{R}^n},\ \mathfrak{T}_{Ln})$)*
Let B be an LG-fuzzy open subset of an LG-fuzzy manifold X and $I_\varepsilon = (-\varepsilon, \varepsilon)$, for $\varepsilon > 0$. Suppose that $f^i(t,\ x^1,\ ...,\ x^n)$, $i = 1, ..., n$, are LG-related functions of class C^r, $r \geq 1$ on $\chi_{I_\varepsilon} \times B$. Then for each $x \in B$ there exists $\delta > 0$ and an LG-neighborhood $\tilde{V}$ of x, $\tilde{V} \subset B$, such that:

(I) For each $a = (a^1,\ ...,\ a^n) \in V$ there exists an n-tuple of C^r fuzzy functions $x(t) = (x^1(t),\ ...,\ x^n(t))$, mapping χ_{I_δ} into $\tilde{V}$, which satisfy the system of first-order differential equations

$$\frac{dx^i}{dt} = f^i(t, x),\ \ i = 1,\ ...,\ n, \tag{3}$$

and the initial conditions

$$x^i(0) = a^i, \;\; i = 1, \; ..., \; n. \tag{4}$$

For each a the fuzzy functions $x(t) = (x^1(t), \; ..., \; x^n(t))$, are uniquely determined.

(II) *These functions being uniquely determined by $a = (a^1, \; ..., \; a^n)$ for every $a \in \tilde{V}$, so we write them $x^i(t, a^1, \; ..., \; a^n)$, $i = 1, \; ..., \; n$, in which case they are of class C^r, in all variables and thus determine a C^r fuzzy map of $\chi_{I_\delta} \times \tilde{V} \to B$.*

Proof Set $U = suppB$. Since all fuzzy functins $f^i(t, \; x^1, \; ..., \; x^n)$, $i = 1, ..., n$, are C^r functions from U to $\mathbb{R}$, hence using existence theorem for ordinary differential equations on $\mathbb{R}^n$, for each $x \in U$ there exists $\delta > 0$ and a neighborhood $V' of \; x, \; V' \subset U$, such that: For each $a = (a^1, \; ..., \; a^n) \in V'$ there exists an n-tuple of C^r functions
$x(t) = (x^1(t), \; ..., \; x^n(t))$, mapping I_δ into U , which satisfy the system of first-order differential equations 3 and 4. If we define a fuzzy subset
$\tilde{V} : \mathbb{R}^n \to L, \; by \; \tilde{V}(a) = B(x(a)) \; for \; each \; a \in V'$ and $\tilde{V}(a) = 0$ elsewhere. Then we have $x \in LGRf(\tilde{V}, \; B)$ and so x is an C^r fuzzy function which satisfies 3 and 4. This completes the proof.

Theorem 3.23 *Let Z be an LG-fuzzy vector field on an LG- fuzzy manifold $(X, \mathfrak{T})$. Then for each $p \in X$, there exists an LG-neighborhood V and real number $\delta > 0$ such that there corresponds an C^∞ LG-related function*

$$\theta^V : \chi_{I_\delta} \times V \longrightarrow A$$

satisfying

$$\dot{\theta}^V(t, q) = Z_{\theta^V(t,q)} \quad and \quad \theta^V(0, q) = q \;\; \forall \, q \in V.$$

If $F(t)$ is an LG-integral curve of Z with $F(0) = q \in V$, then $F(t) = \theta^V(t, q)$ for $|t| < \delta$ and this mapping is unique.

Proof Let $p \in X$ and (A, ψ) be an LG-local coordinate neighborhood of p and $B = \psi[A]$. Then

$$Z|_{suppA} \; (t, p) = \sum_{i=1}^{n} \alpha^i(t, p) \; H_{i_p} = \sum_{i=1}^{m} \alpha^i(t, p) \; \psi_*^{-1}(\partial/\partial x^i),$$

Thus $\tilde{Z} = \psi_*(Z)$ on $suppB \subseteq \mathbb{R}^n$ can be defined by:

$$\tilde{Z}|_U(t, x) = f^1(t, x) \; \frac{\partial}{\partial x^1} + \; ... \; + \; f^n(t, x) \; \frac{\partial}{\partial x^n},$$

$$f^i(t,x) = \alpha^i \circ \psi^{-1}(t,x) = \alpha^i(t,p).$$

Then by Theorem 3.22 there is an LG-neighborhood $\tilde{V}$ of $a = \psi(p)$, $\tilde{V} \subset B$, and real number $\delta > 0$ together an n-tuple of C^r fuzzy functions $x(t) = (x^1(t),\ ...,\ x^n(t))$, mapping χ_{I_δ} into $\tilde{V}$, which satisfy 3 and 4. Therefore we have an unique LG-integral curve $F : \chi_{I_\delta} \times \tilde{V} \longrightarrow B$ defined by $F(t,a) = (x^1(t,a),\ \ldots,\ x^n(t,a))$. Hence by 3 we have

$$\dot{F}(t,a) = \sum_{i=1}^{n} \frac{dx^i}{dt} \frac{\partial}{\partial x^i} = \sum_{i=1}^{n} f^i(t,a) \frac{\partial}{\partial x^i} = \tilde{Z}_{F(t,a)}.$$

We set $V = \psi^{-1}(\tilde{V})$ and define

$$\theta^V : \chi_{I_\delta} \times V \longrightarrow A, \quad \theta^V(t,q) = \psi^{-1}\big(F(t,\psi(q))\big).$$

Since ψ and ψ^{-1} are diffeomorphisms, by chain rull, we have:

$$\begin{aligned}\dot{\theta}^V(t,q) &= \frac{d}{dt}\ \psi^{-1}(F(t,\psi(q)) \\ &= \sum_{i=1}^{n} \frac{\partial \psi^{-1}}{\partial x^i}|_{F(t,\psi(q))}\ \frac{dx^i}{dt}|_{(t,a)} \\ &= \sum_{i=1}^{n} \psi_*^{-1}(\frac{\partial}{\partial x^i})|_{F(t,a)}\ f^i(t,a) \\ &= \sum_{i=1}^{n} H_{iq}\ \alpha^i(t,q) = \ Z_{\theta^V(t,q)}.\end{aligned}$$

Theorem 3.24 *for any C^∞ LG-fuzzy vector field Z on an LG-fuzzy manifold $(X,\mathfrak{T})$, there is an uniquely determined open interval $I(p) = \{t \mid \alpha(p) < t < \beta(p)\}$ containing $t = 0$ and having the property that there exists an unique C^∞ LG-integral curve $F(t)$ defined on $\chi_{I(p)}$ and such that $F(0) = p$.*

Proof By Theorem 3.23, there is an LG-integral curve $F(t)$ of Z with $F(0) = q$ and $F(t) = \theta^{V_1}(t,q)$ for $t \in I_{\delta_1}$. Given any other LG-integral curve $G(t)$ of Z with $G(0) = p$ and $G(t) = \theta^{V_2}(t,q)$ for $t \in I_{\delta_2}$.
From the existence theorem for ordinary equations on $\mathbb{R}^n$, they agree on some open interval I_δ around $t = 0$. Thus $F(t) = G(t)$ on an open set $I_\delta = I_{\delta_1} \cap I_{\delta_2}$ around 0. Therefore $I(p)$ is defined: it is the intersection of the domains of all LG-integral curves which pass through p at $t = 0$; the asserted properties are immediate

We shall use the notation $F(t) = \theta(t,p)$ for the unique LG-integral curve $F(t)$ such that $F(0) = p$.

Theorem 3.25 *For any C^∞ LG- fuzzy vector field Z, the domain W of $\theta(t,p)$ is LG-open in $1_{\mathbb{R}} \times X$ and θ is an C^∞ LG-related function onto X .*

Proof Let $\tilde{W} = \{(t,p) \in \mathbb{R} \times suppX \mid \alpha(p) < t < \beta(p) \}$. According to what has been shown in ordinary geometry, both $\tilde{W}$ and θ are uniquely determined by $\tilde{Z} = \psi_*(Z)$ and $\tilde{W}$ is an open subset of $\mathbb{R} \times suppX$ and $\theta : \tilde{W} \to suppX$ is C^∞ on suppX. We define L-fuzzy subset $W : \mathbb{R} \times M \to L$, by:
$$W(t,p) = \begin{cases} X(p) & (t,p) \in \tilde{W} \\ 0 & elsewhere. \end{cases}$$
Then W is LG-fuzzy open subset of $1_{\mathbb{R}} \times X$ and $\theta : W \to X$ is C^∞ LG-related function on X.

Definition 3.26 *A LG-fuzzy local one-parameter group action or LG-flow on an C^∞ LG-fuzzy manifold X, is an C^∞ LG-related function $\theta : W \to X$ which satisfies the following two conditions:*

(i) $\theta_0(p) = p$ for all $p \in X$,

(ii) $if(s,p) \in W$, then $\alpha(\theta_s(p)) = \alpha(p) - s, \quad \beta(\theta_s(p)) = \beta(p) - s,$

and moreover for any t such that $\alpha(p) - s < t < \beta(p) - s, \quad \theta_{t+s}(p)$ is defined and $\theta_t \circ \theta_s(p) = \theta_{t+s}(p)$.

Definition 3.27 *A fuzzy subgroup H of an LG-fuzzy Lie group G is called an LG-fuzzy Lie subgroup of G if H is an LG-fuzzy submanifold and is an LG-fuzzy Lie group with its LG-fuzzy structure as an LG-immersed submanifold.*

Theorem 3.28 *To each LG-fuzzy local one-parameter group action θ on LG- fuzzy manifold X is associated a unique maximal domain of defnition W. Two LG-fuzzy local one-parameter groups are equal if and only if they have the same LG- fuzzy infnitesimal generator Z; and each LG-fuzzy vector field Z on X determines an LG-fuzzy local one-parameter group θ, W of which it is the LG- fuzzy infnitesimal generator.*

Remark 3.29 *A general nth order ordinary LG-differential equation in the independent variable t and dependent variable x and its derivatives is given by a relation*
$$F(t, x, \frac{dx}{dt}, \ldots, \frac{d^n x}{dt^n}) = 0.$$
We suppose that this is an C^∞ LG-related function defined on some LG-neighborhood in $1_{\mathbb{R}^{n+2}}$ of the point $(0, a_0, a_1, \ldots, a_n)$ and that in an LG-neighborhood U of this point. We can write it in the form
$$\frac{d^n x}{dt^n} = G(t, x, \frac{dx}{dt}, \ldots, \frac{d^{n-1} x}{dt^{n-1}}).$$

Now let $x = x^1, dx/dt = x^2, \ldots, d^{n-1}x/dt^{n-1} = x^n$ *and consider the first order system of ordinary LG-differential equations:*

$$\frac{dx^1}{dt} = x^2, \;\; \frac{dx^2}{dt} = x^3, \;\ldots,\; \frac{dx^n}{dt} = G(t, x^1, x^2, \ldots,\; x^{n-1})$$

with initial conditions:

$$x^i(0) = a^i, \quad i = 1, \ldots,\; n.$$

Therefore Theorem 3.22 *gives the existence and uniqueness of solutions of ordinary LG-differential equations of nth order.*

C^∞ LG-fuzzy covector fields on an LG-fuzzy manifold

Let $(X, \mathfrak{T})$ be an LG-fuzzy manifold of dimension n and $p \in X$. An LG-fuzzy covector σ_p on X is an element of $(LGT_p)^*(X)$. So $\sigma_p : LGT_p(X) \to 1_{\mathbb{R}}$ is a linear LG-related function. Given LG-coordinate frames $H_{1p}, \ldots, H_{np}$ of $LGT_p(X)$, there is a uniquely determined dual basis $\varpi_p^1, \ldots, \varpi_p^n$ of $(LGT_p)^*(X)$ satisfying,

$$\varpi_p^i\ (H_{jp}) = \delta_j^i.$$

Hence

$$\sigma_p = \sum_{i=1}^{n} \sigma_p(H_{ip})\ \varpi_p^i.$$

We call $\varpi^1, \ldots, \varpi^n$ a field of LG-coordinate coframes if $H_1, \ldots, H_n$, are LG-coordinate frames.

Definition 3.30 *An* C^∞ *LG-fuzzy covector field on an LG-fuzzy manifold* $(X, \mathfrak{T})$*, is a function* σ *which assigns to each* $p \in X$ *an* C^∞ *LG-covector* $\sigma_p \in LGT_p^*(X)$ *in such a manner that for any LG-fuzzy coordinate neighborhood* (A, ψ) *with LG-local coordinate frames* $H_{1p}, \ldots, H_{np}$*, the functions* $\sigma(H_{ip})$*,* $i = 1, \ldots, n$*, are* C^∞ *on suppA . For convenience, " LG-fuzzy covector field" will mean* C^∞ *LG-fuzzy covector field.*

Remark 3.31 *For any* C^∞ *LG-fuzzy vector field* Z *on an LG-fuzzy open subset* U *of* X*, the function* $\sigma(Z)$ *is* C^∞ *on* U*. To see this we take a covering of* V *by LG-local coordinate neighborhoods of* X*; let* (A, ψ) *be such an LG-coordinate neighborhood. Then* $Z = \sum_{i=1}^{n} \alpha^i H_i$*, where* α^i *are* C^∞ *on* A *. Thus*

$$\sigma(Z) = \sum_{i=1}^{n} \alpha^i \sigma(H_i)$$

is C^∞ *if* $\sigma(H_1), \ldots, \sigma(H_n)$ *are. Hence an* C^∞ *LG-fuzzy covector field* σ *defines a map of* $L\mathfrak{X}(X)$ *to* $C^\infty(X)$ *which is* $C^\infty(X)$*-linear. More precisely, if* $f, g \in C^\infty(X)$ *and* Z_1 *and* Z_2 *are* C^∞ *LG-fuzzy vector fields on* X*, then*

$$\sigma(f\ Z_1 + g\ Z_2) = \sigma\big(f\ \sum_{i=1}^{n} \alpha_1^i\ H_i + g\ \sum_{i=1}^{n} \alpha_2^i\ H_i\big)$$

$$= \textstyle\sum_{i=1}^{n} f\ \alpha_1^i\ \sigma(H_i) + \sum_{i=1}^{n} g\ \alpha_2^i\ \sigma(H_i)$$

$$= f\,\sigma(Z_1) + g\ \sigma(Z_2).$$

Example 3.32 *If* f *is an* C^∞ *LG-related function from* $U \in supp\mathfrak{T}$ *to* $V \in supp\mathfrak{T}_{L1}$*, then it defines an* C^∞ *LG-fuzzy covector field* df *on* U*, called LG-differential of* f *and is defined by the formula* $df(Z_p) = Z_p f$ *for any* C^∞ *LG-fuzzy vector field* Z *on* X *and each* $p \in X$*.*

$$df(Z_p) = Z_p f = \sum_{i=1}^{n} \alpha^i(p) H_i(f)$$

$$= \textstyle\sum_{i=1}^{n} \alpha^i(p)\ \psi_*^{-1}(\partial/\partial x^i)(f)$$

$$= \textstyle\sum_{i=1}^{n} \alpha^i(p)\ \partial/\partial x^i(f \circ \psi^{-1})$$

$$= \textstyle\sum_{i=1}^{n} \alpha^i(p)\ \partial\hat{f}/\partial x^i.$$

Example 3.33 *Let* $M = \mathcal{M}_n(\mathbb{R})$ *be the set of* $n \times n$ *matrices over* $\mathbb{R}$ *and* $L = \mathbb{N} \cup \{\infty\}$*. There is a bijection map* ψ *from* M *to* $\mathbb{R}^{n^2}$*:*

$$\psi(a_{ij}) = (a_{11}, \ldots, a_{1n}, \ldots, a_{n1}, \ldots, a_{n^2}).$$

Let X *be an L-fuzzy subset of* M *defined by:*

$$X\big((a_{ij})\big) = 2 + sup\{\ \lfloor\ |a_{ij}|\ \rfloor : 1 \leq i, j \leq n\ \}$$

Using ψ *and fuzzy Euclidean topology* τ_{In^2} *on* $1_{\mathbb{R}^{n^2}}$*, we define a natural gradation of openness:* $\mathfrak{T} : I_X^M \to L$ *by*
$$\mathfrak{T}(A) = \begin{cases} \infty & \psi[A] \in \tau_{In^2}, \\ 1 & elsewhere. \end{cases}$$
Therefore $(X, \mathfrak{T})$ *is an LG-fuzzy manifold with the single LG-local coordinate neighborhood* $x = (x_{ij}),\ 1 \leq i, j \leq n$*. Therefore* d/dx_{ij} *and* dx_{ij}*,* $1 \leq i, j \leq n$*, are the fields of LG-frames and LG-coframes on* G *respectively.*

Theorem 3.34 *Let the exponential* e^A *of a matrix* $A \in \mathcal{M}_n(\mathbb{R})$ *be defined to be the matrix given by*

$$e^A = I + A + \frac{1}{2!}\, A^2 + \frac{1}{3!}\, A^3 + \dots . \tag{5}$$

if the series coverges. Then series (5) *converges absolutely for all* $A \in M$ *and uniformly on compact subsets. The mapping* $exp : (X, \mathfrak{T}) \to (X, \mathfrak{T})$ *defined by* $A \to e^A$ *is an* C^∞ *one-to-one LG-related function. If we let* $\rho = \sup\{\ |a_{ij}| : 1 \le i, j \le n\ \}$, *then we have* $X(e^A) \le X(e^{n\rho} I)$.

Proof First we show that $exp\ [X] \le X$ and hence $exp : (X, \mathfrak{T}) \to (X, \mathfrak{T})$ is an LG-related function:

$$exp\ [X](D) = \bigvee \{X(A) \mid A \in exp^{-1}(D)\} = \bigvee \{X(A) \mid e^A = D\}$$

$$X(A) \le \bigvee_{k \ge 0} \{\ X\big(\frac{1}{k!}\ A^k\big)\} = X\big(\sum_{k=0}^{\infty} \frac{1}{k!}\ A^k\big) = X(e^A)$$

$$\Longrightarrow \quad exp\ [X](D) \le X(D).$$

Similarly to the proof of this theorem in ordinary geometry, we denote by $a_{ij}^{(k)}$ the entries of the matrix A^k, then by induction on k we have the inequality

$$|a_{ij}^{(k)}| \le (n\rho)^k.$$

Hence the sequence converges absolutely for every A and converges uniformly on every compact set is contained in a set $K_\rho = \{A \mid |a_{ij}| \le \rho\}$. Therefore the mapping

$$\psi \circ exp \circ \psi^{-1}\big((a_{11},\ \dots,\ a_{1n},\ \dots,\ a_{n1},\ \dots,\ a_{nn})\big)$$

is C^∞, since the entries of the partial sums are polynomials in the variables a_{ij}. Thus exp is C^∞ LG-related function. Since we have $e^A e^{-A} = e^I = I$. Therefore exp is a one-to-one LG-related function. Further more we have

$$\begin{aligned}
X(e^A) &= X\big(\sum_{k=0}^{\infty} \frac{1}{k!}\ A^k\big) \\
&= \ \bigvee_{k \ge 0} \{\ X\big(\frac{1}{k!}\ A^k\big)\} \\
&= \ \bigvee_{k \ge 0} \{\ 2 + sup\ \{\lfloor\ \frac{1}{k!}\ |a_{ij}^{(k)}|\ \rfloor : 1 \le i, j \le n\}\} \\
&\le \ \bigvee_{k \ge 0} \{\ 2 + sup\ \{\lfloor\ \frac{1}{k!}\ (n\rho)^k\ \rfloor : 1 \le i, j \le n\}\} \\
&= \ \bigvee_{k \ge 0} \{\ 2 + \sup\ \{\lfloor\ \frac{1}{k!}\ (n\rho I)_{i,j}^k\ \rfloor : 1 \le i, j \le n\}\}
\end{aligned}$$

$$\begin{aligned}&= \mathrm{X}\ \big(\ \textstyle\sum_{k=0}^{\infty} \dfrac{1}{k!}\ (n\rho I)^k\big)\\ &= \mathrm{X}\ (\ \big(\ \textstyle\sum_{k=0}^{\infty} \dfrac{1}{k!}\ (n\rho)^k\big) I)\\ &= \mathrm{X}\ (\mathrm{e}^{n\rho} I)\end{aligned}$$

C^∞ LG-fuzzy tensor fields on an LG-fuzzy manifold

Now we define the L-tensors and L-alternating k-forms on an L-fuzzy vector space and the exterior product and pullback of them.

Definition 3.35 *An L-tensor Φ on an L-fuzzy vector space (V, η) over $\mathbb{R}$ is defined as a multi-linear map by*

$$\Phi : \underbrace{V \times \ldots \times V}_{k} \times \underbrace{V^* \times \ldots \times V^*}_{l} \to \ \mathbb{R}$$

with V^* denoting the dual space to V, k its covariant order, and l its contravariant order (or (k, l) its order).

We denote the collection of all L-tensors on V of order (k, l), by $L\mathcal{T}_l^k(V)$.

Let $\phi : V \to W$ be a linear map between L-fuzzy real vector spaces. The pullback of an L-alternating k-form $\omega \in \bigwedge^k W^*$ under is the L-alternating k-form $\phi^*\omega \in \bigwedge^k V^*$ defined by

$$(\phi^*\omega)\big(v_1, \ldots, v_k\big) = \omega\big(\phi(v_1), \ldots, \phi(v_k)\big)$$

Let $k, l \in \mathbb{N}$. The set $S_{k,l} \subset S_{k+l}$ of (k, l)-shuffles is the set of all permutations in S_{k+l} that leave the order of the first k and of the last l elements unchanged:

$$S_{k,l} = \{\sigma \in S_{k+l} \mid \sigma(1) < \ldots < \sigma(k),\ \sigma(k+1) < \ldots < \sigma(k+l)\}.$$

Definition 3.36 *The exterior product of $\omega \in \bigwedge^k V^*$ and $\tau \in \bigwedge^l V^*$ is the L-alternating $(k+l)$-form $\omega \wedge \tau \in \bigwedge^{k+l} V^*$ defned by*

$$(\omega \wedge \tau)(v_1, \ldots, v_{k+l}) = \sum_{\sigma \in S_{k,l}} \varepsilon(\sigma)\ \omega\big(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\big)\ \tau\big(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}\big)$$

$$\varepsilon(\sigma) := (-1)^\nu, \quad \nu(\sigma) = \big\{(i, j) \in \{1, \ldots, k\}^2 \mid i < j,\ \sigma(i) > \sigma(j)\big\}$$

The exterior product is super-commutative:

$$\omega \wedge \tau = (-1)^{deg(\omega) deg(\tau)} \tau \wedge \omega$$

Definition 3.37 *Let* $(X,\mathfrak{T})$ *be LG-fuzzy manifold of dimension* n. *Let* $L\mathcal{T}^k_{lp}(X)$ *be the collection of all L-tensors on* $LGT_p(X)$ *of order* (k,l). *A field* Φ *of* C^∞ *LG-multi-linear forms, on* X *consists of a function assigning to each* $p\in X$ *a multilinear form* $\Phi_p\in L\mathcal{T}^k_{lp}(X)$, *such that for any LG-local coordinate neighborhood* (A,ψ), *the functions*

$$\alpha^{j_1,\dots,j_k}_{i_1,\dots,i_l}=\Phi(H_{i_1},\ \dots,\ H_{i_l},\ \varpi^{j_1},\ \dots,\ \varpi^{j_k}),$$

defined by Φ *and the LG-local coordinate frames* $H_1,\ \dots,\ H_n$ *and coframes* $\varpi^1,\ \dots,\ \varpi^n$, *are* C^∞.

Definition 3.38 *An alternating covariant LG-fuzzy tensor field of order* k *on* X *will be called an LG-differential form of degree* k. *The set of all such forms is denoted by* $L\Omega^k(X)$.
If $f:X\to Y$ *is an* C^1 *LG-related function between LG-fuzzy manifolds and* $\omega\in L\Omega^k(Y)$. *Then pullback of* ω *under* f *is the LG-differential k-form* $f^*\omega\in L\Omega^k(X)$ *defined by*

$$(f^*\omega)_p\big(v_1,\dots,v_k\big)=\omega_{f(p)}\big(df(p)v_1,\dots,df(p)v_k\big)$$

for $p\in X$ *and* $v_1,\dots v_k\in LGTp(X)$.

Example 3.39 *Let* $(1_{\mathbb{R}^n},\mathfrak{T}_{Ln})$ *be LG-fuzzy Euclidean manifold. Then*

$$L\Omega^0(1_{\mathbb{R}^n})=\mathbb{R},\quad L\Omega^1(1_{\mathbb{R}^n})=Hom(LTG_p(1_{\mathbb{R}^n}),\mathbb{R})$$

$$\forall\omega\in L\Omega^k(1_{\mathbb{R}^n})\quad \omega=\sum_{I\in\mathfrak{I}}\omega_I dx^I,\quad \omega_I=\omega(e_{i_1},\dots,e_{i_k})$$

$$I=(I_1,\dots,i_k)\quad dx^I=dx^{i_1}\wedge\dots\wedge dx^{i_k}$$

Theorem 3.40 *Consider* $\breve{M}$ *the set of all L-tensors of order* (k,l) *on all point of* M. *So*

$$\breve{M}=L\mathcal{T}^k_l(X)=\bigcup_{p\in X}L\mathcal{T}^k_{lp}(X).$$

Let $\breve{X}:\breve{M}\to L$ *be an L-fuzzy subset of* $\breve{M}$ *defined by* $\breve{X}(\Phi_p)=X(p)$. *We assigne to each* $A\in supp\mathfrak{T}$, *an L-fuzzy subset* $\breve{A}$ *of* $\breve{M}$ *which* $\breve{A}(\Phi_p)=A(p)$. *So we have*

$$supp\breve{A}=\bigcup_{p\in suppA}L\mathcal{T}^k_{lp}(X).$$

Let $\mathfrak{A}=\{(A_i,\psi_i),\ i\in K\}$ *be the* C^∞ *LG-structure of* X, *then we can define an LG-structure on* $\breve{X}$ *by* $\breve{\mathfrak{A}}=\{(\breve{A}_i,\breve{\psi}_i),\ i\in K\}$, *which*

$$\breve{\psi}_i:supp\breve{A}_i\ \to\ \mathbb{R}^{n\times\binom{n}{k}\times\binom{n}{l}},\qquad \breve{\psi}_i(\Phi_p)=(\psi(p),\ \alpha^{j_1,\dots,j_k}_{i_1,\dots,i_l}(p)).$$

Now we define an L-gradation of openness on $\breve{X}$, $\breve{\mathfrak{T}} : L_{\breve{X}}^{\breve{M}} \to L$ *by*

$$\breve{\mathfrak{T}}(\mathcal{A}) = \begin{cases} \mathfrak{T}(A) & \mathcal{A} = \breve{A},\ for\ some\ A \in \mathfrak{T}, \\ 0 & elsewhere. \end{cases}$$

Then $(\breve{X}, \breve{\mathfrak{T}})$ *is an LG-fuzzy manifold of dimension* $n \times \binom{n}{k} \times \binom{n}{l}$.

Proof It was proved for ordinary C^∞ n-manifold X, that for each $p \in X$, the set of all tensors of order (k, l) at p, $T^k_{lp}(X)$ is a vector space over $\mathbb{R}$ of dimension $\binom{n}{k} \times \binom{n}{l}$. In a same way it can be proved that $(L\mathcal{T}^k_{lp}(X),\ \eta)$ is an L-fuzzy vector space over $\mathbb{R}$ where $\eta(\Phi_p) = A(p)$ and (A, ψ) is an LG-local coordinate neighborhood of p. One can show that $\breve{\mathfrak{T}}$ satisfies three conditions of Definition 2.2 and the structure $\breve{\mathfrak{A}}$ is C^∞, and proofs are straightforward.

Let $(X, \mathfrak{T})$ be an C^∞ LG-fuzzy manifold of dimension n, and $p \in X$. Then $L\mathcal{T}^k_{lp}(X)$ is an L-fuzzy topological vector space (L-gftvs).

Definition 3.41 *Let A be an LG-fuzzy open subset of LG-fuzzy manifold X with the LG-structure* $\{(A_\lambda, \psi_\lambda)\}_{\lambda\in\Lambda}$.
Let ω *be an LG-differential k-form on X. We denote the corresponding LG-differential forms in local coordinates by* $\omega_\lambda \in L\Omega^k(suppA_\lambda)$, *so that* $\omega|_{suppA_\lambda} = \psi^*_\lambda\ \omega_\lambda$ *for all* $\lambda \in \Lambda$. *The LG-exterior differential is a linear operator* $d : L\Omega^k(X) \to L\Omega^{k+1}(X)$, *that satisfies*

$$d\omega|_{suppA_\lambda} = \psi^*_\lambda\ d\omega_\lambda$$

Let $(X, \mathfrak{T})$ be an LG-fuzzy manifold.

i) The LG-exterior differential satisfies the Leibnitz rule

$$d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^{deg(\omega)}\omega \wedge d\tau$$

ii) The LG-exterior differential satisfies $d \circ d = 0$.

iii) The LG-exterior differential commutes with pullback: If $F : X \to Y$ is an LG-fuzzy smooth map between LG-fuzzy manifolds then for every $\omega \in L\Omega(Y)$, we have

$$F^*d\omega = dF^*\omega$$

Definition 3.42 *There is an LG-cochain complex*

$$L\Omega^0(X) \longrightarrow^d L\Omega^1(X) \longrightarrow^d L\Omega^2(X \longrightarrow^d \ldots \longrightarrow^d L\Omega^n(X)$$

called the LG-de Rham complex, as Lemma 6.10 *shows. An LG-differential form* $\omega \in L\Omega^K(X)$ *is called closed if* $d\omega = 0$ *and is called exact if there is an*

LG-differential $(k-1)$*-form* τ *such that* $d\tau=\omega$. *Lemma* 6.10(ii) *asserts that every exact LG-differential k-form is closed and the quotient space*

$$LH^k(X)=\frac{ker\ d: L\Omega^k(X)\to L\Omega^{k+1}(X)}{im\ d: L\Omega^{k-1}(X)\to L\Omega^k(X)}$$

is called the kth LG-de Rham cohomology group of X.

Many authors like M. L. Puri, D. A. Ralescu, [15], M. L. Puri [16], R. S. Saadati, S. M .Vaezpour [19], G. Rano, T. Bag [17] and T. Bag, A. K. Samanta [1], have introduced the concept of fuzzy metric and fuzzy norm in different perceptions. We introduce the LG^{P} norm and LG-metric on $LGT_p(X)$ as follows:

Proposition 3.43 *Let* $(X,\mathfrak{T})$ *be an* C^∞ *LG-fuzzy manifold. Then* $LGT_p(X)$ *is an LG-fuzzy metric space.*

Proof Let (A,ψ) be any LG-coordinate neighborhood of p, and $H_{1p},\ \ldots,\ H_{np}$, be LG-coordinate frames of $LGT_p(X)$. Then $Z_p=\sum_{i=1}^n a^iH_{ip}\ with\ a^i\in\mathbb{R},\forall i$. So Z_p determines uniquely a family of LG-fuzzy points $\{a^i_{\lambda}\}$ on $\mathbb{R}$ where $\lambda=A(p)$ and for all $q\in\mathbb{R}$, $a^i_{\lambda}(q)=\begin{cases}\lambda & q=a^i,\\ 0 & elsewhere.\end{cases}$
Thus we have an unique LG-fuzzy vector induced by cartesian product:

$$Z_p\longleftrightarrow a_{\lambda}\qquad a_{\lambda}=\prod_{i=1}^n a^i_{\lambda}\qquad or\qquad a_{\lambda}=\begin{pmatrix}a^1_{\lambda}\\ a^2_{\lambda}\\ \vdots\\ a^n_{\lambda}\end{pmatrix}$$

We can define an LG^{P} norm on these LG-fuzzy vectors by

$$\parallel a_{\lambda}\parallel_P=\left(\sum_{i=1}^n\left(a^i_{\lambda}\lambda\right)^P\right)^{\frac{1}{P}}=\lambda\left(\sum_{i=1}^n\left(a^i_{\lambda}\right)^P\right)^{\frac{1}{P}}\tag{6}$$

If λ is 1, this reduces to the L^{P} norm on a_{λ}. The LG^{P} norm induces an LG-metric on $LGT_p(X)$:
Let $W_p\longleftrightarrow b_\lambda=(b^1_{\lambda},\ldots,b^n_{\lambda})$. Then

$$d_P(Z_p,W_p):=\parallel a_\lambda-b_\lambda\parallel_P$$

Proposition 3.44 $LGT_p(X)$ *is an LG-fuzzy reflexive Banach space with the LG-metric* d_P.

Proof Let $\{Z_p^j\}_{j=1}^{\infty}$ be a sequence of LG-tangent vectors such that $d_{_P}(Z_p^j, Z_p^{j+1}) \longrightarrow 0$.
Since any LG-tangent vector Z_p^j corresponds to an LG-fuzzy vector $a_\lambda^j = (a_\lambda^{j_1},\ \ldots,\ a_\lambda^{j_n})$, therefore we have a sequence of L-fuzzy vectors $\{a_\lambda^j\}_{_{j=1}}^{\infty}$, such that for given $\varepsilon > 0$, there exists a natural number N, such that

$$\| a_{_\lambda}^l - a_{_\lambda}^k \|_{_P} < \varepsilon \quad \forall\ l,\ k \geq N.$$

By (6.1) the $LG^{^P}$ norm of the L-fuzzy vectors $a_{_\lambda}^j$ equals to λ times ordinary $L^{^P}$ norm of them, thus we have a Caushi sequence $\{a^j\}_{_{j=1}}^{\infty}$, where $a^j = (a^{j_1},\ \ldots,\ a^{j_n}) \in \mathbb{R}^n$. Since $\mathbb{R}^n$ is a Banach space with $L^{^P}$ norm, hence there exists a vector $b = (b^1,\ \ldots,\ b^n)$ satisfying $\lim_{j\to\infty}\ a^j = b$. Hence there exists an LG-fuzzy vector b_λ satisfying

$$\lim_{j\to\infty}\ a_{_\lambda}^j = b_{_\lambda}.$$

Therefore $LGT_p(X)$ is an LG-fuzzy Banach space. Since every finite-dimensional normed space is reflexive, hence $LGT_p(X)$ is an LG-fuzzy reflexive Banach space

4 Open Problem

The open problem here is to examine under what conditions, we can construct LG-fuzzy Riemannian or Minkowski or Finsler manifolds as interesting developments of fuzzy knowledge frontiers?

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